ALGEBRAS GENERATED BY CURVATURE FORMS ON $\mathbb{SL}_n/P$ AND SCHUBERT CALCULUS

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Abstract. Below we introduce and study the algebra generated by the Chern forms of an arbitrary flag variety $\mathbb{SL}_n/P$.

1. Introduction

The main goal of this paper is to answer a number questions posed by the third author on the conference "New trends in combinatorics", held in Stockholm in February 2005, see [5]. These questions are meant to extend the earlier results obtained in [6] and [3] by the second, the third and the fourth authors from the case of the variety of complete flags to the case of Grassmannians and varieties of incomplete flags.

Let G be a connected complex semisimple Lie group and P its parabolic subgroup. The quotient space $X = G/P$ is then a compact homogeneous complex manifold. We choose a maximal compact subgroup K of G and let $T := K \cap P$ be the maximal compact subgroup of P. The group K acts transitively on X. Thus X can be identified with the quotient space K/T.

In our main example $G = \mathbb{SL}_n(\mathbb{C})$, $P \subset \mathbb{SL}_n(\mathbb{C})$ is some parabolic subgroup, $K = \mathcal{U}_n$ is the group of unitary matrices, $T = \mathcal{U}_n \cap P$, and X is the flag variety corresponding to P. It is well-known, up to a natural isomorphism, the set of all parabolic subgroups in $\mathbb{SL}_n(\mathbb{C})$ is in $1-1$-correspondence with the set of all compositions of n, i.e. sequences of positive integers $\lambda = (\lambda_1 < \lambda_2 < \dots < \lambda_k = n)$.

Given a composition $\lambda = (\lambda_1 < \lambda_2 < \dots < \lambda_k = n)$, denote by P_λ the corresponding parabolic subgroup. The quotient space $F_\lambda := \mathcal{U}_n/P_\lambda$ consists of all (in)complete flags in $\mathbb{C}^n$ of format λ, i.e. sequences of k enclosed subspaces $V_1 \subset V_2 \subset \dots \subset V_k \simeq \mathbb{C}^n$ of dimensions $\lambda_1, \lambda_2, \dots, \lambda_k = n$ respectively. For each F_λ, there is a sequence of associated tautological vector bundles $E_1 \subset E_2 \subset \dots \subset E_k \simeq \mathbb{C}^n$ over F_λ, where the fiber of E_i over a point $f = (V_1(f) \subset V_2(f) \subset \dots \subset V_k(f)) \in F_\lambda$ is the subspace $V_i(f)$. Denote by $L_1 := E_1, L_2 := E_2/E_1, \dots, L_k := E_k/E_{k-1}$ the sequence of the corresponding quotient bundles. Fixing some Hermitian metric on the original $\mathbb{C}^n$, we equip every E_i and L_i, $i = 1, \dots, k$ with the structure of a Hermitian vector bundle. Moreover, for any composition λ of n, the unitary group $\mathcal{U}_n$ acts on the flag variety F_λ, on each vector bundle E_i, and on each L_i preserving the Hermitian structure. Note that L_i is a vector bundle over F_λ of dimension $\kappa_i := \lambda_i - \lambda_{i-1} > 0$. Denote by $c_1^{(i)}, c_2^{(i)}, \dots, c_{\kappa_i}^{(i)}$ the standard Chern classes of L_i. Since $\mathcal{U}_n$ acts transitively on F_λ,

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then for every i and each $j = 1, \dots, \kappa_i$, there exists a unique $\mathcal{U}_n$-invariant differential form $\phi_j^{(i)}$ on F_λ representing the j-th Chern class $c_j^{(i)}$. In what follows, we will call $\phi_j^{(i)}$ the *Bott-Chern form* corresponding to the Chern class $c_j^{(i)}$.

