AN INVERSE PROBLEM IN PÓLYA–SCHUR THEORY. I.
NON-DEGENERATE AND DEGENERATE OPERATORS

PER ALEXANDERSSON, PETTER BRANDÉN, AND BORIS SHAPIRO

Abstract. Given a linear ordinary differential operator $T$ with polynomial coefficients, we study the class of closed subsets of the complex plane such that $T$ sends any polynomial (resp. any polynomial of degree exceeding a given number) with all roots in a given subset to a polynomial with all roots in the same subset or 0. Below we discuss some general properties of such subsets as well as the problem of existence of a minimal under inclusion subset among those.

If a new result is to have any value, it must unite elements long since known, but till then scattered and seemingly foreign to each other, and suddenly introduce order where the appearance of disorder reigned. Then it enables us to see at a glance each of these elements at a place it occupies in the whole.

— H. Poincaré, Science and method

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1. Introduction

In 1914, generalizing some earlier results of E. Laguerre, G. Pólya and I. Schur [PS14] created a new branch of mathematics now referred to as the Pólya–Schur theory. The main result of [PS14] is a complete characterization of linear operators acting diagonally in the monomial basis of $\mathbb{R}[x]$ and sending any polynomial with all real roots to a polynomial with all real roots (or to 0). Without the requirement of diagonality of the action a characterization of such linear operators was obtained by the second author jointly with late J. Borcea [BB09].
The main question considered in the Pólya–Schur theory [CC04] can be formulated as follows.

**Problem 1.** Given a subset $S \subset \mathbb{C}$ of the complex plane, describe the semigroup of all linear operators $T : \mathbb{C}[z] \to \mathbb{C}[z]$ sending any polynomial with roots in $S$ to a polynomial with roots in $S$ (or to 0).

**Definition 2.** If an operator $T$ has the latter property, then we say that $S$ is a $T$-invariant set, or that $T$ preserves $S$.

So far Problem 1 has only been solved for the circular domains (i.e., images of the unit disk under Möbius transformations), their boundaries [BB09], and recently for strips [BC17]. Even a very similar case of the unit interval is still open at present.

It seems that for a somewhat general class of subsets $S \subset \mathbb{C}$, Problem 1 is out of reach of all currently existing methods.

In this paper, we consider an inverse problem in the Pólya–Schur theory which seems both natural and more accessible. We will restrict ourselves to consideration of closed $T$-invariant subsets.

**Problem 3.** Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, find a sufficiently large class of closed $T$-invariant sets. Ultimately, describe all closed $T$-invariant subsets of the complex plane.

For example, if $T = \frac{d^j}{dx^j}$, then a closed subset $S \subseteq \mathbb{C}$ is $T$-invariant if and only if it is convex. Although it seems too optimistic to hope for a complete solution of Problem 3 for an arbitrary linear operator $T$, we present below a number of relevant results valid for linear ordinary differential operators of finite order. Note that an arbitrary linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$ can be represented as a formal linear differential operator with polynomial coefficients, i.e., $T = \sum_{j=0}^{\infty} Q_j(x) \frac{d^j}{dx^j}$ where each $Q_j(x)$ is a polynomial, see [Pee59]. To move further, we need to introduce some basic notions.

**Definition 4.** Given a linear ordinary differential operator of order $k \geq 1$ $T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j}$ with polynomial coefficients, define its Fuchs index as

$$\rho_T = \max_{0 \leq j \leq k} (\deg(Q_j) - j).$$

Alternatively, the Fuchs index can be defined as the maximum difference between the output and input polynomial, when acted on by $T$:

$$\rho_T = \max_p (\deg(T(p)) - \deg(p)).$$

An operator $T$ is called non-degenerate if $\rho_T = \deg(Q_k) - k$, and degenerate otherwise. In other words, $T$ is non-degenerate if $\rho_T$ is realized by the leading coefficient of $T$.

We say that $T$ is exactly solvable if its Fuchs index is zero.

A few operators are shown in Table I with some of their properties listed.

**Definition 5.** Given a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$, we denote by $\mathcal{I}_n^T$ the collection of all closed subsets $S \subseteq \mathbb{C}$ such that for every polynomial of degree $n$ with roots in $S$, its image $T(p)$ is either 0 or has all roots in $S$. In this situation, we say that $S$ belongs to the class $\mathcal{I}_n^T$.

Similarly, a closed set $S$ belongs to the class $\mathcal{I}_n^T$ if for every polynomial of degree at least $n$ with roots in $S$, its image $T(p)$ is either 0 or has all roots in $S$. Obviously, the class $\mathcal{I}_0^T$ coincides with the class of all $T$-invariant sets. We say that a set
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Operator | Fuchs index | Properties
---|---|---
\((x^3 + 2x) \frac{d^3}{dx^3} + x \frac{d^2}{dx^2} + 1\) | 0 | Exactly solvable, non-degenerate
\((x + 1) \frac{d^3}{dx^3} + x^4 \frac{d^2}{dx^2} + 2x\) | 2 | Degenerate
\(x^2 \frac{d^3}{dx^3} + 4 \frac{d^2}{dx^2}\) | -1 | Non-degenerate

Table 1. Three examples of differential operators.

\(S \in I_T^n\) (resp. \(S \in I_T^{\geq n}\)) is **minimal** if there is no closed proper nonempty subset of \(S\) belonging to \(I_T^n\) (resp. \(I_T^{\geq n}\)).

**Remark 6.** Obviously, for any \(T\) and any \(n\), the whole complex plane \(\mathbb{C}\) is a trivial example of a set belonging to both \(I_T^n\) and \(I_T^{\geq n}\). On the one hand, it is more natural to study the class \(I_T^n\) in case when the operator \(T\) preserves the space of polynomials of degree \(n\). In particular, any exactly solvable operator preserves the degree of polynomials it acts upon (except for possibly finitely low degrees). Thus, for an exactly solvable operator, it makes sense to consider the class \(I_T^n\) and its elements for all \(n\) and study their behavior when \(n \to \infty\). On the other hand, for an arbitrary linear operator \(T\) it is more natural to consider non-trivial subsets of \(\mathbb{C}\) belonging to \(I_T^{\geq n}\), where \(n\) is any non-negative integer. Observe that families of sets belonging to \(I_T^n\) (resp. \(I_T^{\geq n}\)) are closed under taking the intersection.

In the present paper (which is the first part of two) we study the class \(I_T^{\geq n}\) for an arbitrary \(T\) of the form (1.1). The sequel article [ABS19] is devoted to the study of the class \(I_T^n\) and also of the so-called Hutchinson invariant sets for exactly solvable operators and their relation to the classical complex dynamics.

The structure of the paper is as follows. In Section 2, we present and prove some general results about \(I_T^{\geq n}\) for an arbitrary operator \(T\) (with non-constant leading term). In Section 3, we prove all results related to non-degenerate operators. In Section 4 and Section 5, we prove all results related to degenerate differential operators including operators with constant leading term. Finally, Section 7 contains a number of open problems connected to this topic.

**Acknowledgements.** Research of the third author was supported by the grant VR 2016-04416 of the Swedish Research Council. He wants to thank Lionel Lang and August Tsikh for discussions related to tropical algebraic geometry.

2. **General properties of invariant sets**

**Definition 7.** Given an operator \(T\) of the form (1.1) with \(Q_k(x)\) different from a constant, denote by \(\text{Conv}(Q_k) \subset \mathbb{C}\) the convex hull of the zero locus of \(Q_k(x)\). We refer to \(\text{Conv}(Q_k)\) as the **fundamental polygon** of \(T\).

The next proposition contains basic information about invariant sets in \(I_T^{\geq n}\).

**Theorem 8.** The following facts hold:

1. for any operator \(T\) as in (1.1) and any non-negative integer \(n\), every \(S \in I_T^{\geq n}\) is convex;
2. for any operator \(T\) as in (1.1) and any non-negative integer \(n\), if \(S\) is an unbounded closed set belonging to \(I_T^{\geq n}\), then \(S\) is \(T\)-invariant, i.e., \(S\) belongs to \(I_T^{\geq 0}\).
(3) for any $T$ as in (1.1) with $Q_k(x)$ different from a constant and any non-negative integer $n$, every $S \in \mathcal{T}_{\geq n}$ contains the fundamental polygon $\text{Conv}(Q_k)$;
(4) for any $T$ as in (1.1) with $Q_k(x)$ different from a constant and any non-negative integer $n$, the set $\mathcal{T}_{\geq n}$ has a unique minimal (under inclusion) element.

Proof. **Item (i).** Fix $S \in \mathcal{T}_{\geq n}$ and choose $x_1, x_2 \in S$. Take $p(x) = (x - x_1)^m(x - x_2)^m$ for sufficiently large $m$, and consider $p^{(i)}(x)$. Then

$$p^{(i)}(x) = \sum_{j=0}^{\ell} \binom{\ell}{j} \frac{m!}{(m-j)!} \frac{m!}{(m+j-n)!} (x - x_1)^{m-j}(x - x_2)^{m+j-\ell}$$

which implies that

$$q(x) := \frac{p^{(i)}(x)}{(x - x_1)^{m-\ell}(x - x_2)^{m-\ell}} = \sum_{j=0}^{\ell} \binom{\ell}{j} (m)_j (m)_{\ell-j} (x - x_1)^{\ell-j}(x - x_2)^j.$$

Dividing both sides by $m^{\ell}$ and expanding the Pochhammer symbols, we see that

$$m^{-\ell}q(x) = \left( \sum_{j=0}^{\ell} \binom{\ell}{j} (x - x_1)^{\ell-j}(x - x_2)^j \right) + \frac{R_1(x)}{m} + \frac{R_2(x)}{m^2} + \cdots$$

$$= ((x - x_1) + (x - x_2))^\ell + O(m^{-1})R(x).$$

Using the latter expression, we obtain

$$p^{(i)} = m^{\ell}((x - x_1)(x - x_2))^{m-\ell} \left( (2x - x_1 - x_2)^\ell + O(m^{-1})R(x) \right).$$

Therefore,

$$T(p(x)) = Q_k((x - x_1)(x - x_2))^{m-k} \left( (2x - x_1 - x_2)^\ell + O(m^{-1})R(x) \right)$$

$$+ \sum_{j=1}^{k} \binom{k_j}{j}(x - x_1)(x - x_2))^{m-k+j} \left( (2x - x_1 - x_2)^\ell + O(m^{-1})R_j(x) \right).$$

All terms in the above sum go to 0 as $m$ gets large, implying that the roots of $T(p(x))$ are close to that of

$$Q_k((x - x_1)(x - x_2))^{m-\ell}(2x - x_1 - x_2)^\ell.$$

Since $\frac{a_1 + a_2}{2}$ is a root of the latter polynomial, the original set $S$ were convex.

