HARDY-PETROVITCH-HUTCHINSON'S PROBLEM AND PARTIAL THETA FUNCTION

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To the memory of Vladimir Igorevich Arnold

Abstract. In 1907, M. Petrovitch [20] initiated the study of a class of entire functions all whose finite sections (i.e. truncations) are real-rooted polynomials. He was motivated by previous studies [17] of E. Laguerre on uniform limits of sequences of real-rooted polynomials and by an interesting result of G. H. Hardy [10]. An explicit description of this class in terms of the coefficients of a series is impossible since it is determined by an infinite number of discriminantal inequalities, one for each degree. However, interesting necessary or sufficient conditions can be formulated. In particular, J. I. Hutchinson [11] has shown that an entire function $p(x) = a_0 + a_1x + \ldots + a_nx^n + \ldots$ with strictly positive coefficients has the property that all of its finite segments $a_ix^i + a_{i+1}x^{i+1} + \ldots + a_jx^j$ have only real roots if and only if $\frac{a_i^2}{a_{i-1}a_{i+1}} \geq 4$ for $i = 1, 2, \ldots$. In the present paper, we give sharp lower bounds on the ratios $\frac{a_i^2}{a_{i-1}a_{i+1}}$, $(i = 1, 2, \ldots)$ for the class considered by M. Petrovitch. In particular, we show that the limit of these minima when $i \to \infty$ equals the inverse of the maximal positive value of the parameter for which the classical partial theta function belongs to the Laguerre-Pólya class $\mathcal{L} - \mathcal{P}I$. We also explain the relation between Newton's and Hutchinson's inequalities and the logarithmic image of the set of all real-rooted polynomials with positive coefficients.

1. Introduction

In what follows we will use the terms 'real-rooted polynomial' and 'hyperbolic polynomial' as synonyms. Consider the space $\mathcal{P}_n$ of polynomials of the form $p(x) = 1 + a_1x + \ldots + a_nx^n$ with real coefficients. A polynomial $p(x) \in \mathcal{P}_n$ with all positive coefficients is called *section-hyperbolic* if for $i = 1, \ldots, n$, its section $1 + a_1x + \ldots + a_ix^i$ is hyperbolic. Let $\Delta_n \subset \mathcal{P}_n$ be the set of all section-hyperbolic polynomials of degree n, and let $\Delta = \bigcup_n \Delta_n$ be the set of all section-hyperbolic polynomials. (Notice that by a result of E. Laguerre [17] a formal power series whose sections belong to Δ is an entire function lying in the Laguerre-Pólya class $\mathcal{L} - \mathcal{P}I$. Recall that an entire function is said to belong to the Laguerre-Pólya class $\mathcal{L} - \mathcal{P}I$ if it is the local uniform limit in $\mathbb{C}$ of a sequence of polynomials with positive coefficients and negative zeros, see e.g. Ch. 18 of [16]. Such functions can be presented in the form $cx^me^{\sigma x}\prod_{k=1}^{\omega}(1 + x/x_k)$, where $c > 0$, $m \in \mathbb{N} \cup \{0\}$, $\sigma \geq 0$, $\omega \in \mathbb{N} \cup \{\infty\}$, $x_k > 0$ and $\sum_{k=1}^{\omega} 1/x_k < \infty$.)

Date: June 4, 2012.
2010 *Mathematics Subject Classification.* Primary 30C15; Secondary 12D10, 26C05.
Key words and phrases. real-rooted polynomias, section-hyperbolic polynomials, Laguerre-Pólya class.

We will call an entire function with positive coefficients *section-hyperbolic* if all its sections belong to Δ. The following question was posed to the second author in April 2010 by Professors O. Katkova and A. Vishnyakova, who attributed it to Professor I. V. Ostrovskii (see [19], [27]).

Problem 1 (Hardy-Petrovitch-Hutchinson-Ostrovskii[1])**.** *For a given positive integer i, find or estimate*

$$m_i = \inf_{p\in\Delta} \frac{a_i^2}{a_{i-1}a_{i+1}}.$$

Denote by Pol_n the space of all monic real polynomials of degree n, and by $\Sigma_n \subset \mathrm{Pol}_n$ the set of all such polynomials having all roots negative. Given a polynomial $p(x) = a_0+a_1x+\ldots+a_nx^n$ with $a_n \neq 0$, define its *reciprocal polynomial* as $P(x) = a_0x^n+a_1x^{n-1}+\ldots+a_n = x^np(1/x)$. Obviously, the map sending p to its reciprocal P is a diffeomorphism between $\mathcal{P}_n$ and Pol_n. We say that a polynomial $P(x) = x^n + a_1x^{n-1} + \ldots + a_n$ is *reciprocal section-hyperbolic* if its sections $x^n + \ldots + a_ix^{n-i}$ from the back are hyperbolic for $i = 1,\ldots,n$. The reciprocal send diffeomorphically the set $\Delta_n \subset \mathcal{P}_n$ of all section-hyperbolic polynomials onto the set of all reciprocal section-hyperbolic polynomials. By a slight abuse of notation, we denote the latter set by $\Delta_n \subset \mathrm{Pol}_n$ and will freely use both interpretations. (In fact, the second interpretation can already be found in the original paper [20].) Observe that the quantities $\frac{a_i^2}{a_{i-1}a_{i+1}}$ are preserved (up to a change of indices) by taking the reciprocal.

Notice that since the natural projection $\pi_i : \Delta_{i+1} \to \Delta_i$ 'forgetting' the leading monomial is surjective, one has

$$m_i = \inf_{p\in\Delta} \frac{a_i^2}{a_{i-1}a_{i+1}} = \inf_{p\in\Delta_{i+1}} \frac{a_i^2}{a_{i-1}a_{i+1}}.$$

Thus to determine m_i, it suffices to consider Δ_{i+1}. Moreover, $m_i \leq 4$ by Hutchinson's theorem; see [11]. (Hutchinson's theorem was rediscovered 70 years later in [12].)

Petrovitch knew that $m_1 = 4$, $m_2 = \frac{27}{8} = 3.375$ and $m_3 \approx 3.264$ (see pp. 42–43 of [20]). On p. 331 of [11], Hutchinson writes that the sequence $\{m_i\}$ should be strictly decreasing to some unknown limit m_∞ of which it is known that it should exceed 2.

The next result was proved by O. Katkova and A. Visnyakova around 2006.

Theorem 1. *The minima m_i are greater than or equal to 3 for $i = 1, 2, \ldots$.*

Remark 1. Hutchinson's result implies that the quantities $\frac{a_i^2}{a_{i-1}a_{i+1}}$ can attain arbitrarily large values on Δ_n, so it is their minimal values which are important. Also notice that if one considers these quantities on the set Σ_n of all monic polynomials with negative roots, then the famous Newton inequalities show that

$$\frac{a_i^2}{a_{i-1}a_{i+1}} \geq \frac{(n-i+1)(i+1)}{(n-i)i} > 1 \quad (i = 1,\ldots,n-1). \tag{1}$$

[1]The order of the names is chronological in accordance with the years of their contributions to the topic under consideration. According to our former advisor, the late Vladimir Igorevich Arnold, the main reason why a mathematical concept is named after a certain mathematician is that this particular person has **never** considered this particular concept. Although Arnold's statement seems rather extreme it turns out to be true in a surprisingly large number of cases.

(See [18] and Proposition 7.) Moreover, the equality in (1) is attained exactly on polynomials with all real and coinciding roots. In particular, for appropriate choices of n and i, the quantity $a_i^2/a_{i-1}a_{i+1}$ can be arbitrarily close to 1.

1.1. **Introductory results.** Below we solve the Hardy-Petrovitch-Hutchinson-Ostrovskii problem by presenting explicitly the entire function which simultaneously realizes all the above m_i. Namely, consider the following sequence of polynomials

$$\begin{aligned} p_1(x) &= 1+x\ , \\ p_2(x) &= p_1(x)+\tfrac{x^2}{4} &&= 1+x+\tfrac{x^2}{4}\ , \\ p_3(x) &= p_2(x)+\tfrac{x^3}{54} &&= 1+x+\tfrac{x^2}{4}+\tfrac{x^3}{54}\ , \\ p_4(x) &= p_3(x)+\tfrac{(69-11\sqrt{33})x^4}{13824} &&= 1+x+\tfrac{x^2}{4}+\tfrac{x^3}{54}+\tfrac{(69-11\sqrt{33})x^4}{13824}\ , \end{aligned}$$

given by the inductive procedure:

$$p_n(x) = p_{n-1}(x) + A_n x^n,$$

where A_n is the maximal positive number such that $p_n(x)$ is hyperbolic. (The fact that such $A_n > 0$ exists is not completely obvious and will be proven in Lemma 9.) Set

$$p_\infty(x) = 1 + x + \sum_{n=2}^{\infty} A_n x^n = \lim_{n\to\infty} p_n(x),$$

where the limit is understood in the sense of formal power series. (This series appears on p. 42 of [20]. One can show that $p_\infty(x)$ is an entire function.)