Given a composition λ of n, the main object of study of the present paper is the graded subalgebra $\mathcal{A}_\lambda$ of the algebra $\Lambda_{inv}(F_\lambda)$ of all $\mathcal{U}_n$-invariant complex-valued differential forms on F_λ, where $\mathcal{A}_\lambda$ is generated by all the Bott-Chern forms $\phi_j^{(i)}$ of all non-trivial Chern classes of all quotient bundles L_i, $i = 1, \dots, k$. The case of the variety of complete flags, i.e. $\lambda = (1, 2, 3, \dots, n)$ was considered in substantial detail in [6] and [3]. The explicit description of the Bott-Chern form $\phi_j^{(i)}$ is given below. It is based on the formula for the curvature matrix of G/P discovered by P. Griffiths and W. Schmid in [1].

Namely, let $\lambda := \lambda_1 < \lambda_2 < \dots < \lambda_k = n$ be an arbitrary composition of n. Consider the index set $[n] = \{1, 2, \dots, n\}$ whose elements will be called *vertices*. By A_i we denote the group of vertices whose indices lie in the interval $[\lambda_{i-1}+1, \lambda_i]$, where $\lambda_0 := 0$.

Proposition 1. *For any $i = 1, 2, \dots, k$, the curvature matrix of the $\mathcal{U}_n$-invariant Hermitian quotient bundle L_i over F_λ is given by*

$$Q_i = \left\{ \sum_{j \notin A_i} \omega^{\alpha j} \wedge \omega^{j\beta} \right\}_{\alpha,\beta \in A_i}. \tag{1}$$

Proof. According to formula (4.2x) of [1]. For $\alpha, \beta \in A_i$, the element $\Theta_{\alpha,\beta}$ of the curvature matrix is given by[1]

$$\begin{aligned}\Theta_{\alpha,\beta} &= \sum_{j \le \lambda_{i-1}} \omega^{j\beta} \wedge \bar{\omega}^{j\alpha} - \sum_{j > \lambda_i} \omega^{\alpha j} \wedge \bar{\omega}^{\beta j} = \\ &= - \sum_{j \le \lambda_{i-1}} \omega^{j\beta} \wedge \omega^{\alpha j} + \sum_{j > \lambda_i} \omega^{\alpha j} \wedge \omega^{j\beta} = \sum_{j \notin A_i} \omega^{\alpha j} \wedge \omega^{j\beta}.\end{aligned} \tag{2}$$

□

Remark 1. Observe that formula (2) does not depend on an order of parts of the composition λ, but only on the partition defined by λ. This observation means that similarly to the cohomology rings, the algebras $\mathcal{A}_\lambda$ depend only on the underlying partition of n. Additionally, since every $\mathcal{U}_n$-invariant form on F_λ is closed, there is a natural surjective map $\pi_\lambda : \mathcal{A}_\lambda \to H^*(F_\lambda, \mathbb{C})$ sending every differential form in $\mathcal{A}_\lambda$ to its cohomology class. This map π_λ is an isomorphism if and only if F_λ is a symmetric space, i.e. a Grassmannian.

Let $\mathcal{E}_i$ ($\mathcal{C}_i$) be the set of Eulerian directed subgraphs (oriented cycles) of the complete bipartite digraph on sets A_i and $[n] \setminus A_i$, such that indegree of each vertex $a \in A_i$ is at most 1. Denote by $\mathcal{E}_i^{(j)}$ is the subset of $\mathcal{E}_i$ of Eulerian digraphs of size $2j$. For a cycle $C = (t_1, \alpha_1, t_2, \alpha_2, \dots, t_\ell, \alpha_\ell) \in \mathcal{C}_i$, define the exterior monomial $m_i(C)$ given by

$$m_i(C) := \omega^{t_1\alpha_1} \wedge \omega^{\alpha_1 t_2} \wedge \omega^{t_2\alpha_2} \wedge \dots \wedge \omega^{\alpha_\ell t_1}.$$

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[1] Observe that there a mistake in indices in the formula from Proposition 2 of [8]

For a graph $G \in \mathcal{E}_i$, we define the exterior monomial $m_i(G)$ as the exterior product

$$m_i(G) := m_i(C_1) \wedge \ldots \wedge m_i(C_r),$$

where $C_1, \ldots, C_r$ is a partition of G into simple directed cycles.