**Item (ii).** Assume that $S$ is an unbounded set belonging to $\mathcal{T}_{\geq n}$ for some positive $n$. Take some polynomial $p$ of degree less that $n$ with roots in $S$. Consider a 1-parameter family of polynomials of degree $n$ of the form $P_t := (x - \alpha(t))^{n-\deg p(x)}$, $t \in [0, +\infty)$, where $\alpha(t)$ is a variable point in $S$ which continuously depends on $t$ and escapes to $\infty$ when $t \to +\infty$. (Such a family obviously exists since $S$ is convex and unbounded.) Consider the polynomial family $T(P_t)$. Since $S \in \mathcal{T}_{\geq n}$, the roots of $T(P_t)$ belong to $S$ for any finite $t$ and continuously depend on $t$. Since $S$ is closed the same holds for the limit of the roots of $T(P_t)$ which do not escape to infinity. Notice that the set of finite limiting roots exactly coincides with the set of roots of $T(p)$ which finishes the proof of item (ii).

**Item (iii).** Choose arbitrary $T$ with $Q_k(x)$ different from a constant, any non-negative integer $n$, and an arbitrary set $S \in \mathcal{T}_{\geq n}$. Take $p(x) = (x - \alpha)^m$, where $\alpha \in S$. Then

$$\frac{T(p(x))}{(m)_k} = \sum_{j=0}^{k} \binom{k}{j} (m)_j (x - \alpha)^{m-j}.$$
If we let $m \to \infty$, then $(m_j)_{j<k} \to 0$ for $j<k$. Hence, the roots of $T(p(x))$ approach that of $Q_k(x)(x-\alpha)^{m-k}$ as $m$ grows.

**Item (iv).** Observe that for any differential operator $T$ as above, the set $I_{\geq n}^T$ is non-empty since it at least contains the whole $\mathbb{C}$. Now notice that by items (i) – (ii), the intersection of all sets in $I_{\geq n}^T$ is non-empty. Indeed each of them contains all roots of $Q_k(x)$. Since this intersection is convex it also contains the convex hull $\text{Conv}(Q_k)$ of the roots of $Q_k(x)$. Since $I_{\geq n}$ consists of closed convex sets with a nonempty common intersection, there is the unique minimal set in $I_{\geq n}^T$.

The unique minimal element in $I_{\geq n}^T$ whose existence is given in Theorem 8(iv) is denoted $M_{\geq n}^T$. The following consequence of Theorem 8 is straightforward.

**Corollary 9.** (i) Under the assumption that $Q_k(x)$ is not constant, one has the sequence of inclusions of closed convex sets

$$M_{\geq 0}^T \supseteq M_{\geq 1}^T \supseteq \cdots. \quad \text{(2.1) \{eq:limit\}}$$

(ii) Under the same assumption, if for some $n$, there exists a compact set $S \in I_{\geq n}^T$, then $M_{\geq m}^T$ is compact for all $m \geq n$ and there exists a well-defined limit

$$M_{\infty}^T := \lim_{n \to \infty} M_{\geq n}^T. \quad \text{(2.2) \{eq:limit2\}}$$

Obviously, $M_{\infty}^T$ is a closed convex compact set.

**Remark 10.** The assumption that $Q_k(x)$ is different from a constant is important for the existence of the unique minimal under inclusion element in $I_{\geq n}^T$. Many operators with a constant leading term violate this property. For example, for $T = \frac{d}{dx}$, every convex closed subset of $\mathbb{C}$ belongs to $I_{\geq n}^T$ for every non-negative integer $n$. In fact, every point in $\mathbb{C}$ is a minimal set for $T = \frac{d}{dx}$. More details about operators with a constant leading term can be found in Section 4.

**Remark 11.** Corollary 16 of the next section implies that for a non-degenerate $T$, the minimal set $M_{\geq n}^T$ is compact for any sufficiently large $n$. However this compactness property might fail for small $n$. As we will show later in Proposition 21 of §4 for any degenerate operator $T$ and non-negative integer $n$, every set in $I_{\geq n}^T$ and, in particular, $M_{\geq n}^T$ is unbounded implying that compact invariant sets exist for non-degenerate operators only. there exists a compact set $S \in I_{\geq n}^T$ for some non-negative integer $n$, if and only if $T$ is non-degenerate.

### 3. Non-degenerate Operators

The main result of this section is Corollary 16 claiming that for a fixed non-degenerate differential operator $T$, there exists a nonnegative integer $n$ such that $I_{\geq n}^T$ contains all sufficiently large disks. This implies compactness of the minimal set $M_{\geq n}^T$ for large $n$. Unfortunately, at present we do have an explicit description the boundary of $M_{\geq n}^T$ for a given $T$ and $n$. Our best result in this direction is Proposition 21 which describes the boundary of the limit $M_{\infty}^T$.

The next example shows that Corollary 16 is the best we can hope for, as there exist non-degenerate exactly solvable operators for which $M_{\geq n}^T$ is non-compact for small values of $n$.

**Example 12.** Consider the non-degenerate exactly solvable operator given by

$$T = \left( -\frac{x^2}{4} + \frac{x}{4} \right) \frac{d}{dx^2} + \left( \frac{x}{4} - \frac{1}{2} \right) \frac{d}{dx} + 1. \quad \text{(3.1)}$$
We have chosen $T$ in such a way that for every $z \in \mathbb{C}$,
\[
T \left( (x - z)^j \right) = (x - (2z)) \left( x - \left( \frac{x}{2} + \frac{1}{2} \right) \right).
\]
(3.2)\{eq:exactlySolvNonCompact\}

Take any closed subset $S \in T_{\geq 2}$. The first factor in (3.2) ensures that if $z \in S$, then we also have $2z \in S$. The second factor ensures that if $z \in S$, then $\frac{1}{2}(z + 1) \in S$. These two facts imply that $S$ must contain the interval $[1, \infty)$ of the real axis. In particular, the minimal set $M^0_{\geq 2}$ cannot be bounded.

### MAYBE A PICTURE HERE!

#### 3.1. Existence of invariant disks.

Define the $n^{th}$ Fuchs index of a linear operator $T : \mathbb{C}[x] \to \mathbb{C}[x]$ to be
\[
\rho = \rho_n = \max_{0 \leq j \leq n} (\deg T(x^j) - j),
\]
and call $T$ non-degenerate if $\deg T(x^n) - n = \rho_n$. Set $G_T(x,y) := T[(1+xy)x^n]$ and note that there are polynomials $P_{\ell}^n, \ell = -n, \ldots, \rho$, of degree at most $n$, such that
\[
G_T(x,y) = \sum_{-n \leq \ell \leq n} x^\ell P_{\ell}^n(xy).
\]
(3.4)

Thus $T$ is non-degenerate if and only if the degree of $P^\rho$ is $n$. When $T = \sum_{j=0}^k \xi_j \frac{d^j}{dx^j}$ is a differential operator of order $k$,
\[
G_T(x,y) = \sum_{j=0}^k j! x^{-j} Q_j(x) \binom{n}{j} (xy)^j (1 + xy)^{n-j},
\]
and it follows that
\[
P_{\ell}^n(x) = \sum_{j=0}^k j! a_{\ell,j} \binom{n}{j} x^j (1 + x)^{n-j},
\]
(3.5)\{prn\}

where $a_{\ell,j}$ is the coefficient of $x^{j+\ell}$ in $Q_j(x)$.

In what follows, $D_R$ denotes the open disk $\{ z \in \mathbb{C} : |z| < 1 \}$, and $\bar{D}_R$ is the closure of $D_R$. We also define $\Omega_R$ as the open set $\{ (z,w) \in \mathbb{C}^2 : |z| > R \text{ and } |w| > 1/R \}$.

**Proposition 13** ([B199] Thm.7). Let $T : \mathbb{C}[x] \to \mathbb{C}[x]$ be a linear operator of rank greater than one. The disk $\bar{D}_R$ is $n$-invariant if and only if $G_T(z,w) \neq 0$ for all $(z,w) \in \Omega_R$. \prop{bb09}

**Theorem 14.** Suppose $T : \mathbb{C}[x] \to \mathbb{C}[x]$ is a non-degenerate linear operator with $n^{th}$ Fuchs index $\rho$. Let $g(x)$ be the greatest common divisor of $\{ P_{\ell}^n(x) \}_{\ell}$. Then the closed disk $\bar{D}_R = \{ z : |z| \leq R \}$ is $n$-invariant for all sufficiently large $R > 0$ if and only if

1. all zeros of $g(x)$ lie in $\{ z : |z| \leq 1 \}$, and
2. all zeros of $P_{\rho}^n(x)/g(x)$ lie in $\{ z : |z| < 1 \}$.

**Proof.** Suppose $T : \mathbb{C}[x] \to \mathbb{C}[x]$ is a non-degenerate linear operator. We first prove that conditions (1) and (2) are sufficient. Assume (1) and (2). Since $\deg P_{\rho}^n \leq \deg P_{\rho}^n = n$ for all $j$, and the zeros of $P_{\rho}^n(x)/g(x)$ lie in the open unit disk, there is a positive constant $C$ such that $|P_{\ell}^n(z)/P_{\rho}^n(z)| < C$ for all $|z| \geq 1$ and all $\ell$. Hence, for sufficiently large $R$: If $(z,w) \in \Omega_R$, then
\[
|G_T(z,w)| = \left| \prod_{\ell=-n}^{\rho-1} z^{-\ell} P_{\ell}^n(zw) \right| \leq \prod_{\ell=-n}^{\rho-1} R^{-(\rho-\ell)}C < \frac{C}{R-1} < 1.
\]
(3.6)\{trett\}

For such $R$, the disk $\bar{D}_R = \{ z : |z| \leq R \}$ is $n$-invariant by Proposition 13.