The first result of this paper is as follows.

Theorem 2. *For any positive integer i, one has*

$$m_i = \inf_{p\in\Delta} \frac{a_i^2}{a_{i-1}a_{i+1}} = \frac{A_i^2}{A_{i-1}A_{i+1}}\ ,$$

where A_i are the above coefficients. In other words, m_i is attained at $p_{i+1} \in \Delta_{i+1}$, or, equivalently, the function $p_\infty(x)$ minimizes all m_i simultaneously. Moreover, (up to a scaling of the independent variable x), p_{i+1} is the unique polynomial in Δ_{i+1} minimizing the quantity $a_i^2/a_{i-1}a_{i+1}$.

By the above remark, one can also conclude that $\frac{a_i^2}{a_{i-1}a_{i+1}}$ attains its minimum at the monic polynomial $P_{i+1} \in \mathrm{Pol}_{i+1}$ which is the reciprocal polynomial to p_{i+1}. We call the inequalities of the form

$$\frac{a_i^2}{a_{i-1}a_{i+1}} \geq m_i, \tag{2}$$

with m_i defined above, *Petrovitch's inequalities,* and the inequalities of the form

$$\frac{a_i^2}{a_{i-1}a_{i+1}} \geq 4 \tag{3}$$

Hutchinson's inequalities; see footnote above. Using Theorem 1, one can show that the m_i are algebraic numbers and calculate them on a computer with arbitrary precision. For example, the first 10 decimals of the first 18 m_i's are as follows:

m_1	4	m_{10}	3.2336374426
m_2	$\frac{27}{8}$	m_{11}	3.2336368370
m_3	$\frac{2(69+11\sqrt{33})}{81}$	m_{12}	3.2336367032
m_4	3.2403064116	m_{13}	3.2336366736
m_5	3.2351101647	m_{14}	3.2336366671
m_6	3.2339623707	m_{15}	3.2336366656
m_7	3.2337086596	m_{16}	3.2336366653
m_8	3.2336525783	m_{17}	3.2336366652
m_9	3.2336401824	m_{18}	3.2336366652

Further calculations show that the next ten m_i have the same first 10 decimals as m_{17}, they are monotonically decreasing, and every second time the next decimal stabilizes. Our next result confirms this behavior.

Theorem 3. *The sequence $\{m_i\}_{i\geq 1}$ is strictly monotone decreasing.*

Recall that P_i denotes the reciprocal polynomial for p_i introduced above. (Since the constant term of p_i equals 1, P_i is monic.) In Lemma 9 below, we show that each P_i has only simple negative roots except for a single double root which has the minimal absolute value among the roots of P_i. Since each root of p_i is the inverse of the corresponding root of P_i, one gets that p_i also has all roots simple and negative except for a single double root which has the maximal absolute value among the roots of p_i. Denote by ξ_i the unique double root of P_i. For a positive integer i define the *scaled reciprocal polynomial* $\widetilde{P}_i(x) = P_i(-\xi_i x)/P_i(0)$. The scaling of P_i is performed in such a way that its double root is placed at -1 and its constant term equals 1.

1.2. **Main result.** From Theorems 1 and 3 we get that the sequence $\{m_i\}$ has a limit, which we denote by m_∞. The main result of this paper is the explicit description of m_∞ and the related entire function. To obtain this, we define the formal power series $\Psi(q,u)$, which we, by a small abuse of notation, call the *partial theta function*:

$$\Psi(q,u) = \sum_{j=0}^{\infty} q^{\binom{j+1}{2}} u^j. \tag{4}$$

This function already appears on p. 330 of [11].

Remark 2. The standard partial theta function is usually defined by the series $\Theta(q,u) = \sum_{j=0}^{\infty}(-1)^j q^{\binom{j}{2}} u^j$; see e.g. [1], [2], [24]. It is reasonable to refer both to Ψ and Θ as partial theta functions, since they satisfy the obvious relation

$$\Psi(q,u) = \Theta(q,-qu), \tag{5}$$

which allows us to translate properties from one to the other. Several beautiful identities satisfied by $\Psi(q,u)$ were stated without proofs in Ramanujan's "lost" notebook. The latter was found by G. E. Andrews, who subsequently put significant effort into proving these identities. New results about sums and products of partial theta functions can be found in e.g. [4]. This function is also of interest in statistical physics and combinatorics, see [25]; in Ramanujan type q-series, see [26]; in asymptotic analysis, see [5]; and in the theory of (mock) modular forms, see [6].

An excellent recent survey can be found in Chapter 6 of the edited version of the lost notebook; see [3].

In what follows, we always consider q as a parameter and u as the main variable. One can easily see that $\Psi(q,u)$ has a positive radius of convergence as a function of u if and only if $|q| \leq 1$. If $|q| = 1$, then $\Psi(q,u)$ has a radius of convergence equal to 1; while for any q with $|q| < 1$ the function $\Psi(q,u)$ is entire. Moreover for small positive q, the series $\Psi(q,u)$, considered as a function of u belongs to the Laguerre-Pólya class $\mathcal{L}-\mathcal{P}I$, i.e. it has all roots negative; see e.g. [16], Ch. 8. (The well-known characterization of these functions was obtained almost one hundred years ago in [22].) Notice that for $\Psi(q,u)$, the quotient

$$\frac{a_i^2}{a_{i-1}a_{i+1}} = q^{2\binom{i}{2}-\binom{i-1}{2}-\binom{i+1}{2}} = q^{-1}\ .$$

The main result of our paper is as follows.

Theorem 4. *The limit $m_\infty = \lim_{i\to\infty} m_i$ exists and coincides with $1/\widetilde{q}$, where $\widetilde{q} > 0$ is the maximal positive number for which the series $\Psi(q,u)$ belongs to the Laguerre-Pólya class $\mathcal{L}-\mathcal{P}I$ as a function of u. Moreover, the sequence $\{\widetilde{P}_i\}$ of the scaled reciprocal polynomials (introduced after Theorem 3 above) converges to $\Psi(\widetilde{q}, -\tilde{u}x)$, where $\tilde{u}$ is the unique real double root of $\Psi(\tilde{q},u)$.*

Notice that the constant $1/\widetilde{q} \approx 3.2336$ has earlier appeared in the papers [13] and [14], where the authors studied functions closely related to the above partial theta function; one of these in paper [13] is the function

$$g_a(x) := \sum_{k=0}^{\infty} \frac{x^k}{a^{k^2}},\ a > 1.$$

Part 3 of Theorem 4 of [13] claims that $g_a(x)$ has only real zeros if and only if $a^2 \geq 1/\widetilde{q}$, and that all but some finite number of its sections are hyperbolic, if and only if $a^2 > 1/\widetilde{q}$. (Similar statements can be found in the recent preprint [9].) Again, notice that there exists a simple relation between $g_a(x)$ and $\Theta(q,u)$, namely

$$g_{1/\sqrt{q}}(u) = -\Theta(q, \sqrt{q}u).$$

Comparing the latter equation with (5) and using Part 3 of Theorem 4 of [13], we conclude that the functions $g_{1/\sqrt{q}}(u)$, $\Psi(q,u)$ and $\Theta(q,u)$ belong to $\mathcal{L}-\mathcal{P}I$ exactly on the same interval $(0,\widetilde{q})$ of values of q. The approximate value of $\widetilde{q}$ to 10 decimal places is 0.3092493386; see Figure 1. We will later show that $\tilde{q}$ is a root of the transcendental equation (12).

Theorem 2 of [14] claims the following.