Lemma 2.
- *$m_i(G)$ does not depend on a choice of a cycle partition;*
- *(?) For any π, $(-1)^\pi \omega^{\alpha_1 t_1} \wedge \omega^{t_1 \alpha_{\pi(1)}} \wedge \ldots \wedge \omega^{\alpha_j t_j} \wedge \omega^{t_j \alpha_{\pi(j)}} = (-1)^j m_i(G)$.*
- *if $G \in \mathcal{E}_{i_1} \cap \mathcal{E}_{i_2}$, then $m_{i_1}(G) = \pm m_{i_2}(G)$*

Proof. **BLA** □

Theorem 3. *For $i = 1, 2, \ldots, k$, the j-th Bott-Chern form is given by*

$$\phi_j^{(i)} = (-1)^j \sum_{G \in \mathcal{E}_i^{(j)}} cp(G) m_i(G),$$

where $cp(G) = \prod_{i \in [n]} indeg_G(i)!$ is the number of cycle partitions of G.

Proof. From Proposition 1, we have

$$\phi_j^{(i)} = tr\left(Q_i^{\wedge j}\right) = \sum_{\lambda_{i-1} < \alpha_1 < \ldots < \alpha_j \leq \lambda_i} \sum_{\pi \in S_j} (-1)^\pi \prod_{\ell \in j} \left(\sum_{t \notin A_i} \omega^{\alpha_\ell t} \wedge \omega^{t \alpha_{\pi(i)}} \right) =$$

$$= \sum_{\lambda_{i-1} < \alpha_1 < \ldots < \alpha_j \leq \lambda_i} \sum_{\pi \in S_j} (-1)^\pi \sum_{t_1, \ldots, t_j \notin A_i} \omega^{\alpha_1 t_1} \wedge \omega^{t_1 \alpha_{\pi(1)}} \wedge \ldots \wedge \omega^{\alpha_j t_j} \wedge \omega^{t_j \alpha_{\pi(j)}}.$$

□

Theorem 4. *For any $i = 1, 2, \ldots, k$,*

$$\sum_{j=0} \phi_j^{(i)} t^j = \prod_{C \in \mathcal{C}_i} \left(1 + m_i(C)(-t)^{\frac{|C|}{2}} \right).$$

Proof. **BLA** □

Define $\Omega_{i,j} = \omega^{ij} \wedge \omega^{ji}$. Note that $\Omega_{ij} = -\Omega_{ji}$ and have degree 2, i.e., commute with all ω^{pq}.

Example 1. *Take $\lambda := (\lambda_1, \lambda_2, \lambda_3) = (1, 3, 4)$. Then $A_1 = \{1\}$, $A_2 = \{2, 3\}$ and $A_3 = \{4\}$. We get the following explicit expressions for the Bott-Chern formulas:*

$$\phi_1^{(1)} = -(\Omega_{21} + \Omega_{31} + \Omega_{41});$$

$$\phi_1^{(2)} = -(\Omega_{12} + \Omega_{13} + \Omega_{42} + \Omega_{43}) = (\Omega_{21} + \Omega_{31}) - (\Omega_{42} + \Omega_{43});$$

$$\begin{aligned}\phi_2^{(2)} &= 2\Omega_{12}\Omega_{13} + 2\Omega_{42}\Omega_{43} + \Omega_{12}\Omega_{43} + \Omega_{42}\Omega_{13} + \\ &\quad + + \omega^{12} \wedge \omega^{24} \wedge \omega^{43} \wedge \omega^{31} + \omega^{13} \wedge \omega^{34} \wedge \omega^{42} \wedge \omega^{21} = \\ &= 2\Omega_{21}\Omega_{31} + 2\Omega_{42}\Omega_{43} - \Omega_{21}\Omega_{43} - \Omega_{42}\Omega_{31} + \\ &\quad + \omega^{12} \wedge \omega^{24} \wedge \omega^{43} \wedge \omega^{31} + \omega^{13} \wedge \omega^{34} \wedge \omega^{42} \wedge \omega^{21};\end{aligned}$$

$$\phi_1^{(3)} = -(\Omega_{14} + \Omega_{24} + \Omega_{34}) = \Omega_{41} + \Omega_{42} + \Omega_{43}.$$