If $g(x)$ has a zero in $\{ z : |z| \leq 1 \}$, then $G_T(z,w) = 0$ for some $(z,w) \in \Omega_R$, and by Proposition 13 the disk $\bar{D}_R$ is not invariant. Suppose $(P_{\rho}^n/g)(\xi) = 0,$ where
\(|\xi| \geq 1\), and that \(g(z)\) has no zeros in \(\{z : |z| \leq 1\}\). Consider a sequence \(\{\xi_j\}_{j=1}^\infty\), where \(P^\rho_n(\xi_j) \neq 0\), \(|\xi_j| > 1\), and \(\lim_{j \to \infty} \xi_j = \xi\). Let now
\[
B_j(z) := G_T(z, \xi_j/z)/g(\xi_j) = \sum_{\ell \leq \rho} z^\ell (P^\rho_\ell / g)(\xi_j).
\]
Since \((P^\rho_\ell / g)(\xi) = 0\), we see that at least one zero, say \(z_j\), of \(B_j(z)\) tends to \(\infty\) as \(j \to \infty\). Hence for \(R_j := \frac{1}{2} |z_j|(1 + |\xi_j|)\) we have \((z_j, \xi_j/z_j) \in \Omega_{R_j}\), while \(G_T(z_j, \xi_j/z_j) = 0\).

Recall that the map \(z \mapsto z/(1 + z)\) sends the set \(\{z \in \mathbb{C} : \text{Re}(z) \geq -1/2\}\) to the unit disk.

Theorem 15. Suppose \(T = \sum_{j=0}^k Q_j(x) \frac{d^j}{dx^j}\) is a non-degenerate differential operator of order \(k\). Let \(n \geq k\), let \(\rho\) be the \(n\)th Fuchs index of \(T\), and \(a_{\rho,j}\) be the coefficient of \(x^{n+j}\) in \(Q_j\). Consider
\[
f^n_\omega(x) := \sum_{j=0}^k j! a_{\rho,j} \binom{n}{j} x^j. \tag{3.7} \label{eq:thmPoly}
\]
If \(f^n_\omega(1) \neq 0\), and all zeros of \(f^n_\omega(x)\) have real part greater than \(-1/2\), then \(D_R\) is \(n\)-invariant for all sufficiently large \(R\).

Proof. The condition \(f^n_\omega(1) \neq 0\) guarantees that \(P^\rho_n(\xi)\) has degree \(n\), by (3.5). Note that \((1 + x)^{n-k}\) divides \(P^\rho_n(\xi)\) for all \(\ell\). Since \(T\) is non-degenerate, \(a_{\rho,k} \neq 0\). Now,
\[
P^\rho_n(x)/(1 + x)^{n-k} = \sum_{j=0}^k j! a_{\rho,j} \binom{n}{j} \left(\frac{x}{1 + x}\right)^j (1 + x)^k,
\]
has all its zeros in \(\{z : |z| < 1\}\) if and only if all zeros of \(f^n_\omega(x)\) have real part greater than \(-1/2\). The proof now follows from Theorem 14.

Corollary 16. If \(T\) is a non-degenerate differential operator, then there is an integer \(N_0\) and a positive number \(R_0\) such that \(D_R := D(0, R)\) is \(n\)-invariant whenever \(n \geq N_0\) and \(R \geq R_0\).

Proof. Note that the zeros of \(f^n_\omega\) approach 0 as \(n \to \infty\). Since \(a_{\rho,k} \neq 0\), we see by (3.5) that there are positive numbers \(N_0\) and \(C\) such that
\begin{itemize}
  \item \(|P^\rho_n(z)/P^\rho_n(\xi)| < C\) for all \(|z| \geq 1\), all \(\ell\), and all \(n \geq N_0\),
  \item \(\sum_{j=0}^k j! a_{\rho,j} \binom{n}{j} \neq 0\),
  \item the zeros of \(f^n_\omega(z)\) are in \(D(0, 1/2)\).
\end{itemize}
Hence the estimate in (3.6) can be made uniform in \(n\). Indeed, we may choose \(R_0 = C + 1\).

Remark 17. Note that by item II of Theorem 8 if \(T\) is a linear operator and \(\Omega \subseteq \mathbb{C}\) is closed and unbounded, then \(\Omega\) is \(n\)-invariant if and only if it is \(\ell\)-invariant for all \(\ell \leq n\). Indeed if \(f(z)\) has degree \(\ell \leq n\) we may take a sequence \(\{w_j\}_{j=1}^\infty\) in \(\Omega\) for which \(|w_j| \to \infty\) as \(j \to \infty\). Then the zeros of
\[
T(f) = \lim_{j \to \infty} T[(1 - x/w_j)^{n-\ell} f(z)]
\]
is in \(\Omega\) by Hurwitz’ theorem.

The following important notion can be found in [BB09, Def. 1].

Definition 18. A polynomial \(f(z_1, \ldots, z_\ell) \in \mathbb{C}[z_1, \ldots, z_\ell]\) is called stable if for all \(\ell\)-tuples \((z_1, \ldots, z_\ell) \in \mathbb{C}^\ell\) with \(\text{Im}(z_j) > 0, 1 \leq j \leq \ell\), one has \(f(z_1, \ldots, z_\ell) \neq 0\).
Proposition 19. Let $H$ be a closed half-plane given by $H = \{az + b : \text{Im}(z) \leq 0\}$ and let $T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j}$ be a differential operator. The following are equivalent:

1. The set of positive integers for which $H$ is $n$-invariant is unbounded,
2. $H$ is $n$-invariant for all $n \geq 0$,
3. The polynomial $\sum_{j=0}^{k} Q_j(ax+b)(-y/a)^j$ as an element in $\mathbb{C}[x, y]$ is a stable polynomial in $(x, y)$.

Proof. By Remark 17 we see that (1) and (2) are equivalent. Now, (2) is equivalent to that the operator $S : \mathbb{C}[x] \to \mathbb{C}[x]$ defined by

$$S(f)(x) = T(f(\phi^{-1}(x)))(\phi(x)),$$

where $\phi(x) = ax + b$, preserves stability. The operator $S$ is again a differential operator, so the equivalence of (2) and (3) now follows from □

Example 20. Consider the operator $T : \mathbb{C}_n \to \mathbb{C}$ given by

$$T = (x^2 - x^3) \frac{d^3}{dx^3} + (x + x^2) \frac{d^2}{dx^2} + (2x) \frac{d}{dx} - 6.$$ (3.8)

When $n = 3$, we have that for every $z \in \mathbb{C}$,

$$T [(x - z)^3] = 12 (x - z^2) (x - z/2).$$ (3.9)

In particular, if $z$ is in an invariant set, then $z^2$ is also in the set. Thus, there cannot be any large invariant disk. However, this does not violate Theorem 17: the 3rd Fuchs index of $T$ is 0, but $P_0^n(x) = -6(1 + 2x)$. Hence, the operator is degenerate and the theorem does not apply.

3.2. Description of the limit minimal set $M^T_{\infty}$. Recall that in Corollary 8 we showed that whenever the leading coefficient $Q_k(x)$ in an operator $T$ is non-zero, then there is a minimal invariant set $M^T_{\infty}$ containing the convex hull of the roots of $Q_k(x)$. Furthermore, if $T$ is non-degenerate, Corollary 16 implies that $M^T_{\infty}$ is compact.

Theorem A (See [Sha10, Thm. 9]). Given a non-degenerate operator $T$ as in (1.1) and $\epsilon > 0$, there exists a positive integer $n_\epsilon$ such that for any $n > n_\epsilon$ and any polynomial $p$ of degree $n$ with all roots in Conv($Q_k$), all roots of $T(p)$ lie in the $\epsilon$-neighborhood of Conv($Q_k$).

Conjecture 21. For any non-degenerate $T$, $M^T_{\infty} = \text{Conv}(Q_k)$.

Proof. By (iii)–(iv) of Theorem 8 we know that for any non-degenerate operator $T$, the minimal sets $M^T_{\geq n}$ exist and are compact for all sufficiently large $n$; they contain Conv($Q_k$), and satisfy the sequence of inclusions

$$M^T_{\geq n} \supseteq M^T_{\geq n+1} \supseteq M^T_{\geq n+2} \supseteq \cdots$$

The fact that $M^T_{\infty} := \lim_{n \to \infty} M^T_{\geq n}$ equals Conv($Q_k$) follows from Theorem A □

Let us now describe a special class of non-degenerate operators for which all $M^T_{\geq n}$ coincide.
Proposition 22. Take a non-degenerate operator of the form $T = Q_k(x) \frac{\partial^k}{\partial x^k} + Q_{k-1}(x) \frac{\partial^{k-1}}{\partial x^{k-1}}$ satisfying the condition
\[
\frac{Q_{k-1}(x)}{Q_k(x)} = \sum_{i=1}^{\deg Q_k} \frac{\kappa_i}{x - x_i},
\] \tag{3.10}
where $\kappa_i \geq 0$ and $\{x_1, \ldots, x_{\deg Q_k}\}$ is the set of all roots of $Q_k(x)$. Then,
\[
M^T = M_{-1}^T = M_{-2}^T = \cdots = M_1^T = \text{Conv}(Q_k).
\]

Proof. By (iii) of Theorem 8, it suffices to show that under our assumptions on $T$, Conv$(Q_k)$ is a $T$-invariant set. Moreover by Gauss–Lucas theorem, for $T = Q_k(x) \frac{\partial^k}{\partial x^k} + Q_{k-1}(x) \frac{\partial^{k-1}}{\partial x^{k-1}}$ satisfying (3.10), it suffices to show that Conv$(Q_k)$ is $\hat{T}$-invariant where $\hat{T} = Q_k(x) \frac{\partial^k}{\partial x^k} + Q_{k-1}(x)$. Assume now that $p(x)$ is an arbitrary polynomial of some degree $n$ whose roots $r_1, \ldots, r_n$ lie in Conv$(Q_k)$ and consider $q = \hat{T}(p)$. We want to show that $q(z) \neq 0$ for any $z \in \mathbb{C} \setminus \text{Conv}(Q_k)$. Assume $q(z) = 0$ which is equivalent to
\[
Q_k(z)p'(z) + Q_{k-1}(z)p(z) = 0 \iff \frac{p'(z)}{p(z)} = -\frac{Q_{k-1}(z)}{Q_k(z)}.
\] \tag{3.11}

The latter expression is equivalent to
\[
\sum_{j=1}^{n} \frac{1}{z - r_j} = -\sum_{i=1}^{\deg Q_k} \frac{\kappa_i}{z - x_i},
\]
where $\{x_1, \ldots, x_{\deg Q_k}\}$ is the set of roots of $Q_k$ and $\kappa_i \geq 0$. Assuming that $z \notin \text{Conv}(Q_k)$, choose a line $L$ separating $z_1$ from Conv$(Q_k)$. By our assumptions, $L$ separates $z$ from all $r_j$’s and all $x_i$’s. Because of this and taking into account the signs, one can easily conclude that the left-hand side of the latter expression is a complex number pointing from $z_1$ to the halfplane not containing $z$ and the right-hand side does the opposite. Therefore, (3.11) can not hold if $z \notin \text{Conv}(Q_k)$.