Theorem 5. *Let $f(x) = \sum_{j=0}^{\infty} a_j x^j$, $a_j > 0$ be an entire function and $S_n(x) = \sum_{j=0}^{n} a_j x^j$ be its sections. Suppose that there exists a subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ such that $S_{n_j}(x)$ is hyperbolic for $j = 1, 2, \ldots$. If*

$$\delta_\infty(f) = \lim_{n\to\infty} \frac{a_n^2}{a_{n-1}a_{n+1}}$$

exists, then for any positive integer m, $\displaystyle\sum_{j=0}^{m} \frac{x^j}{j!(\sqrt{\delta_\infty})^{j^2}}$ *is section hyperbolic.*

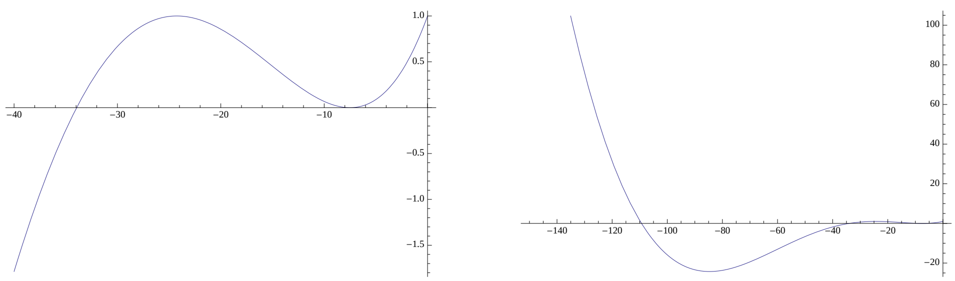

FIGURE 1. $\Psi(\widetilde{q}, u)$ in the intervals $[-40, 0]$ and $[-150, 0]$. (The value $\widetilde{u}$ of the negative double root of $\Psi(\widetilde{q}, u)$ to 10 decimal places is -7.5032559833.)

As a corollary of Theorem 5 one obtains that if for sufficiently large n all sections $1 + a_1 x + \cdots + a_n x^n$ of f are hyperbolic hyperbolic and $\delta_\infty(f)$ exists, then $\delta_\infty(f) \geq 1/\widetilde{q}$.

1.3. **Further directions and miscellanea.** Notice that $\Psi(q, x)$ belongs to $\mathcal{L} - \mathcal{P}I$ if and only if $q \in (0, \widetilde{q}]$ where $\widetilde{q}$ is the constant appearing in Theorem 4, see Theorem 4 of [13].

Definition. We call the set $\mathfrak{S}$ of all $q^* \in \mathbb{C}$, $|q^*| < 1$ such that the partial theta function $\Psi(q^*, u)$ (as a function of u) has a double root the *spectrum* of $\Psi(q, u)$.

The following additional statement is proven in the forthcoming paper [15].

Theorem 6. *The spectrum* $\mathfrak{S}$ *contains infinitely many positive values* $0 < \widetilde{q} = \hat{q}_1 < \hat{q}_2 < \cdots < \hat{q}_N < \cdots < 1$, *i.e. there exists an infinite sequence* $\{\hat{q}_i\}_{i \geq 1}$ *of numbers in* $(0, 1)$ *such that for each positive integer* i *the function* $\Psi(\hat{q}_i, u)$ *has a negative double root in the variable* u.

Interesting numerical information about the positive part of $\mathfrak{S}$ was recently obtained by two talented high school students A. Broms and I. Nilsson together with a Ph.D. student of the second author P. Alexandersson. Namely, they calculated the first 40 values of $\hat{q}_i$ presented below to 10 decimal places.

$\hat{q}_1$	0.3092493386	$\hat{q}_{11}$	0.8723052796	$\hat{q}_{21}$	0.9296892842	$\hat{q}_{31}$	0.9514695286
$\hat{q}_2$	0.5169593598	$\hat{q}_{12}$	0.8819492604	$\hat{q}_{22}$	0.9327105022	$\hat{q}_{32}$	0.9529270933
$\hat{q}_3$	0.6306283161	$\hat{q}_{13}$	0.8902367613	$\hat{q}_{23}$	0.9354824750	$\hat{q}_{33}$	0.9542995777
$\hat{q}_4$	0.7012650701	$\hat{q}_{14}$	0.8974353356	$\hat{q}_{24}$	0.9380348437	$\hat{q}_{34}$	0.9555942239
$\hat{q}_5$	0.7492689316	$\hat{q}_{15}$	0.9037465007	$\hat{q}_{25}$	0.9403927267	$\hat{q}_{35}$	0.9568174745
$\hat{q}_6$	0.7839844578	$\hat{q}_{16}$	0.9093249273	$\hat{q}_{26}$	0.9425775510	$\hat{q}_{36}$	0.9579750799
$\hat{q}_7$	0.8102506509	$\hat{q}_{17}$	0.9142913049	$\hat{q}_{27}$	0.9446077062	$\hat{q}_{37}$	0.9590721890
$\hat{q}_8$	0.8308155811	$\hat{q}_{18}$	0.9187411894	$\hat{q}_{28}$	0.9464990647	$\hat{q}_{38}$	0.9601134256
$\hat{q}_9$	0.8473534437	$\hat{q}_{19}$	0.9227512224	$\hat{q}_{29}$	0.9482653982	$\hat{q}_{39}$	0.9611029540
$\hat{q}_{10}$	0.8609417724	$\hat{q}_{20}$	0.9263835914	$\hat{q}_{30}$	0.9499187138	$\hat{q}_{40}$	0.9620445345

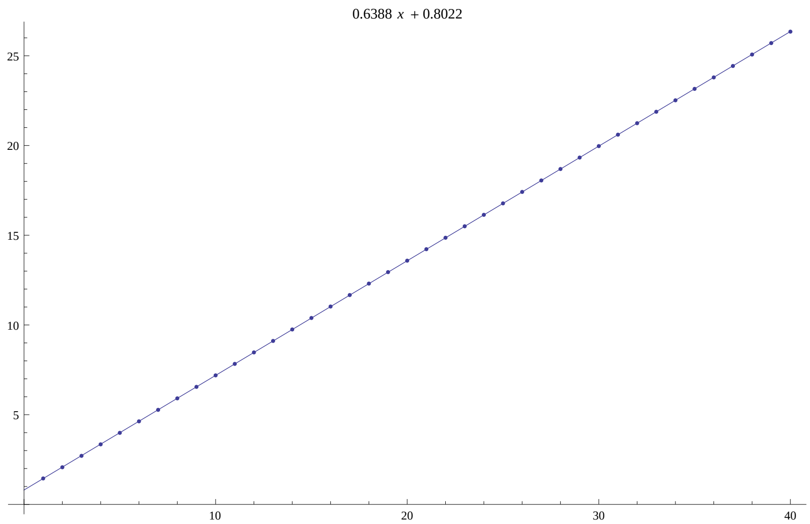

FIGURE 2. The first 40 values of $\frac{1}{1-\hat{q}_i}$.

The plot of the values of $1/(1-\hat{q}_i)$ against the number i as given in Fig. 2 suggests that

$$\hat{q}_i \approx 1 - \frac{1}{0.6388i + 0.8022}.$$

Problem 2. What is the asymptotics of the sequence $\{\hat{q}_i\}$ when $i \to \infty$?

Let us also mention the following question posed by Professor A. Sokal.

Problem 3. Is it true that $\mathfrak{S}$ is empty within the open disk $|q| < \widetilde{q}$?

Finally let us mention a technical result of independent interest very much in the spirit of the modern study of amoebas of complex hypersurfaces, and, in particular, of discrimininants, see e.g. [8]. Denote by $L\Sigma_n \subset \mathbb{R}^n$ (respectively $L\Delta_n \subset \mathbb{R}^n$) the image of $\Sigma_n \subset \mathrm{Pol}_n$ (respectively of $\Delta_n \subset \mathrm{Pol}_n$) under taking coefficientwise logarithms.

Proposition 7. (i) *The polyhedral cone described by Hutchinson's inequalities* (3) *(coinciding with the logarithmic image of the set of two-sided section-hyperbolic polynomials) is the maximal polyhedral cone contained in $L\Delta_n$. The same cone is the maximal polyhedral cone contained in $L\Sigma_n$.*
(ii) *The minimal polyhedral cone containing $L\Delta_k$ is given by Petrovitch's inequalities* (2) *while the minimal polyhedral cone containing $L\Sigma_n$ is given by Newton's inequalities* (1)*; see Remark 1.*

Notice that Hutchinson's cone is, on the other hand, the recession cone of the logarithmic image of the set of sign-invariant hyperbolic polynomials; see [21]. A fact similar to Proposition 7 is proven in Theorem F of [14].