The Hilbert series of the algebra $\mathcal{A} := \mathcal{A}_{(1,3,4)}$ *equals*

$$\mathcal{H}_{(1,3,4)}(t) = 1 + 2t + 4t^2 + 5t^3 + 3t^4 + t^5 = (1+t)(1+t+3t^2+2t^3+t^4).$$

Namely,

(0) $\dim \mathcal{A} = 1$ *in degree* 0*;*
(1) $\dim A = 2$ *in degree* 1 *because* $\phi_1^{(1)} + \phi_1^{(2)} + \phi_1^{(3)} = 0$*;*
(2) $\dim \mathcal{A} = 4$ *in degree* 2*, the generators are* $(\phi_1^{(1)})^2$*,* $\phi_1^{(1)}\phi_1^{(3)}$*,* $(\phi_1^{(3)})^2$ *and* $\phi_2^{(2)}$*;*
(3) $\dim \mathcal{A} = 5$ *in degree* 3*, the generators are* $(\phi_1^{(1)})^3$*,* $(\phi_1^{(1)})^2\phi_1^{(3)}$*,* $\phi_1^{(1)}(\phi_1^{(3)})^2$*,* $(\phi_1^{(3)})^3$ *and* $\phi_2^{(1)}c_2^{(2)}$*;*
(4) $\dim \mathcal{A} = 3$ *in degree* 4*, the generators are* $(\phi_1^{(1)})^3\phi_1^{(3)}$*,* $(\phi_1^{(1)})^2(\phi_1^{(3)})^2$*,* $\phi_1^{(1)}(\phi_1^{(3)})^3$*;*
(5) $\dim \mathcal{A} = 1$ *in degree* 5*.*

$$1 + \phi_1^{(2)}t + \phi_2^{(2)}t^2 = \\ = \prod_{\substack{i=1 \textit{ or } 4\\ j=2 \textit{ or } 3}} (1 - \Omega_{ij}t) \prod_{\substack{i_1,i_2=1 \textit{ or } 4\\ j_1,j_2=2 \textit{ or } 3}} \left(1 + \omega^{i_1j_1} \wedge \omega^{j_1i_2} \wedge \omega^{i_2j_2} \wedge \omega^{j_2i_1}t^2\right).$$

Our first results describes the Hilbert series of $\mathcal{A}_\lambda$ for an arbitrary composition λ.

Namely, the total dimension of $\mathcal{A}_\lambda$ as the vector space over $\mathbb{C}$ equals the number of integer points in the permutahedron $\mathcal{P}_\lambda$, where ...

2. Grassmann case

For $\lambda = (m,n)$, the corresponding F_λ is the Grassmannian $G_{m,n}$ of m-planes in $\mathbb{C}^n$. Since $G_{m,n}$ is a symmetric space, its algebra $\mathcal{A}_{(m,n)}$ is isomorphic to the $H^\star(G_{m,n},\mathbb{C})$. It is well-known that a natural additive basis of $H^\star(G_{m,n},\mathbb{C})$ is given by the Schur polynomials S_μ where μ runs over the set of all partitions which fit in the rectangular box of sizes $m \times (n-m)$. Our Chern forms correspon to the elementary symmetric functions, or in other words, to the square partitions of size $1, 2, ..., \min(m, n-m)$.

Our next goal is to find a representation of each such Schur polynomial similar to that

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