A special case of Proposition 22 when $Q_k(x)$ is a real-rooted polynomial follows from more general results of Bra10.

4. DEGENERATE OPERATORS: PRELIMINARIES

In our study of degenerate operators we will use some classical results about root asymptotics of bivariate polynomials which is modern terminology belong to the tropical geometry. We start by introducing the domination partial order on points in $\mathbb{R}^2$. Namely, we say that a point $p = (u, v) \in \mathbb{R}^2$ dominates a point $p' = (u', v')$ if $u \geq u'$ and $v \geq v'$. Given a subset $S \subseteq \mathbb{R}^2$, we call by its northeastern border $\text{NE}_S$ the set of all points in $S$ which are not dominated by other points in $S$. Observe that $\text{NE}_S$ can be empty if $S$ is non-compact, but for compact $S$, $\text{NE}_S$ is always nonempty. Furthermore, if $S$ is both compact and convex then $\text{NE}_S$ is contractible.

4.1. (Tropical) algebraic preliminaries and three types of Newton polygons. Given a bivariate polynomial $R(u, v) = \sum_{(i, j) \in \Theta} a_{i,j} u^i v^j$, denote by Conv$(R) \subset \mathbb{R}^2$ its Newton polygon, i.e. the convex hull of the set of exponents $(i, j) \in \Theta$. The northeastern border of Conv$(R)$ will be denoted by $\text{NE}_R$, see examples in Fig. 1 and Fig. 2. By the above, $\text{NE}_R$ is connected and contractible. The point of $\text{NE}_R$ with the maximal value of $u$ will be called the \textit{eastern vertex} and denoted by $V_e$ and the point of $\text{NE}_R$ with the maximal value of $v$ will be called the \textit{northern vertex} and denoted by $V_n$. The set $\text{NE}_R$ coincides with a point if and only if $V_e = V_n$. Notice that every edge of the boundary of Conv$(R)$ included in $\text{NE}_R$ has a negative slope. Finally, denote by $R^{ne}(u, v)$ the restriction of $R(u, v)$ to the subset $\Theta^{ne} \subseteq \Theta$. 

\section*{References}

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consisting of all monomials whose exponents are the vertices of $\text{NE}_R$. We will call $R^{\text{ne}}(u,v)$ the \textit{essential part} of $R(u,v)$.

\textbf{Remark 23.} Observe that for any bivariate $R(u,v)$ and $\alpha, \beta \in \mathbb{C}$, the change of variables $u = \tilde{u} + \alpha, v = \tilde{v} + \beta$ does not change neither $\text{NE}_R$ nor $R^{\text{ne}}(u,v)$.

Given an arbitrary bivariate polynomial

$$R(u,v) = \sum_{(i,j) \in \Theta} a_{i,j} u^i v^j = \sum_{j=0}^{m} R_j(v) u^j$$

and some number $w \in \mathbb{C}$, denote by $U_R(w)$ the set of zeros of the equation $R(u,w) = 0$ in the variable $u$ considered as the divisor in $\mathbb{C}$, i.e. zeros are counted with multiplicities. Here $m$ is the degree of $R$ w.r.t. $u$. Assume that the parameter $w$ runs over the portion of the positive half-axis $[\kappa, +\infty)$ which contains no root of $R_m(v)$; one can always choose $\kappa$ sufficiently large so that the latter condition is satisfied. (Obviously, for all $w \in [\kappa, +\infty)$, the degree of the divisor $U_R(w)$ equals $m$.) We define the subdivisor $U_R^\infty(w) \subset U_R(w)$ as the set of all roots $u(w)$ whose absolute values tend to $+\infty$ when $w$ tends to $+\infty$ along the positive half-axis. Notice that $U_R^\infty(w)$ is well-defined for all sufficiently large positive $\kappa > \kappa$ since there exists $\tilde{\kappa}$ such that for any $w \in [\tilde{\kappa}, +\infty)$, the absolute value of every root in $U_R^\infty(w)$ will be strictly larger than the absolute value of any root in the complement $U_R(w) \setminus U_R^\infty(w)$.

Our next goal is to describe $U_R^\infty(w)$ in terms of $R^{\text{ne}}(u,v)$. In what follows we will frequently use the following statement.

Given an arbitrary bivariate polynomial $R(u,v)$ whose $\text{NE}_R$ is not a single point, decompose $\text{NE}_R$ into the (disjoint) union of consecutive edges $\text{NE}_R = \cup_{s=1}^{h} e_s$ covering $\text{NE}_R$ from north to east. That is $e_1$ starts at $V_n$, $e_h$ ends at $V_e$, and each $e_s$ is adjacent to $e_{s+1}$, see Fig. 1. The absolute values of the slopes of $e_1, \ldots, e_h$ are strictly increasing. The following statement can be easily deduced from the known results of [Č48, Section 38, Th. 63–66], and [Wal78, Ch. 4, Sections 3 and 4]. (To use the latter results, one has to substitute $u$ and $v$ by $u^{-1}$ and $v^{-1}$ respectively.)

\textbf{Proposition 24.} The degree of the divisor $U_R^\infty(w)$ is equal to $i_e - i_n$, where $V_e = (i_e, j_e)$ and $V_n = (i_n, j_n)$. In other words, $\deg U_R^\infty(w)$ equals the length of the projection of $\text{NE}_R$ onto the $u$-axis.

Additionally, $U_R^\infty(w)$ splits into $h$ subdivisors $U_s^\infty(w), \ldots, U_h^\infty(w)$ corresponding to the edges $e_1, \ldots, e_h$ respectively; the degree of $U_s^\infty(w)$, $s = 1, \ldots, h$ equals the length of the projection of $e_s$ on the $u$-axis. All zeros in the divisor $U_s^\infty(w)$ have the asymptotic growth $u \sim e^{sl_s}$ where $l_s$ is the absolute value of the slope of $e_s$.

Possible values of $\epsilon$ can be found by substituting $e^{sl_s}$ in the restriction of $R(u,v)$ to the monomials contained in the edge $e_s$ and finding the non-vanishing roots of this restriction.

\textbf{Definition 25.} Given an arbitrary bivariate polynomial $R(u,v)$ whose northeastern border $\text{NE}_R$ is not a single point, we will call the slopes of edges in $\text{NE}_R$ the \textit{characteristic exponents} of $R(u,v)$. For a given edge $e_s \in \text{NE}_R$, all possible values of $\epsilon$ corresponding to the restriction of $R(u,v)$ to this edge will be called the \textit{leading constants} corresponding to (the characteristic exponent of) $e_s$. The union of all leading constants of $R(u,v)$ will be denoted by $\Upsilon_R$.

\textbf{Example 26.} To illustrate Proposition 24 and Definition 25 take

$$R(u,v) = u^8 + u^7 v^2 + u^5 v^4 + (5 + 7\sqrt{-1})u^3 v^6 - 23uv^7.$$  

One can easily check that all monomials in $R(u,v)$ belong to $\text{NE}_R$ which consists of three edges $e_1, e_2, e_3$ connecting $(1,7)$ with $(3,6)$, $(3,6)$ with $(7,2)$, and $(7,2)$ with $(8,0)$ resp. (The exponent $(5,4)$ of the second monomial belongs to $e_2$.)
Corollary 27. In the above notation, for a given bivariate polynomial \( R(u, v) \), the family of convex hulls of \( U^\infty_R(w) \) converges to \( \mathbb{C} \) when \( w \to +\infty \) if and only if the convex hull of \( \Upsilon_R \) contains 0 as its internal point.

Proof. (Sketch) This statement is rather obvious since if 0 is an interior point of the convex hull of \( \Upsilon_R \), then the roots in \( U^\infty_R(w) \) will be asymptotically moving to infinity when \( w \to +\infty \) in the directions prescribed by all values of \( \epsilon \in \Upsilon_R \) and their convex hull will contain the centered at 0 disk of any given radius for sufficiently large \( w \).}

Let us fix a connected contractible piecewise linear curve \( \text{NE} \subset \mathbb{R}^2 \) with integer vertices consisting of pairwise non-dominating points, see Fig. [1]. In other words, \( \text{NE} \) is a piecewise linear path with integer vertices whose edges have negative slopes and their absolute values increase when moving down the path. Denote by \( \text{Pol}(\text{NE}) \) the set of all bivariate polynomials whose northeastern border coincides with \( \text{NE} \). (In particular, we assume that all coefficients at the corners/endpoints of \( \text{NE} \) are non-vanishing. \( \text{Pol}(\text{NE}) \) is a Zariski-open subset of a finite-dimensional linear space of bivariate polynomials.) Recall that the \textit{integer length} of a closed straight interval \( I \subset \mathbb{R}^2 \cap \mathbb{Z}^2 \) is the number of points from \( \mathbb{Z}^2 \) contained in \( I \), i.e. the number of integer points belonging to \( I \).
Definition 28. Given \( NE \subset \mathbb{R}^2 \) as above, we call it
(i) defining if there exists an edge in \( NE \) with the slope \(-\alpha/\beta\) where \( \alpha \) and \( \beta \) are coprime positive integers and \( \beta \geq 3 \);
(ii) almost defining if there are no edges as in (i), but there either
   a) exists at least one edge in \( NE \) with the slope \(-\alpha/2\) and whose integer length is larger than 2, or
   b) there exist at least two edges with the slope \(-\alpha/2\) and integer length at least 2;
(iii) non-defining in the remaining case i.e., when either all edges of \( NE \) or all edges but one have negative integer slopes and the remaining edge has a negative half integer slope and integer length 2.