Acknowledments. The authors want to thank Professors O. Katkova and A. Vishnyakova of Kharkov National University for the formulation of the problem and their proof of Theorem 1, which they kindly allowed us to include in the present paper. They also gave us important hints and contributed to the proof of Lemma 18. We are sincerely grateful to Professors A. Eremenko and A. Sokal and, especially, Professor G. E. Andrews for valuable information about partial theta functions. The second author wants to acknowledge the hospitality of Laboratoire de Mathématiques, Université de Nice during his visit in April-May 2011, when this project was carried out. Finally, we thank our anonymous referees for their useful remarks which allowed us to considerably improve the quality of exposition.

2. Proving Theorems 1 - 3

We start with Theorem 1.

Proof. Take some polynomial $r_n(x) = a_0 + a_1x + \ldots + a_nx^n$ belonging to $\mathcal{P}_n$ and set $r_i(x) = a_0 + a_1x + \ldots + a_ix^i$ $(i = 2, 3, \ldots, n)$. By our assumption, the polynomials $r_i(x)$ are hyperbolic for all i. Set

$$\gamma_i := \frac{a_{i-1}}{a_i}, \quad \delta_i := \frac{\gamma_i}{\gamma_{i-1}} = \frac{a_{i-1}^2}{a_{i-2}a_i}.$$

Let us fix an arbitrary $i = 3, 4, \ldots, n$. Denote by $0 > x_1^{(i)} \geq x_2^{(i)} \geq \ldots \geq x_i^{(i)}$ the zeros of $r_i(x)$ and by $N_j(X^{(i)})$ their Newton power sum $\sum_{\nu=1}^{i}(x_\nu^{(i)})^j$. Using the Cauchy inequality with $|x_j^{(i)}|^{1/2}$ and $|x_j^{(i)}|^{3/2}$ we get

$$N_1(X^{(i)})N_3(X^{(i)}) \geq (N_2(X^{(i)}))^2 \ .$$

From the standard identities for (elementary) symmetric functions, we get

$$\begin{array}{rcl} N_1(X^{(i)}) & = & -\gamma_i \\ N_2(X^{(i)}) & = & \gamma_i(\gamma_i - 2\gamma_{i-1}) \\ N_3(X^{(i)}) & = & -\gamma_i(\gamma_i^2 - 3\gamma_i\gamma_{i-1} + 3\gamma_{i-1}\gamma_{i-2}) \ . \end{array}$$

Substituting these identities in the above inequality and dividing by $\gamma_i^2\gamma_{i-1}$, we obtain

$$\gamma_i - 4\gamma_{i-1} + 3\gamma_{i-2} \geq 0.$$

Dividing the latter inequality by γ_{i-2} and using δ_j's, we get the following inequality:

$$\delta_i\delta_{i-1} - 4\delta_{i-1} + 3 \geq 0. \tag{6}$$

Since $i = 3, 4, \ldots, n$ is an arbitrary index, we get from (6) the following system of inequalities:

$$\delta_i\delta_{i-1} - 4\delta_{i-1} + 3 \geq 0 \quad (i = 3, 4, \ldots, n). \tag{7}$$

Since $r_2(x)$ is hyperbolic, we have $\delta_2 \geq 4$. Suppose that the statement of the theorem is not true, and denote by j the smallest index such that $\delta_j < 3$, so that $\delta_{j-1} \geq 3$ and $\delta_j < 3$ $(j = 3, 4, \ldots, n)$. We rewrite (7) for $i = j$ in the form

$$(\delta_j - 4)\delta_{j-1} + 3 \geq 0.$$

Since $\delta_j - 4 < 0$, and $\delta_j < \delta_{j-1}$, the above inequality implies that

$$(\delta_j - 4)\delta_j + 3 > 0;$$

whence $\delta_j \in (-\infty, 1) \bigcup (3, +\infty)$. By our assumption, $r_j(x)$ is a hyperbolic polynomial; thus $\delta_j \in (-\infty, 1)$ is impossible. We conclude that $\delta_j \geq 3$. □

To prove Theorem 2, we need some preliminaries. Observe that rescaling of the independent variable x by an arbitrary positive constant acts on all spaces of polynomials we introduced above, preserving the quantities $a_i^2/a_{n-1}a_n$. This action allows us to normalize $a_1 = 1$ in Pol_n, and analogously $a_1 = 1$ in $\mathcal{P}_n$, and therefore to reduce the number of parameters by one. Define $\mathcal{P}_n^1$ as the space of polynomials of the form $p(x) = 1 + x + a_2x^2 + \ldots + a_nx^n$, and Pol_n^1 as the space of polynomials of the form $P(x) = x^n + x^{n-1} + a_2x^{n-2} + \ldots + a_n$. Notice that taking the reciprocal sends $\mathcal{P}_n^1$ onto Pol_n^1 and that our main polynomials p_i belong to $\mathcal{P}_i^1$, while their reciprocal polynomials P_i belong to Pol_i^1. From now on, instead of working in $\Delta_n \subset \mathrm{Pol}_n$, we will work in $\Delta_n^1 \subset \mathrm{Pol}_n^1$ which is the restriction of Δ_n to Pol_n^1. The above group action carries our proofs from one space to the other.

Define the standard embedding

$$\mathrm{em}_{j,n} : \mathrm{Pol}_j^1 \to \mathrm{Pol}_n^1, \ j < n$$

(respectively $\mathrm{Pol}_j \to \mathrm{Pol}_n$), given by multiplication of a monic polynomial of degree $j < n$ by x^{n-j}. Obviously, the image $\mathrm{em}_{j,n}(\mathrm{Pol}_j^1) \subset \mathrm{Pol}_n^1$ coincides with the coordinate subspace of all monic polynomials having all coefficients of degree less than $n - j$ vanishing. Denote by $\mathcal{D}_j \subset \mathrm{Pol}_j^1$ the standard discriminant consisting of all monic polynomials of degree j having at least one real root of multiplicity at least 2. Embedding $\mathcal{D}_j$ into Pol_n^1 using $\mathrm{em}_{j,n}$, let us define the discriminant $\mathcal{D}_{j,n} \subset \mathrm{Pol}_n^1$ by taking the trivial $(n-j)$-dimensional cylinder over $\mathrm{em}_{j,n}(\mathcal{D}_j)$ along all coefficients of degree less than $(n - j)$. Define $\Delta_n^1 \subset \mathrm{Pol}_n^1$ and $\Sigma_n^1 \subset \mathrm{Pol}_n^1$ as the restrictions of Δ_n and Σ_n to Pol_n^1. Finally, consider the closure $\overline{\Delta}_n^1 \subset \mathrm{Pol}_n^1$ of the set $\Delta_n^1 \subset \mathrm{Pol}_n^1$.

Lemma 8. (i) *The set $\overline{\Delta}_n^1$ has a natural stratification of an $(n-1)$-dimensional simplex with vertices $P_1 = x^n + x^{n-1}$, $P_2 = x^n + x^{n-1} + x^{n-2}/4$, $P_3, \ldots, P_n$. Distinct $(n-2)$-dimensional (boundary) faces of $\overline{\Delta}_n^1$ belong to distinct $\mathcal{D}_{j,n}, \ j = 0, 1, 2, \ldots, n$; see Figure 3.*

(ii) *The natural projection π_n 'forgetting' the constant term sends $\overline{\Delta}_n^1$ onto $\overline{\Delta}_{n-1}^1$.*

(iii) *Any polynomial in $\overline{\Delta}_n^1$ can be connected to P_n by a smooth path along which all coefficients are non-decreasing.*

Remark 3. The original set Δ_n (respectively $\overline{\Delta}_n$) is the cylinder over Δ_n^1 (respectively $\overline{\Delta}_n$) obtained by the group action of rescaling of x by positive constants.

Proof of Lemma 8. The first two statements are rather obvious and proved by induction. The inductive step of statement (i) looks like this: denote by $F \subset \overline{\Delta}_{n-1}^1$ a face of maximal dimension such that the hyperbolic polynomials defined by its points have all roots distinct. For $P \in F$, the values of $c \geq 0$ for which the polynomial $P + c$ is hyperbolic is a segment or a point. Thus F gives rise to a face $F^* \subset \overline{\Delta}_n^1$ which is a one-dimensional cylinder over F. Distinct faces F of $\overline{\Delta}_{n-1}^1$ give rise to distinct faces F^* of $\overline{\Delta}_n^1$. The interior Δ_{n-1}^1 of $\overline{\Delta}_{n-1}^1$ is a face F_0 of $\overline{\Delta}_n^1$. There is exactly one face F_1 of $\overline{\Delta}_n^1$ which is not created in this way. It is a graph of a continuous function defined in Δ_{n-1}^1. Hence F_0 and F_1 are distinct from all other $(n-2)$-dimensional faces of $\overline{\Delta}_n^1$ (and do not coincide with each other).

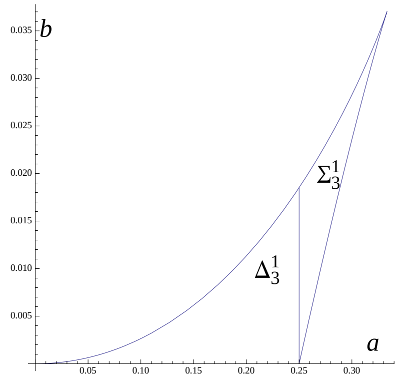

FIGURE 3. Domains Σ_3^1 and Δ_3^1 for the family x^3+x^2+ax+b. (Σ_3^1 is the larger curvilinear triangle containing the smaller curvilinear triangle Δ_3^1.)

To prove (ii) just observe that the projection π_n sends each of the three sets Δ_n^1, F_0 and F_1 onto Δ_{n-1}^1. It also sends F^* to F for the rest of the $(n-2)$-dimensional faces of $\overline{\Delta}_n^1$.

$\overline{\Delta}_{n-1}^1$ is naturally embedded in the hyperplane $a_n = 0$ of Pol_n^1 using the multiplication of polynomials of degree $n-1$ by x. Then $\overline{\Delta}_n^1$ is fibered over the image of $\overline{\Delta}_{n-1}^1$ in Pol_n^1 along the constant term. To prove (iii) we show that the face $\mathcal{D}_n$ of the boundary of $\overline{\Delta}_n^1$ can be expressed as $a_n = a_n(a_2, \ldots, a_{n-1})$, where $(a_2, \ldots, a_{n-1}) \in \overline{\Delta}_{n-1}^1$. Moreover, one has $\partial a_n/\partial a_i > 0$ for $i = 2, \ldots, n-1$ in the whole open set Δ_{n-1}^1. Indeed, denote by $x_i < 0$ the roots of $P(x)$. The double root is denoted by $x_{n-1} = x_n$. Since $a_i > 0$ for $i = 1, \ldots, n$ it will be convenient to consider them as elementary symmetric functions in the positive quantities $-x_j$. One has

$$\frac{\partial a_n}{\partial a_i} = \sum_{j=1}^{n-1} \frac{\partial a_n/\partial(-x_j)}{\partial a_i/\partial(-x_j)}.$$

The quantities a_n, $\partial a_n/\partial(-x_j)$ and $\partial a_i/\partial(-x_j)$ are given by homogeneous polynomials with positive coefficients in all $-x_j$. This fact implies that the directional derivative of the function a_n is non-negative along any vector in Δ_{n-1}^1 with all non-negative coordinates. Using this statement together with induction on n, we get that any polynomial in Δ_n^1 can be connected to P_n by a smooth path with non-decreasing coordinates. □

Remark 4. In Lemma 9 below, we will prove that each polynomial in Δ_n^1 has either simple negative roots or at most one double root (in which case it belongs to $\mathcal{D}_n$), which is the rightmost among all roots of the considered polynomial.

Proof of Theorem 2. Given some reciprocal section-hyperbolic polynomial $P(x) = x^n+x^{n-1}+a_2x^2+\ldots+a_n$ of degree $n \geq 4$ consider the function $\delta_n := a_{n-1}^2/a_{n-2}a_n$. We want to show that $m_{n-1} = \min_{\Delta_n^1} \delta_n$ is attained at $P_n(x)$, which is the reciprocal polynomial to $p_n(x)$ defined in the Introduction. For fixed $a_2, \ldots, a_{n-1}$, the function

δ_n is minimal when a_n is maximal, in which case the polynomial $P(x)$ belongs to $\mathcal{D}_n$. Thus, we can restrict our consideration to $\Delta_n^1 \ni P(x) \in \mathcal{D}_n$. Since each $P(x)$ can be connected to $P_n(x)$ by a smooth path along which each coefficient is non-decreasing it is enough to show that for $i = 2, \dots, n-1$ the partial derivative $\partial\delta_n/\partial a_i$ is negative when δ_n is restricted to $\mathcal{D}_n$.

Recall that $a_{n-1} = (-1)^{n-1}e_{n-1}(x_1, \dots, x_n)$ and $a_n = (-1)^n e_n(x_1, \dots, x_n)$, where x_i are the zeros of P and e_i is the ith elementary symmetric polynomial. There are three different cases to consider: (1) $i < n-2$; (2) $i = n-2$ and (3) $i = n-1$. For $i < n-2$ one has

$$\frac{\partial\delta_n}{\partial a_i} = \frac{\partial\delta_n}{\partial a_n}\frac{\partial a_n}{\partial a_i} = -\frac{a_{n-1}^2}{a_{n-2}a_n^2}\frac{\partial a_n}{\partial a_i}.$$

Since all $a_i > 0$ and $\partial a_n/\partial a_i > 0$ on $\mathcal{D}_n$, by Lemma 8, case (1) is settled. Analogously, we have

$$\frac{\partial\delta_n}{\partial a_{n-2}} = -a_{n-1}^2\frac{a_n + a_{n-2}(\partial a_n/\partial a_{n-2})}{(a_{n-2}a_n)^2}\ .$$

Again since all $a_i > 0$ and $\partial a_n/\partial a_{n-2} > 0$ on $\mathcal{D}_n$, case (2) is settled.

Finally, one has

$$\frac{\partial\delta_n}{\partial a_{n-1}} = \frac{2a_{n-1}}{a_{n-2}a_n} - \frac{a_{n-1}^2(\partial a_n/\partial a_{n-1})}{a_{n-2}a_n^2} = \frac{a_{n-1}(2a_n - a_{n-1}(\partial a_n/\partial a_{n-1}))}{(a_{n-2}a_n^2)}\ ,$$

where

$$a_{n-1}\frac{\partial a_n}{\partial a_{n-1}} = a_{n-1}\sum_{i=1}^{n-1}\frac{\partial a_n}{\partial(-x_i)}\bigg/\frac{\partial a_{n-1}}{\partial(-x_i)}\ .$$

For $i \neq n-1$ one has $a_{n-1} = f_i - x_i g_i$, where f_i and g_i are homogeneous polynomials in the positive variables $-x_k$, $k \neq i$, having all their coefficients positive. Therefore for $i \neq n-1$ one has

$$(8)\quad a_{n-1} > -x_i g_i = -x_i\frac{\partial a_{n-1}}{\partial(-x_i)} \quad \text{and} \quad a_{n-1}\frac{\partial a_n}{\partial(-x_i)}\bigg/\frac{\partial a_{n-2}}{\partial(-x_i)} > -x_i\frac{\partial a_n}{\partial(-x_i)}\ .$$

For $i = n-1$ one has $a_{n-1} = -x_{n-1}v + x_{n-1}^2 w$ and

$$-x_{n-1}\frac{\partial a_{n-1}}{\partial(-x_{n-1})} = -x_{n-1}v + 2x_{n-1}^2 w.$$

Here v and w are homogeneous polynomials with positive coefficients in $-x_j$, $j \neq n-1$. (Their explicit formulas are unnecessary for our purposes.) Therefore, one has

$$a_{n-1} > -\frac{x_{n-1}}{2}\frac{\partial a_{n-1}}{\partial(-x_{n-1})}.$$

Thus

$$2a_n - a_{n-1}\frac{\partial a_n}{\partial a_{n-1}} < 2a_n + \frac{1}{2}\sum_{i=1}^{n-1}x_i\frac{\partial a_n}{\partial(-x_i)} < a_n\left(2 - \frac{n}{2}\right) \leq 0\ .$$

For the homogeneous polynomial a_n of degree n we used Euler's identity:

$$na_n = -\sum_{i=1}^{n-1} x_i \frac{\partial a_n}{\partial(-x_i)}.$$

By the above argument, any directional derivative $\partial\delta_n/\partial\vec{u}$ is non-positive if $\vec{u}$ is an arbitrary vector in Δ^1_{n-1} with all non-negative coordinates. Moreover since any polynomial $P \in \Delta^1_n$ can be connected with P_n by a smooth path with nondecreasing (and on some subintervals strictly increasing) coordinates, the value of δ_n at P_n is strictly smaller than at any other such $P \neq P_n$. In the bigger set Δ_n this means that only polynomials obtained from P_n by scaling of the variable x can have the same value of $\delta_n = m_{n-1}$ as P_n has. The result follows. □

Now we settle Theorem 3 by induction.