Figure 2. Examples of defining/almost defining/non-defining Newton polygons which are listed in Definition 28. The slopes of the edges of the northeastern boundary have been expressed as fractions, such that the length of the projection is in the denominator.

Definition 29. A Newton polygon \( N \subset \mathbb{R}^2 \) is called defining/almost defining/non-defining if its northeastern border contains at least one edge and is defining/almost defining/non-defining respectively.

In Fig. 2 we show Newton polytopes illustrating Definition 28 and Definition 29.

Proposition 30. Given \( NE \subset \mathbb{R}^2 \) as above, the convex hull of \( U_R^\infty(w) \) converges to \( C \), when \( w \to +\infty \)
(i) for any \( R \in \text{Pol}(NE) \) if \( NE \) is defining;
(ii) for almost any \( R \in \text{Pol}(NE) \) if \( NE \) is almost defining;
(iii) if \( NE \) is non-defining there is a full-dimensional subset of \( \text{Pol}(NE) \) for which the convex hull of \( U_R^\infty(w) \) converges to \( C \) when \( w \to +\infty \) and the complement of the latter set in \( \text{Pol}(NE) \) is also full-dimensional.

Remark 31. In case (ii), the condition of nongenericity is given by the fact that all \( \epsilon \in \Upsilon_R \) are real proportional to each other (i.e. they lie on the same real line in \( \mathbb{C} \) passing through the origin);
   In case (iii) if one forces the next to the leading coefficient for some edge with integer slope and length of projection larger than 2 to vanish, i.e. one forces the sum of the respective \( \epsilon \) to vanish, then the conclusion of Corollary 27 will be valid for a generic choice of the remaining coefficients at the vertices belonging to this edge.
If the convex hull of $\mathcal{U}_R^\infty(w)$ does not tend to $\mathbb{C}$, but $i_n > 0$ which means that $\mathcal{U}_R(w) \setminus \mathcal{U}_R^\infty(w)$ is nonempty, then the convex hull of $\mathcal{U}_R(w)$ will tend to the convex cone with apex at 0 spanned by the elements of $\Upsilon_R$.

**Proof of Proposition 30.** By Corollary 27 we need to prove that the convex hull of $\Upsilon_R$ contains 0 as its internal point

(i) for any $R \in \text{Pol}(\text{NE})$ if $\text{NE}$ is defining;
(ii) for almost any $R \in \text{Pol}(\text{NE})$ if $\text{NE}$ is almost defining;
(iii) if $\text{NE}$ is non-defining, polynomials $R \in \text{Pol}(\text{NE})$ for which $\Upsilon_R$ contains 0 as an internal point form a full-dimensional set with the full-dimensional complement.

Indeed, assume that $\text{NE}$ is defining. Then it contains an edge $e_s$ with the slope $-\alpha/\beta$ where $\alpha$ and $\beta$ are coprime positive integers and $\beta \geq 3$. Take any polynomial $R(u, v) \in \text{Pol}(\text{NE})$ and denote by $R_s(u, v)$ the restriction of $R$ to $e_s$. Substituting $u = e^{\alpha/\beta}$ in the equation $R_s(u, v) = 0$ and factoring out a power of $v$, we get a univariate algebraic equation defining $\epsilon$ which only involves powers of $\epsilon$ which are multiples of $b \geq 3$. Since every nonvanishing $\epsilon$ appears in $\Upsilon_R$ together with all $\epsilon \cdot e^{2\pi\epsilon \cdot \ell/b}$ for $\ell = 1, \ldots, b - 1$ one obtains that 0 lies in the interior of the convex hull of $\Upsilon_R$.

Assume now that $\text{NE}$ is almost defining. Then it either contains an edge $e_s$ with the slope $-\alpha/2$ and length greater than 2 or two edges with half integer slopes and length 2 each. (All the remaining edges have integer slopes.) In the former case, the algebraic equation satisfied by epsilon has an even degree exceeding 2 and contains only even powers of $\epsilon$. Its non-vanishing solutions come in pairs of numbers of the form $(\alpha, -\alpha)$. If at least two such pairs are non-proportional over $\mathbb{R}$ (which happens generically) then 0 is the inner point of $\Upsilon_R$. Similar in the latter case we have two second order equations without linear terms defining $\epsilon$. Again typically their pairs of solutions are non-proportional over $\mathbb{R}$ and the result follows.

Finally, assume that $\text{NE}$ is non-defining. Then all edges but possible one have integer slopes which means that the corresponding equations for $\epsilon$ will have all possible monomials present and their non-trivial roots can both contain 0 inside their convex hull and also lie in a halfplane in $\mathbb{C}$ bounded by a real line passing through the origin. If there is one edge of length 2 and half-integer slope in $\text{NE}_R$, then it produces one pair of opposite values for $\epsilon$.

**4.2. Preliminaries on exactly solvable operators.** In the rest of this section we will need the following information, see e.g. [Ber07].

Given an exactly solvable operator $T$, observe that for each non-negative integer $j$,

$$T(x^j) = \lambda_j^T x^j + \text{lower order terms}. \tag{4.1}$$

Define the **spectrum** of an exactly solvable $T$ as the sequence of complex numbers $\Lambda^T := \{\lambda_j^T\}_{j=0}$. \[\text{lm: egenpolys}\]

**Lemma A** (See [MS01]). For any exactly solvable operator $T$ and any sufficiently large positive integer $n$, there exists a unique (up to a constant factor) eigenpolynomial $p_n^T(x)$ of $T$ of degree $n$. Additionally, the eigenvalue of $p_n^T$ equals $\lambda_n^T$, where $\lambda_n^T$ is given by \[\text{4.1}.\]

One can easily show that for any exactly solvable operator $T$ and any sufficiently large positive integer $n$, $|\lambda_n^T| > |\lambda_j^T|$ for $0 \leq j < n$.

**Remark 32.** In addition to Lemma A, observe that for any exactly solvable operator $T$ as in \[\text{4.1}\] and any non-negative integer $n$, $T$ has a basis of eigenpolynomials in the linear space $\mathbb{C}_n[x]$ consisting of all univariate polynomials of degree at most $n$. This follows immediately from e.g., the fact that $T$ is triangular in the monomial basis \{1, $x$, \ldots, $x^n$\}. In other words, even if $T$ has a multiple eigenvalue it has now...
Proposition 33. Given an exactly solvable operator \( T \) as in (1.1) and any invariant set \( S \in \mathcal{I}^T_n \), \( S \) must contain the union of all roots of the eigenpolynomial \( p^m_n \), where \( m \geq n \) and \( |\lambda^m_n| > |\lambda^j_T| \) for \( 0 \leq j < m \). The latter fact implies that \( S \) contains the union of all roots of all eigenpolynomials of sufficiently large degrees.

Proof. Indeed, as we mentioned above the absolute values of \( \{\lambda^m_n\} \) will be strictly increasing from some degree \( \kappa T \). Choose some \( m \geq n \) such that \( |\lambda^m_n| > |\lambda^j_T| \) for \( 0 \leq j < m \) and a basis \( \{p^0_T, p^1_T, \ldots, p^m_T\} \) of the space \( \mathbb{C}[x] \) of all polynomials of degree at most \( m \) consisting of \( T \)’s eigenpolynomials. Pick a polynomial \( q \) of degree \( m \) whose roots belong to \( S \) and expand it as \( q(x) = \sum_{j=0}^m a_j p^j_T(x) \) with \( a_m \neq 0 \). Repeated application of \( T \) to \( q \) gives

\[
T^\ell(q) = \sum_{j=0}^m a_j \lambda^j m p^j_T(x) = \lambda^m m \sum_{j=0}^m a_j \left( \frac{\lambda^j}{\lambda^m} \right)^\ell p^j_T(x). \tag{4.2} \]

Since \( S \in \mathcal{I}^T_n \) all roots of \( T^\ell(q) \) belong to \( S \). By our assumption and disregarding the common factor \( \lambda^m_n \), the polynomial in the right-hand side of (4.2) equals \( a_m p^m_T(x) \) plus some polynomial of degree smaller than \( m \) whose coefficients tend to 0 as \( \ell \) tends to infinity. Since \( a_m \neq 0 \) the roots of the polynomials in the right-hand side of (4.2) tend to those of \( p^m_T \) implying that the latter must necessarily belong to \( S \). \( \square \)

4.3. General results about degenerate operators. An important although not very complicated result about degenerate operator which partially follows from our previous considerations is as follows.

Proposition 34. If \( T \) is a degenerate operator, then for any non-negative \( n \), every set \( \mathcal{I}^T_{\geq n} \) is unbounded and, therefore, is \( T \)-invariant.

\( T \)-invariance follows from the unboundedness by item (ii) of Theorem 8.

Proof. Let us start with the special case of degenerate exactly solvable operators. (They and their invariant sets are the main object of study of our sequel paper [ABS19].)

Any exactly solvable operator \( T \) preserves the degree of a generic polynomial it acts upon and has a unique (up to a constant factor) eigenpolynomial \( p_n^m(x) \) of any sufficiently large degree \( n \), see Lemma [A] and [Ber07] Lemma 1. Moreover, if \( r_n \) denotes the maximum of the absolute value of the roots of \( p_n(x) \), then for any degenerate exactly solvable \( T \), \( \lim_{n \to \infty} r_n = +\infty \), see [Ber07] Theorem 1.