Proof. The base of induction is $4 = m_1 > m_2 = \frac{27}{8}$. Assume now that the statement is proved for m_{i-1}; we want to show that $m_{i-1} > m_i$. By Theorem 2, m_{i-1} is attained as the quotient $a^2_{i-1}/a_{i-2}a_i$ at the polynomial P_i which is a monic polynomial of degree i. Moreover, up to scaling of x, P_i is the unique polynomial in Pol_i, where this minimum is attained.

Set $P_i(x) = \sum_{j=0}^{i} \gamma_j x^j$, $\gamma_i = 1$. The quotient $\gamma_1^2/\gamma_2\gamma_0$ coincides with m_{i-1}. Given a polynomial R, denote by $R^{[k]}$ the result of the k-th truncation of R from the back, i.e. the polynomial obtained by removing all terms of R of degree smaller than k.

Consider a perturbation $R(x) := P_i(x) + \varepsilon Q(x)$, where $\varepsilon > 0$ and Q is a monic polynomial of degree $i+1$. We choose Q such that for $k = 0, \dots, i-2$, the truncation $Q^{[k]}$ has a root at the unique negative double root of ${P_i}^{[k]}$. (Notice that ${P_i}^{[i-1]}$ has a single negative real root, which we do not have to worry about.)

Setting $Q := x^{n+1} + \sum_{j=0}^{n-2} \alpha_j x^j$, one can easily see that the latter condition yields a triangular linear system for the undetermined coefficients α_j. Hence it has a unique real solution, which we denote by Q^*.

For $\varepsilon > 0$ small enough, all coefficients of the polynomial $R^*(x) = P_i(x) + \varepsilon Q^*(x)$ are positive. All roots of all polynomials ${R^*}^{[k]}$, $k = 0, \dots, i-1$, are distinct and negative. Moreover, for the perturbation R^* the quantity

$$(\gamma_1 + \varepsilon\alpha_1)^2/(\gamma_2 + \varepsilon\alpha_2)(\gamma_0 + \varepsilon\alpha_0)$$

tends to m_{i-1} as $\varepsilon \to 0$. Therefore $m_i \leq m_{i-1}$. To prove that the latter inequality is strict we argue as follows.

The quantity $\gamma_1^2/\gamma_2\gamma_0$ does not change when one performs a linear change of the variable x. Perform such a change, after which the polynomial R^* (up to a constant factor) becomes $x^{i+1} + x^i + \dots$. The linear change and the subsequent multiplication by a positive number increase the coefficient of x^{i+1} and decrease the coefficients of x^l for $l < i$. The latter tend to 0 as $\varepsilon \to 0$.

Now consider P_{i+1}. The foregoing means that one can find $\delta > 0$ and a sequence Z of reciprocal section-hyperbolic polynomials of degree $i+1$ remaining outside the ball B_δ centered at P_{i+1} and of radius δ, for which the quantity $\gamma_1^2/\gamma_2\gamma_0$ tends to m_i. Indeed, all coefficients of P_{i+1} are positive, while some of the coefficients of polynomials belonging to $\{Z\}$ tend to 0.

One knows that (up to a scaling) the minimal value of the quantity $\gamma_1^2/\gamma_2\gamma_0$ in Pol_{i+1} is attained only at P_{i+1}. Therefore there exists $\eta > 0$ such that for all reciprocal section-hyperbolic polynomials from ∂B_δ the quantity $\gamma_1^2/\gamma_2\gamma_0$ exceeds $m_i + \eta$.

On the other hand, similarly to what we did while proving part (iii) of Lemma 8, one can define a procedure of continuously changing a polynomial Z into the polynomial P_{i+1} so that the quantity $\gamma_1^2/\gamma_2\gamma_0$ strictly decreases. The continuous deformation intersects B_δ. Hence B_δ contains section-hyperbolic polynomials of degree $i+1$ whose quantity $\gamma_1^2/\gamma_2\gamma_0$ is both greater than $m_i+\eta$ and less than some number arbitrarily close to m_{i-1}. Hence $m_i < m_{i-1}$. □

In Lemma 8, we proved that Δ_n^1 is a curvilinear $(n-1)$-dimensional simplex with vertices $P_1 = x^n + x^{n-1}$, $P_2 = x^n + x^{n-1} + x^{n-2}/4$, P_3, …, P_n. Next we present a corollary of Theorem 3 and Lemma 8 describing the behavior of $\delta_n = a_{n-1}^2/a_{n-2}a_n$ on $n-1$ edges of this simplex connecting the (most important) new vertex P_n with already existing vertices P_1, …, P_{n-1}.

Corollary 1. *For $i = 1, \ldots, n-2$, the restriction of $\delta_n = a_{n-1}^2/a_{n-2}a_n$ onto the edge $e_{i,n}$ of Δ_n^1 connecting P_i to P_n is monotone decreasing from the value m_{n-i-1} to m_{n-1}. On the remaining edge $e_{n-1,n}$, the function δ_n decreases from $+\infty$ to m_{n-1}.*

Notice that formally δ_n is undefined at P_1, …, P_{n-1}, so the claim

$$\delta_n(P_i) = m_{n-i-1}$$

in Corollary 1 should be understood as a limit.

3. Proving Theorem 4

We now proceed with Theorem 4, whose proof requires a number of intermediate steps. Its main claim is that the sequence $\{\widetilde{P}_i\}$ of the scaled reciprocal polynomials converges to $\Psi(\tilde{q}, -\tilde{u}x)$.

For technical reasons it will be convenient, in parallel with the sequence $\{P_i\}$, to work with the sequence $\{S_i\}$ of polynomials given by

$$S_i = xP_i,$$

(hence $\deg S_i = i+1$). In order to prove Theorem 4 we have to study in detail the behavior of the roots and the critical points in the sequences $\{P_i\}$ and $\{S_i\}$ as well as of their scaled versions.

We will first prove that each P_i has all its zeros negative. Moreover, exactly one of these roots, ξ_i, the rightmost one, is a double root the rest being simple. This circumstance implies that, for $i \geq 3$, the point ξ_i is the rightmost critical point of S_{i-1}, and it is a local minimum. Moreover, the critical values of S_{i-1} at all other local minima are smaller than $S_{i-1}(\xi_i)$, i.e. the absolute values of all other local minima are larger than the one at ξ_i; see Lemma 9 below.

This will imply the recurrence:

$$A_i = -S_{i-1}(\xi_i), \quad P_i(x) = S_{i-1}(x) + A_i, \quad \text{and} \quad S_i(x) = x(S_{i-1}(x) + A_i). \tag{9}$$

Formula (9) is well-defined if $i := \deg P_i \geq 2$. In the exceptional case $\deg P_1 = 2$ we set $S_2 = xS_1$.

Lemma 9. *The following holds:*

(1) *The polynomials P_i have all roots negative.*
(2) *Exactly one of these roots (namely, the one at ξ_i) is a double root and the rest are simple.*
(3) *For $i \geq 3$ the point ξ_i is the rightmost critical point of S_{i-1}; it is a local minimum; the critical values of S_{i-1} at all other local minima are smaller than $S_{i-1}(\xi_i)$, i.e. the absolute values of all other local minima are larger than the one at ξ_i.*
(4) *For $i \geq 5$ the critical values of S_{i-1} at its maxima form a strictly decreasing sequence and the critical values at its minima form a strictly increasing sequence when the critical points are listed in the increasing order.*

Proof. Recall that polynomials p_1, p_2, p_3 and p_4 are listed in the Introduction and one can check that the current Lemma holds for $i = 1$, 2, 3 and 4 directly. Additionally, item (1) follows from the definition of the polynomials p_i, $i \geq 1$ and the fact that P_i is the reciprocal of p_i.

We will use induction. Suppose that Lemma 9 holds for $i-1$ ($i \geq 5$). When obtaining S_i from S_{i-1} (resp. P_i from P_{i-1}) using (9), one first subtracts $S_{i-1}(\xi_i)$. This operation preserves the order of the critical values at the local minima (resp. maxima) when the critical points are listed in the increasing order. The rightmost minimum becomes a double zero of P_i. All other roots of P_i are simple and smaller than this zero. This follows from part (3) of the present lemma which holds for S_{i-1}. Hence parts (1) and (2) of the lemma hold for P_i.