By Proposition 33 for any exactly solvable operator \( T \), any \( S \in \mathcal{I}^T_{\geq n} \) must contain the union of all roots of all eigenpolynomials \( p^m_n(x) \) for all sufficiently large \( m \), we conclude that any such \( S \) is necessarily unbounded.

Assume now that \( T \) has a positive Fuchs index \( \rho := \rho_T > 0 \). Consider the operator \( T' = \frac{d^\rho}{dx^\rho} \circ T \). If \( T \) is degenerate, then \( T' \) is a degenerate exactly solvable operator. By the Gauss–Lucas theorem, every \( S \in \mathcal{I}^T_{\geq n} \) belongs to \( \mathcal{I}^{T'}_{\geq n} \). Since every subset \( S' \in \mathcal{I}^T_{\geq n} \) is unbounded by the above argument, we have settled the case \( \rho > 0 \).

Assume finally, that \( T \) is a degenerate operator with \( \rho < 0 \). Consider a family of operators

\[
T'_a = (x - a)^{-\rho} \cdot T, \tag{5.1}
\]

where \( a \in \mathbb{C} \). Since under our assumptions, \( -\rho \) is a positive integer, \( T_a \) is a degenerate exactly solvable operator for any \( a \). Given \( S \in \mathcal{I}^T_{\geq n} \), choose \( a \in S \). Then \( S \in \mathcal{I}^{T'_a}_{\geq n} \) and is therefore unbounded by the previous reasoning. \( \square \)
5. Application of algebraic results to degenerate operators

In what follows, we need to consider the action of \( T = \sum_{j=0}^{k} Q_j(x) \frac{d^j}{dx^j} \) on polynomials of the form \((x - t)^n\) for sufficiently large \( n \). One has

\[
T(x - t)^n = (x - t)^{n-k} \sum_{j=0}^{k} (n)_j (x - t)^{k-j} Q_j(x) = (x - t)^{n-k} \psi_T(x, n, t)
\]

where \( \psi_T(x, n, t) \) is a trivariate polynomial. The important circumstance is that the essential part \( \psi_T^e(x, n) \) of \( \psi_T(x, n, t) \) is independent of \( t \). We will apply to \( \psi_T^e(x, n) \) the results of the previous section and discuss how its zeros w.r.t \( x \) behave when \( n \to +\infty \). Denote by \( a_jx^{d_j} \) the leading monomial of \( Q_j(x) \) and consider the polynomial

\[
\tilde{\psi}_T(x, n) = \sum_{j=0}^{k} a_j n^{j} x^{d_j + k-j}.
\]

(It contains much fewer monomials than \( \psi_T(x, n, t) \) but with the same coefficients.) Notice that the essential part \( \psi_T^e(x, n) \) is obtained from \( \tilde{\psi}_T(x, n) \) by removing those monomials which do not belong to \( \text{NE}(\psi_T) \). Taking the symbolic polynomial \( G_T(x, w) = \sum_{j=0}^{k} Q_j(x)w^j \) of \( T \), we introduce its truncation \( \tilde{G}_T(x, w) = \sum_{j=0}^{k} a_j w^j x^{d_j} \) and observe that \( \tilde{\psi}_T(x, n) \) is obtained from \( \tilde{G}_T(x, w) \) by substituting \( w \) by \( n \) and adding \( k - j \) to the powers \( d_j \) in the \( x \) variable to the respective monomial. Thus the Newton polygon of \( \tilde{\psi}_T(x, n) \) is obtained from the Newton polygon of \( \tilde{G}_T(x, w) \) by the affine transformation \( A \) sending \((i, j)\) to \((i + k - j, j)\). Therefore \( \text{NE}(\psi_T) \) is obtained from the part of the boundary of the Newton polygon of \( \tilde{\psi}_T(x, w) \) under the latter affine transformation, see Fig. 3 for an example.

![Figure 3](fig:neBoundaryShift)

**Figure 3.** The affine transformation \( A \) sending \( N_T \) to \( N_\psi \). Here \( T = (x^3 + \ldots) \frac{d^2}{dx^2} + (x^5 + \ldots) \frac{d^3}{dx^3} + (x^7 + \ldots) \frac{d^4}{dx^4} + (x + \ldots) \frac{d}{dx} \), \( \tilde{G}_T(x, w) = x^3w^3 + x^5w^6 + x^7w^6 + x^3w + x^3 \), and \( \psi_T(x, n) = n^7x^3 + n^6x^7 + n^5x^2 + n^2x^{12} + x^{10} \).

Denote the Newton polygon of \( \tilde{G}_T(x, w) \) by \( N_T \) and the Newton polygon of \( \tilde{\psi}_T(x, w) \) by \( N_\psi \). We have that \( N_\psi = A \circ N_T \). The relation between the slopes of edges before and after the affine transformation \( A \) is as follows.

If the slope \( sl \) of an edge of \( N_T \) equals \( sl = \frac{\mu}{\nu} \) where \( \mu \) and \( \nu \) are coprime integers and \( \nu > 0 \), then the slope of its image \( \text{asl} \) equals \( \text{asl} = \frac{\nu\mu}{\nu + \mu \nu} \), which implies that \( \text{asl} = \frac{sl}{1 + \nu \mu} \) or, equivalently, \( sl = \frac{\text{asl}}{1 + \nu \mu} \). Therefore if \( \text{asl} \) is a negative integer then we get

\[
\text{asl} = -J, J > 0 \iff sl = \frac{J}{J-1}.
\]
Obviously any $sl$ of the above form is positive (or $+\infty$). 

Analogously, if $asl$ is a negative half-integer then we get 

$$asl = -\frac{J}{2}, J > 0 \text{ and odd} \iff sl = \frac{J}{J - 2}$$

Again any $sl$ of the above form is positive with the only exception $J = 1$ for which $sl = -1$.

It is easy to describe $A^{-1}(\mathbf{NE}_T)$ as the part of the boundary $N_T$ starting at $V_n$ and going southeast till we either reach the lowest point of the polygon or till the slope of the next edge becomes smaller than or equal to 1. Denote $A^{-1}(\mathbf{NE}_T)$ as $\mathfrak{B}_T$ and call it the shifted northeastern border of $N_T$.

One can easily check that for $T = \sum_{j=0}^{k} Q_k(x) \frac{d^j}{dx^j}$, the corresponding $\mathbf{NE}(\psi_T)$ is a single point if and only if $T$ is nondegenerate. So for any degenerate $T$, its $\mathbf{NE}(\psi_T)$ contains at least one edge. Additionally, $asl < 0$ if and only if $\frac{1}{sl} < 1$ which means that either $sl < 0$ or $sl > 1$.

Observe that the vertex $V_n$ of $\psi$ coincides with that of $\tilde{G}$. The following notion is important for the rest of the paper.

**Definition 35.** A degenerate operator $T$ is called defining/almost defining/non-defining if its Newton polygon $N_T$ is defining/almost defining/non-defining resp. In terms of the Newton polygon $N_T$ this means that its shifted northeastern border $\mathfrak{B}_N$ is not a single point and in the defining case it contains an edge with the slope of the form $\frac{J}{j}$ with $\beta \geq 3$, in the almost defining case all edges of $\mathfrak{B}_N$ have slopes $\frac{1}{j}$ but there exists either one edge with slope $\frac{1}{j+2}$, $J$ odd and length greater than 2 or two such edges with length 2; and in the non-defining case contains edges of arbitrary integer length with slopes $\frac{1}{j+1}$, $J$ being a positive integer, except for possibly one edge of integer length 2 whose slope is $\frac{1}{j+2}$, $J$ odd.

The following result is an easy consequence of our previous considerations.

**Theorem 36.** For any nonnegative integer $n$ and (almost) any degenerate operator $T$ whose $N_T$ is (almost) defining, the only set contained in $I_{\geq n}^T$ is $C$.

### 5.1. Degenerate operators with non-defining Newton polygons.

As we have seen above the convex hull of the set $\Upsilon_T$ of all leading constants for (almost) every degenerate $T$ with (almost) defining $N_T$ contains 0 as its interior point.

For degenerate $T$ with non-defining $N_T$ whose northeastern border we will denote by $\mathbf{NE}_T$, it might still happen that 0 is the interior point of the latter convex hull in which case the conclusion of Theorem 36 holds. However for a full-dimensional subset of $Pol(\mathbf{NE})$ with a given non-defining $\mathbf{NE}$, their leading constants belong some half-plane in $C$ bounded by a line passing through 0 and therefore 0 lies on the boundary of their convex hull. In this situation the conclusion of Theorem 36 fails and we will discuss this case below.

**Definition 37.** Given a finite set $\mathcal{U} = \{u_1, \ldots, u_k\}$ of (not necessarily distinct) complex numbers, we define the cone $\mathcal{C}^+ \mathcal{U} \subseteq \mathbb{C}$ generated by $\mathcal{U}$ as given by

$$\mathcal{C}^+ \mathcal{U} := \{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k\}, \text{where } \alpha_j \geq 0, j = 1, \ldots, k.$$ 

We say that a set $S \subseteq \mathbb{C}$ is closed with respect to $\mathcal{C}^+ \mathcal{U} \subseteq \mathbb{C}$ if for any complex number $z \in S$ and any $v \in \mathcal{C}^+ \mathcal{U}$, $z + v$ belongs to $S$.

Obviously, 0 is the interior point of the convex hull of $\mathcal{U} = \{u_1, \ldots, u_k\}$ if and only if $\mathcal{C}^+ \mathcal{U} = \mathbb{C}$.

Given a degenerate operator $T$ with non-defining polygon $N_T$, define $\Upsilon_T := \{\epsilon_1, \ldots, \epsilon_m\}$ as the collection of all its leading constants and set $\mathcal{C}^+_T := \mathcal{C}^+ (\Upsilon_T)$. As we mentioned above, if $\mathcal{C}^+_T = \mathbb{C}$, then the conclusion of Theorem 36 holds. Let us
assume now that $C_T^+$ is a closed sector in the plane with positive angle $\leq \pi$. (We are then missing two remaining cases: $C_T^+$ being a line through the origin and $C_T^+$ being a half-line through the origin.)

Lemma 38. In the above notation, any set $S \in T_{\geq n}^1$ is $C_T^+$-invariant.

Proof. (Sketch) Indeed, take a point $t \in S$ and consider the sequence of polynomials $T(x - t)^n$ when $n$ increases. The roots of $\psi_T(x, n, t)$ whose absolute values tend to infinity will be spreading out to infinity approximately along the rays whose directions are given by the elements of $T_T$. Since every $S$ must be convex the result follows.

Theorem 39. In the above notation, if the leading coefficient $Q_k(x)$ of the operator $T$ is not a constant, then $\lim_{n \to \infty} M_{\geq n}^1 = \text{Conv}(Q_k) \oplus C_T^+$, where $\text{Conv}(Q_k) \oplus C_T^+$ is the subset of $\mathbb{C}$ obtained as the union of all possible shifts of $C_T^+$ for which its apex lie in $\text{Conv}(Q_k)$.

Proof. Write proof

5.2. Degenerate operators with non-defining Newton polygon and constant leading term. The remaining case of a constant leading term is discussed below. One can easily check that degenerate operators

$$T = \frac{d^k}{dx^k} + Q_{k-1}(x) \frac{d^{k-1}}{dx^{k-1}} + \cdots + Q_k(x)$$

with non-defining $N_T$ split into two subclasses:

(A) operators with constant coefficients;

(B) operators satisfying the following three conditions:

(i) $\deg Q_{k-1} = 1$;

(ii) $\deg Q_j \leq 1$ for $j = 0, \ldots, k - 2$;

(iii) if $j_{\min}$ is the smallest value of $j$ for which $\deg Q_j = 1$, then $Q_{\ell}$ must vanish for all $\ell \leq j_{\min} - 2$.

For the more interesting subclass (B) the northeastern border of such operator $T$ can consist of 1, 2 or three edges, see Fig. below. If it consists of 1 edge then after an affine change of $x$ we can reduce such an operator to

$$T = \frac{d^k}{dx^k} - x \frac{d^{k-1}}{dx^{k-1}} + \alpha \frac{d^{k-2}}{dx^{k-2}}, \alpha \in \mathbb{C}.$$

If it consists of 2 edges then after an affine change of $x$ we can reduce such an operator to

$$T = \frac{d^k}{dx^k} - x \left( \frac{d^{k-1}}{dx^{k-1}} + \sum_{i=1}^{\ell} \alpha_i \frac{d^{k-1-i}}{dx^{k-1-i}} \right) + \sum_{i=1}^{\ell} \beta_i \frac{d^{k-2-i}}{dx^{k-2-i}}$$

where $\ell \leq k - 1$ is a positive integer and all $\alpha_i$ and $\beta_i$ are arbitrary complex numbers with the only restriction $\alpha_{i+1} \neq 0$.

Finally, if it consists of 3 edges then after an affine change of $x$ we can reduce such an operator to

$$T = \frac{d^k}{dx^k} - x \left( \frac{d^{k-1}}{dx^{k-1}} + \sum_{i=1}^{\ell} \alpha_i \frac{d^{k-1-i}}{dx^{k-1-i}} \right) + \sum_{i=1}^{\ell} \beta_i \frac{d^{k-2-i}}{dx^{k-2-i}} + \beta_{\ell+1} \frac{d^{k-3-\ell}}{dx^{k-3-\ell}}$$

where $\ell \leq k - 3$ is a positive integer, all $\alpha_i$ and $\beta_i$ are arbitrary complex numbers with the restrictions $\alpha_{\ell} \neq 0$ and $\beta_{\ell+1} \neq 0$. We will discuss these subcases below.
Then a convex set $S$ is closed with respect to translation invariance, we can additionally assume that either all roots of $p(x)$ are real or among these roots there is at least one with a positive imaginary part and at least one with the negative imaginary part. For any natural $n$, all roots of $((x+n)^n p(x))'$ lie in the convex hull of all roots of $p$ appended with $-n$. When $n \to \infty$, we get the required statement. In other words, all roots of $p'(x) + p(x) = e^{-x} (p(x)e^x)'$ lie in the infinite polygon (or halfline) formed by the parallel translation of the convex hull of all roots of $p$ to infinity in the direction $-1$. \hfill \Box
Variation 1: invariant sets for roots of polynomials of a fixed degree.

Instead of looking for a set which is invariant for roots of polynomials of degree at least \( n \), we can relax the requirement and ask that a set is only invariant for roots of polynomials of degree exactly \( n \). In other words, given an operator \( T \) as above, a set \( S \subset \mathbb{C} \) is called \( T \)-invariant in degree \( n \) if every polynomial of degree exactly \( n \) with all roots in \( S \) has all roots in \( S \); has the property that \( T(P) \) has all roots in \( S \) (or is constant).

Given \( T \) and \( n \), we denote by \( I^T_n \) the family of \( T \)-invariant sets in degree \( n \) and we denote by \( M^T_n \) the corresponding unique minimal closed invariant set (if it exists). Note that \( M^T_n \subset M^T_{\geq n} \). It makes most sense to study \( M^T_n \) for exactly solvable operators \( T \) since in this case they preserves the degrees of polynomials they act upon.

As we will show below, in many cases \( M^T_n \) can have a complicated structure — in particular, it does not need to be convex, and it can be a fractal etc. An illustration...
can be found in Fig. 5. Detailed study of $T$-invariant sets in degree $n$ is carried out in the sequel paper [ABS19].

**Example 44.** The minimal invariant set $M_T^1$ for the differential operator $T = (x^2 - x + i) \frac{d}{dx} + 1$ coincides with the classical Julia set associated with $f(x) = x^2 + i$. 

![Figure 5](fig:julia)  
**Figure 5.** The minimal set $M_T^1$ for the operator $(x^2 - x + i) \frac{d}{dx} + 1$. This set has the property that if $t \in M_T^1$, then $\pm \sqrt{t - i}$ is also in $M_T^1$. 

**Variation 2: Hutchinson-invariant sets.** A set $S \subset \mathbb{C}$ is called *Hutchinson-invariant in degree* $n$ if every polynomial of the form $P(t) = (x-t)^n$ with $t \in S$, has the property that $T(P)$ has all roots in $S$ (or is constant). In particular, a $T$-invariant set in degree 1 is a Hutchinson-invariant set in degree 1 and vice versa. However, for $n > 1$, $T$-invariant and Hutchinson-invariant sets in degree $n$ in general do not coincide. We denote by $H_T^n$ the collection of all Hutchinson-invariant sets in degree $n$ and by $H_{n,T}^M \in H_T^n$ the unique minimal under inclusion closed Hutchinson-invariant set in degree $n$ (if it exists). Notice that 

$$H_{n,T}^M \subseteq M_T^n \subseteq M_{\geq n}^T.$$ 

In particular, if $H_{n,T}^M$ exists, then $M_T^n$ and $M_{\geq n}^T$ exist as well.

Let us explain our choice of terminology. *Hutchinson-operator* defined by a finite collection of univariate functions $\phi_1, \ldots, \phi_m$ and its invariant sets were introduced and studied in [Hut81] as well as a large number of follow-up papers. In our situation, assume that the action of $T$ on $(x-t)^n$ factorizes as:

$$T((x-t)^n) = (x - (a_1 t + b_1)) \cdots (x - (a_m t + b_m)),$$ 

(6.1)  

see e.g. (3.2). Then we have that if $S \subset \mathbb{C}$ is Hutchinson-invariant in degree $n$, then $f_i(S) \subseteq S$ for all $i = 1, 2, \ldots, m$, where $f_i(t) = a_i t + b_i$. If all these $f_i$ are contractions, that is, $|a_i| < 1$, one can show that there is a unique minimal non-empty closed Hutchinson-invariant set $S$, and it is exactly the invariant set associated with the *Hutchinson-operator* defined by $f_1, \ldots, f_m$, see [Hut81]. (One can also consider other types of factorizations similar to (6.1) with e.g. polynomial or rational factors.) This observation implies that one can obtain many classical fractal sets such as the Sierpinski triangle, the Cantor set, the Lévy curve and the
Koch snowflake as Hutchinson-invariant sets, see Example 45. In particular, $M_T^n$ does not have to be connected.

Julia sets associated with rational functions can also be realized as Hutchinson-invariant sets of appropriately chosen operators $T$, see [ABS19]. Let us illustrate the situation with Example 44 and Example 46.

**Example 45.** For the differential operator $T = x(x + 1) \frac{d^2}{dx^2} + i \frac{d}{dx} + 2$, the set $HM^T_2$ is a Lévy curve. The roots of $T((x - t)^2)$ are given by

$$x = \frac{1 + i}{2} t \quad \text{and} \quad x = \frac{1 - i}{2} (t - i).$$

The two maps

$$t \mapsto \frac{1 + i}{2} t \quad \text{and} \quad t \mapsto \frac{1 - i}{2} (t - i) \quad (6.2)$$

are both affine contractions which together produce a fractal Lévy curve as their invariant set, see Fig. 6. In particular, every member of $I_T^n$ must contain $HM^T_2$ given by the latter curve which also implies that $M_T^n$ exists.

**Figure 6.** The Hutchinson-invariant set $HM^T_2$ for the operator $T = x(x + 1) \frac{d^2}{dx^2} + i \frac{d}{dx} + 2$. The two colors indicate the image of the set under the two maps in (6.2).

**Example 46.** The differential operator $T = x(x - 1) \frac{d}{dx} + 1$ admits two minimal sets $HM^T_1$, one of which is the one-point set $\{0\}$ and the other is the unit circle. This fact is in line with known properties of the Julia sets; some very special rational functions admit several completely invariant sets containing one or two points. The reason why the above case is exceptional, is that $T$ maps the polynomial $x$ to $x^2$, which has the same zeros as $x$. In general, such exceptional invariant sets only show up in the situation when there exists some $t$ such that $T(x - t) = c(x - t)^k$, see [Bea00].