Then in order to get S_i one multiplies $P_i := S_{i-1}(x) - S_{i-1}(\xi_i)$ by x. If P_i has a maximum (minimum) between its zeros $u < v$, then $S_i = xP_i$ has a minimum (maximum) on (u, v).

Consider four consecutive zeros $u < v < w < h$ of P_i. Denote by μ_1 and μ_2 (resp. M_1 and M_2) the critical values of $S_{i-1}(x) - S_{i-1}(\xi_i)$ (resp. S_i) on (u, v) and (w, h) respectively. Then $|\mu_1| > |\mu_2|$ and $|M_1| \geq |\mu_1||v|$, $|M_2| \leq |\mu_2||w|$, hence $|M_1| > |M_2|$.

Recall that $P_i(\xi_{i-1}) = P_i(0)$ (because $S_{i-1}(\xi_{i-1}) = S_{i-1}(0) = 0$). Consider the minimum of S_i at $\xi_{i+1} \in (\xi_i, 0)$. Denote by μ' the value of S_i at the second minimum from the right, i.e. the next to the left w.r.t. ξ_{i+1}. The corresponding critical point is $< \xi_{i-1}$. Indeed, $(xP_i)'|_{x=\xi_{i-1}} = P_i(\xi_{i-1}) > 0$. Thus

$$|\mu'| > |\xi_{i-1}||P_i(\xi_{i-1})| > |\xi_i||P_i(\xi_{i-1})| \qquad \text{and}$$

$$|S_i(\xi_{i+1})| < |\xi_{i+1}||P_i(0)| = |\xi_{i+1}||P_i(\xi_{i-1})| < |\mu'| \ .$$

The next to last inequality follows from the fact that P_{i-1} is monotone increasing on $(\xi_i, 0)$. Hence items (3) and (4) of Lemma 9 hold for S_i. □

Our further plan is as follows. Lemmas 10, 11, 12 and 13 summarize additional properties of the polynomials S_i. Lemmas 10 and 13 give upper and lower bounds for the ratio ξ_{i+1}/ξ_i. Lemma 11 gives an estimation of how fast the quantities $-S_{i-1}(\xi_i)$ decrease; this estimate is then used in the proof of Lemma 12. The latter states that for i fixed the sequence $S_i(\xi_s)$ is sign alternating.

Lemma 10. *The ratio ξ_{i+1}/ξ_i is less than or equal to $1/3$ for $i \geq 2$. Moreover, the equality takes place only for $i = 2$.*

Proof. Recall that $\deg S_i = i+1$ and $P_i := S_{i-1}(x) - S_{i-1}(\xi_i)$. The quantity ξ_{i+1} satisfies the equation $\xi_{i+1}P_i'(\xi_{i+1}) + P_i(\xi_{i+1}) = 0$. Changing the variable x to $-\xi_i x$ we can w.l.o.g. assume that $\xi_i = -1$. Hence

$$-\frac{1}{\xi_{i+1}} = \frac{P_i'(\xi_{i+1})}{P_i(\xi_{i+1})} = \frac{2}{\xi_{i+1}+1} + \sum_{j=1}^{i-2} \frac{1}{\xi_{i+1}-\alpha_j} \ , \tag{10}$$

where α_j are the roots of P_i smaller than -1 (enumerated in the increasing order).

The equation

$$-\frac{1}{\xi_{i+1}} = \frac{2}{\xi_{i+1}+1} \tag{11}$$

has the unique solution $\xi_{i+1} = -1/3$. One can easily check that $-1/3$ is the solution of (10) only for $i = 2$ in which case (up to rescaling of x)

$$P_1 = (x+2), \quad S_1 = x(x+2), \quad S_2 = x(x+1)^2.$$

For larger i the presence of the additional summand $\sum_{j=1}^{i-2} 1/(\xi_{i+1}-\alpha_j)$ in the right-hand side implies that the graphs of the l.h.s. and the r.h.s. of (10) intersect each other closer to the origin than $-1/3$. Indeed, the function $-1/\xi_{i+1}$ is increasing on $(-1,0)$ while the functions $2/(\xi_{i+1}+1)$ and $\sum_{j=1}^{i-2} 1/(\xi_{i+1}-\alpha_j)$ are decreasing there. Each of these functions takes positive values on $(-1,0)$. The functions $-1/\xi_{i+1}$ and $2/(\xi_{i+1}+1)$ tend to $+\infty$ when their arguments tend to 0 and -1 respectively. $\square$

Recall that $A_i = -S_{i-1}(\xi_i)$, i.e. A_i is the absolute value of the largest minimum of S_{i-1}.

Lemma 11. *The inequality $A_l \leq A_m(4|\xi_m|)^{l-m}/3^{(l-m)(l-m+5)/2}$ holds for $l > m > 1$.*

Proof. By Taylor's formula applied at ξ_m we get

$$S_{m-1}(x) - S_{m-1}(\xi_m) = (x-\xi_m)^2 b(t_x),$$

where $x \in (\xi_m, 0]$, $t_x \in (\xi_m, x)$ and $b = S''_{m-1}/2$. Hence $S_m(x) = x(x-\xi_m)^2 b(t_x)$.

The function b is non-decreasing on $[\xi_m, 0]$ and for $m > 2$ it is strictly increasing. Indeed, $b(t)$ is the second derivative of the hyperbolic polynomial S_{m-1} having all its nonzero roots smaller than ξ_m. Therefore, $b(\xi_m) > 0$.

For $m > 2$ the quantity t_x is an increasing function of x. Indeed, consider the functions $F := (x-\xi_m)^2 b(t_x)$ and $G := (x-\xi_m)^2 b(t_{x_1})$. For $0 \geq x_2 > x_1 > \xi_m$ we get $F(x_1) = G(x_1)$ and $F(x_2) > G(x_2)$. Therefore $b(t_{x_2}) > b(t_{x_1})$ hence $t_{x_2} > t_{x_1}$.

Consider the quantity $|S_m(x)| = |x(x-\xi_m)^2 b(t_x)|$. Its maximum on $[\xi_m, 0]$ equals $A_{m+1} = |S_m(\xi_{m+1})|$. Set $R(x) := |x(x-\xi_m)^2|$. Hence

$$\max_{x\in[\xi_m,0]} R(x) = R\left(\frac{\xi_m}{3}\right) = \frac{4}{27}|\xi_m|^3 \ .$$

On the other hand,

$$\begin{aligned} A_{m+1} &= |S_m(\xi_{m+1})| = R(\xi_{m+1})b(t_{\xi_{m+1}}) < R\left(\tfrac{\xi_m}{3}\right)b(t_0) \\ &= \tfrac{4}{27}|\xi_m|\xi_m^2 b(t_0) = \tfrac{4}{27}|\xi_m|A_m \ . \end{aligned}$$

This is the required inequality for $l = m+1$. To obtain it for $l = m+2$ recall that $|\xi_{m+1}| \le |\xi_m|/3$, by Lemma 10. Hence

$$A_{m+2} < \frac{4}{27}|\xi_{m+1}|A_{m+1} \le \frac{1}{3}\left(\frac{4}{27}\right)^2 \xi_m^2 A_m \ .$$

Suppose that $A_l \le A_m(4|\xi_m|^{l-m})/3^{(l-m)(l-m+5)/2}$. Then $|\xi_l| \le |\xi_m|/3^{l-m}$ and

$$\begin{aligned} A_{l+1} &< \tfrac{4}{27}|\xi_l|A_l \le \tfrac{4}{27}A_m(|\xi_m|/3^{l-m})(4|\xi_m|)^{l-m}/3^{(l-m)(l-m+5)/2} \\ &= A_m(4|\xi_m|)^{l-m+1}/3^{(l-m+1)(l-m+6)/2} \ , \end{aligned}$$

which proves Lemma 11 by induction on l. □

Lemma 12. *The sign of $S_l(\xi_m)$ equals $(-1)^{l-m}$ for $2 \le m \le l-1$.*

Proof. By definition $S_{m+1}(x) = x(S_m(x) + A_{m+1})$. Hence for $l > m$

$$S_l(x) = x^{l-m}S_m(x) + \sum_{j=m+1}^{l} A_j x^{l-j+1} \ .$$

As $S_m(\xi_m) = 0$, we get $S_l(\xi_m) = \sum_{j=m+1}^{l} A_j \xi_m^{l-j+1}$. The signs of the terms in this sum alternate (because $\xi_m < 0$ and $A_j > 0$). By Lemma 11 their absolute values decrease rapidly, and the sign of $A_{m+1}\xi_m^{l-m}$ defines the sign of $S_l(\xi_m)$. Indeed, compare this term with the quantity

$$B := \left|\sum_{j=m+2}^{l} A_j\xi_m^{l-j+1}\right| \le \sum_{j=m+2}^{l} A_j|\xi_m^{l-j+1}| \le A_{m+1}|\xi_m^{l-m}| \sum_{j=m+2}^{l} \frac{4^{j-m-1}}{3^{(j-m-1)(j-m+4)/2}} \ .$$