**Remark 47.** There are at least two advantages in studying Hutchinson-invariant sets compared to the set-up of the present paper. The first one is that the occurring types of fractal sets have already been extensively studied which connects this topic to the existing classical complex dynamics, comp. e.g. [Bar93, Fal04]. The second

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1Minimal here means that no proper closed subset is an invariant set.
Lemma 48. For any operator $T$ and a real number $n_0 \geq 0$, the set $\text{CHM}_{\geq n_0}^T$ consists of finitely many path-connected components.

Proof. By minimality of $\text{CHM}_{\geq n_0}^T$, if $t \in \text{CHM}_{\geq n_0}^T$, we may assume that there exist $t_0 \in \text{CHM}_{\geq n_0}^T$ and $n \geq n_0$, such that $t$ is a root of $\psi(x, t_0, n) = 0$. Since roots of a polynomial depend continuously on the coefficients, there exists a curve $\gamma(s)$, with $\gamma(n) = t$ and $\gamma(\infty) = x^*$, such that

$$\psi(\gamma(s), t_0, s) = 0$$

for all $s \in [n, \infty)$, where $x^*$ is a zero of the leading coefficient $Q_k(x)$ in $T$.

Approximations of $\text{CHM}_{\geq n_0}^T$ can also be obtained via the same Monte–Carlo method as above, but in this case the parameter $n$ has to be chosen according to some probability distribution supported on $[n_0, \infty)$. For example, to produce Figs. 7 and 8, we used $n = (u/(1-u))^r$ where $u$ is uniformly distributed on the unit interval, and $r > 0$ is a fixed parameter.

In general, it is unclear what the relation between $\text{CHM}_{\geq n}^T$ and $\text{M}_{\geq n}^T$ is, but for large $n$, we expect the inclusion $\text{CHM}_{\geq n}^T \subseteq \text{M}_{\geq n}^T$, since extending the domain of $n$ from the set of large integers to the set of large real numbers does not seem to make a big difference. Note that Conjecture 21 and Proposition 49 suggest that these sets coincide in the limit $n \to \infty$.

The following proposition shows that as $n_0$ grows, the minimal continuous Hutchinson-invariant set converges to the zero locus of the leading coefficient $Q_k$ of $T$. 

Variation 3: Hutchinson-invariant sets with continuous parameter. Given $T$ and $n$ as above, consider

$$\psi(x, t, n) := T((x-t)^n)/(x-t)^{n-k},$$

where $k$ is the order of the operator $T$. Then $\psi(x, t, n)$ is a polynomial in $\mathbb{C}[x, t, n]$. Given $n_0 \geq 0$, we say that a set $S$ is Hutchinson-invariant with parameter $\geq n_0$ if for every real number $n \geq n_0$, we have that

$$\psi(x, t_0, n) = 0$$

(considered as a polynomial in $\mathbb{C}[x]$) has all roots in $S$, whenever $t_0 \in S$. We denote by $\text{CHM}_{\geq n_0}^T$ the collection of all Hutchinson-invariant with parameter $\geq n_0$ and by $\text{CHM}_{\geq n_0}^T$ the minimal non-empty closed such set $S$ (it if exists). It is easy to verify that, for all integers $m \geq 1$,

$$\text{HM}_m^T \subseteq \text{CHM}_{\geq m}^T \subseteq \text{CHM}_{\geq 0}^T.$$

Properties of the minimal invariant set $\text{CHM}_{\geq n_0}^T$ seem to substantially depend on whether $n_0 = 0$ or $n_0 > 0$, see Fig. 7 and Fig. 8. Namely, the boundary of $\text{CHM}_{\geq 0}^T$ looks rectifiable, while the boundary of $\text{CHM}_{\geq 1}^T$ seem to have a fractal (and non-rectifiable) character. However, in contrast with Hutchinson-invariant sets which can be fractal, $\text{CHM}_{\geq n_0}^T$ always has a finite number of connected components.
Proposition 49 (Convergence to the zero locus of $Q_k$). Given a non-degenerate operator $T = \sum_{j=0}^{k} Q_j \frac{d^j}{dx^j}$, $R > 0$ and $\delta > 0$, then there exists $n_0 = n_0(R, \delta)$ such that for all $t \in \mathbb{C}$, with $|t| < R$ we have that each root of

$$T[(x-t)^n] = 0$$

different from $t$ lies at a distance at most $\delta$ from some root of $Q_k(x)$.

In particular, for any $\delta > 0$, there exists an $n_0 = n_0(\delta)$ such that the $\delta$-neighborhood of the union of roots of $Q_k(x)$ is Hutchinson-invariant in degree $n$, for all $n \geq n_0$. The same holds for the Hutchinson-invariant sets with parameter exceeding $n$.

**Proof.** Fix $R > 0$ and $\delta > 0$. A straightforward calculation shows that

$$\psi(x,t,n) = Q_k(x) + \sum_{j=1}^{k} \frac{Q_{k-j}(x-t)^j}{(n-k+1)(n-k+2)\cdots(n-k+j)}.$$

Hence, the zeros $\psi(x,t,n) = 0$ tend to the zeros $Q_k(x)$ as $n \to \infty$, provided that $|t| < R$. Thus, for some $n_0 = n_0(\delta)$, all roots of $\psi(x,t,n) = 0$ lie at a distance at most $\delta$ from the fundamental polygon of $T$. \qed

**Variation 4: two-point Hutchinson invariant sets.**

Our last variation of the notion of invariance is inspired by the convexity property of the original invariant sets considered in this paper.
Figure 8. The minimal continuously Hutchinson-invariant set for $T = x \frac{d}{dx} + (x + 1)^2$, with $n_0 = 1$. That is, we now ensure that $n \geq 1$. Recall that the case $n = 1$ gives $M_T^1$, which in this case is a Julia fractal. This explains the fractal boundary.

Set $P(x) := (x - t_1)^{n_1}(x - t_2)^{n_2}$ and consider
\[ \phi(x, t_1, n_1, t_2, n_2) := \frac{T(P)}{(x - t_1)^{n_1 - k}(x - t_2)^{n_2 - k}}, \]
where $k$ is the order of the operator $T$. Again, $\phi(x, t_1, n_1, t_2, n_2)$ is a polynomial in $\mathbb{C}[x, t_1, n_1, t_2, n_2]$. Given $n_0 \geq 0$, a set $S \subset \mathbb{C}$ is called two-point Hutchinson invariant with parameters $\geq n_0$ if for every pair of real number $n_1, n_2 \geq n_0$, we have that
\[ \phi(x, t_1, n_1, t_2, n_2) = 0 \quad (\text{considered as a polynomial in } x) \]
has all roots in $S$, whenever $t_1, t_2 \in S$. We denote by $C_2HM_{\geq n_0}^T$ the minimal under inclusion non-empty closed set $S$ which is two-point Hutchinson invariant with parameters $\geq n_0$ (if it exists).

We have that $CHM_{\geq n_0}^T \subseteq C_2HM_{\geq n_0}^T$. Moreover, we may apply the same technique as in Theorem 8 to show that two-point continuous invariant sets are convex.

In Fig. 9 we compare the minimal Hutchinson-invariant set with a non-negative parameter and the minimal two-point Hutchinson invariant set with non-negative parameters for the operator $T = (x - i) \frac{d}{dx} + \Re(x) - 1$. It is tempting to conjecture that the boundary curve (connecting $-1$ and $1$, and passing through $i$) is a cycloid, and that the boundary curve in Fig. 7 is a cardioid. However, computer comparisons suggest that this might not be the case.

Remark 50. The linear operators which factor as in (6.1) allow us to produce a large class of fractal sets associated with Hutchinson operators, where each map is an affine contraction from $\mathbb{C}$ to $\mathbb{C}$. These minimal invariant sets $HM_n^T$ are fractals,
Figure 9. The 1-point and 2-point Hutchinson-invariant set for the operator $T = (x - i) \frac{d}{dx} + x^2 - 1$. As expected, the second set is convex and is indeed a super-set of the first set. Note: The sporadic pixels below the main convex set seem to be artifacts of the Monte-Carlo method used to produce the image.

and therefore might be hard to study. Can one instead give a (simple) description of the boundaries of the associated continuous Hutchinson invariant set $\text{CHM}_T^{\geq 0}$ or its larger convex cousin $C_2\text{HM}_T^{\geq 0}$? Remember that we have the set of inclusions

$$\text{HM}_n^T \subseteq \text{CHM}_n^T \subseteq C_2\text{HM}_n^T,$$

so a simple description of $\text{CHM}_n^T$ may provide some additional insight in the nature of $\text{HM}_n^T$.

7. Final remarks and problems

1. The major open question related to the present paper is how to describe the boundary of $\text{M}_n^T$ for non-degenerate or degenerate operators with non-defining Newton polygons and $Q_k$ different from a constant. At the moment we only know what happens with $\text{M}_n^T$ when $n \to \infty$. Our numerical experiments indicate that the boundary of $\text{M}_n^T$ is piecewise analytic. Already for no-degenerate operators of order 1 this problem seems to be quite non-trivial.

2. Another important issue to study is the dependence of $\text{M}_n^T$ on the coefficients of operator $T$. It seems that even in the case when $T$ is non-degenerate and $n$ is such that $\text{M}_n^T$ is compact, it is loose compactness under small deformation of $T$ with the space of non-degenerate operators of the same order. Even for operators of order one the question is non-trivial. For example, consider the space of pairs of polynomials $(U, V)$ where $\deg U = k$.
and $\deg V = k - 1$. Describe the boundary of the space of such pairs for which $M_{\leq n}^T$ is compact?

3. Can one say more about operators with constant leading term and $\deg Q_{k-1} = 1$?

4. Can one say more about operators with $\Upsilon_T$ being a line or a halfline?

5. Could there be a compact invariant set of a non-degenerate operator $T$ violating the assumptions of Theorem 14? It guarantees that all sufficiently large disks centered at the origin are $T$-invariant. But is this necessary for the existence of compact invariant sets?

References


Department of Mathematics, Stockholm University, S-10691, Stockholm, Sweden
E-mail address: per@math.su.se

Department of Mathematics, Royal Institute of Technology, SE-100 44, Stockholm, Sweden
E-mail address: pbranden@kth.se

Department of Mathematics, Stockholm University, S-10691, Stockholm, Sweden
E-mail address: shapiro@math.su.se