The sum in the right-hand side is bounded from above by $\sum_{\nu=1}^{\infty} 4^{\nu}/3^{3\nu} = 4/23$. Thus

$$B \le \frac{4}{23}A_{m+1}|\xi_m^{l-m}| \ , \ |S_l(\xi_m)| \ge \frac{19}{23}A_{m+1}|\xi_m^{l-m}|$$

implying sgn $S_l(\xi_m) =$ sgn $\xi_m^{l-m} = (-1)^{l-m}$. □

Lemma 13. *$\xi_{i+1}/\xi_i > 0.2864887043$ for $i > 1$.*

Proof. The claim can be checked directly for $i = 2, 3$ (see polynomials p_1, p_2, p_3, p_4 in the Introduction). Assume that $i \ge 4$ and use the same notation as in the proof of Lemma 10. The right-hand side of equation (10) is bounded from above by $2/(\xi_{i+1}+1) + \sum_{j=1}^{i-2} 1/(\xi_{i+1}+3^j)$ on the interval $(-1, 0)$. Indeed, by Lemma 12, we get $\alpha_j \in (\xi_j, \xi_{j+1})$. To bound from above the r.h.s. of (10) one can first replace $1/(\xi_{i+1} - \alpha_j)$ by $1/(\xi_{i+1} - \xi_j)$. As $\xi_j < \xi_{i+1} < 0$, one has $1/(\xi_{i+1} - \xi_j) > 0$.

Recall that $-1 < \xi_{i+1} < 0$. Additionally, $|\xi_j| \ge 3^{i-j}|\xi_i| = 3^{i-j}$ by Lemma 10. Hence $1/(\xi_{i+1} - \xi_j) < 1/(3^{i-j} - 1)$ and

$$\sum_{j=1}^{i-2} \frac{1}{\xi_{i+1} - \xi_j} < \sum_{j=1}^{i-2} \frac{1}{3^{i-j} - 1} < \sum_{j=1}^{\infty} \frac{1}{3^j - 1} \ .$$

Since $3^j - 1 > 2(3^{j-1} - 1)$ for $j \ge 2$ we get

$$\sum_{j=1}^{\infty}\frac{1}{3^j-1}=\sum_{j=1}^{4}\frac{1}{3^j-1}+\sum_{j=5}^{\infty}\frac{1}{3^j-1}<\frac{1}{2}+\frac{1}{8}+\frac{1}{26}+\frac{1}{80}+\frac{1}{121}<\frac{11}{16}\ ,$$

where

$$\sum_{j=5}^{\infty}\frac{1}{3^j-1}<\frac{1}{3^5-1}\sum_{j=0}^{\infty}\frac{1}{2^j}=\frac{1}{121}.$$

Therefore ξ_{i+1} will be bounded from above by the solution of the equation

$$-\frac{1}{\xi_{i+1}}=\frac{2}{\xi_{i+1}+1}+\frac{11}{16}$$

which belongs to $(-1,0)$. The latter equals $-0.2864887043\ldots$ implying that $\xi_{i+1}/\xi_i>0.2864887043$. □

Remark 5. Lemmas 10 and 13 imply

$$0.2864887043<\frac{\xi_{i+1}}{\xi_i}\leq\frac{1}{3}.$$

The above upper and lower bounds for ξ_{i+1}/ξ_i are quite close to one another and imply that the quantities $|\xi_i|$ decrease approximately as a geometric progression with a common ratio belonging to the interval $[0.2864887043, 1/3]$.

Set

$$\Phi(x):=-\frac{1}{x}-\frac{2}{x+1},\quad \psi(r):=\sum_{j=1}^{\infty}\frac{r^j}{1-r^{j+1}}.$$

The next result is central in the proof of Theorem 4.

Theorem 14. (i) *The limit* $\lambda=\lim_{i\to\infty}\xi_{i+1}/\xi_i$ *exists;*
(ii) λ *is the unique solution of the equation*

$$\Phi(-\lambda)=\psi(\lambda), \tag{12}$$

belonging to $(0,1)$.

Proof. Set $l_0:=0.2864887043\ldots$ and $r_0=1/3$. Lemmas 10 and 13 imply that if λ exists, then it belongs to $I_0:=[l_0,r_0]$. As in the proof of Lemma 10 rescale the variable x to obtain $\xi_i=-1$.

We construct a sequence of closed intervals $I_k:=[l_k,r_k]$, where $l_k\leq l_{k+1}\leq r_{k+1}\leq r_k$, such that for each k fixed the interval I_k contains all accumulation points of the sequence $\{\xi_i\}$.

Consider equation (10) and set

$$U:=\sum_{j=1}^{i-2}\frac{1}{\xi_{i+1}-\alpha_j}.$$

One can view ξ_{i+1} as a variable and α_1, …, α_{i-2} as parameters. Lemma 12 implies the inclusion $\alpha_j\in(\xi_j,\xi_{j+1})$ for $j\geq1$, where α_j are the roots of P_i in the increasing order. We can decrease the value of U for every $\xi_{i+1}\in(-1,0)$ fixed by assuming that for $j=1$, …, $i-2$ one has $\alpha_j=\xi_j$, and then by requiring ξ_j to be as small as possible. The last condition means that $\xi_{m+1}/\xi_m=l_0$ for $m=1,\ldots,i-2$.

Set $Z(c) := \sum_{j=1}^{i-2} 1/(\xi_{i+1} + c^j)$. Equation (10), when modified as above, looks like this:

$$-\frac{1}{\xi_{i+1}} = \frac{2}{\xi_{i+1}+1} + Y \ , \tag{13}$$

where $Y = Z(1/l_0)$. For each i fixed the solution ξ^*_{i+1} to the last equation is $> -1/3$ (by a reasoning analogous to the proof of Lemma 10). The solutions ξ^*_{i+1} increase with i because more and more terms are added to $Z(1/l_0)$ when i increases. Denote by $-r_1$ the limit of these solutions as $i \to \infty$. We get $r_1 < r_0 = 1/3$ (the equality $r_1 = r_0$ is impossible because the sequence $\{\xi^*_{i+1}\}$ is increasing).

It is clear that if the limit λ exists, then it must belong to the interval $[l_0, r_1]$. Moreover, all accumulation points of the sequence of solutions to the original equation (10) belong to $[l_0, r_1]$ when $i \to \infty$.

Analogously, the value of U increases on $(-1, 0)$ if one sets $\alpha_j = \xi_{j+1}$ $(j \geq 1)$ and requires ξ_{j+1} to be the maximal possible. In this case $\xi_{m+1}/\xi_m = r_1$ for $m = 1, \ldots, i-2$ and the modified equation (10) is equation (13) with $Y = Z(1/r_1)$.

Hence for each i fixed the solution to (the modified as above) equation (10) will be smaller than $-l_0$ because $-l_0$ was obtained when the ratios ξ_{m+1}/ξ_m were equal to $r_0 > r_1$. Denote by $-l_1$ the limit of these solutions as $i \to \infty$. We get that $l_1 \geq l_0$ (we do not need to prove that actually $l_1 > l_0$). Thus $l_0 \leq l_1 \leq r_1 \leq r_0$.

The construction of all further quantities l_k and r_k follows a similar pattern. Namely, consider for $\varepsilon > 0$ the interval $I_k(\varepsilon) := [l_k - \varepsilon, r_k + \varepsilon]$. Suppose that for $i > i_k(\varepsilon)$ the solutions to equation (10) belong to the interval $I_k(\varepsilon)$. (This is true for $k = 0$ and 1.) Hence all accumulation points of the sequence of solutions to equation (10) belong to the interval $J_{k+1}(\varepsilon) := [l_{k+1}(\varepsilon), r_{k+1}(\varepsilon)]$, where $r_{k+1}(\varepsilon)$ (resp. $l_{k+1}(\varepsilon)$) is the limit as $i \to \infty$ of the solutions to equation (13) with $Y = Z(1/(l_k - \varepsilon))$ (resp. with $Y = Z(1/(r_k - \varepsilon))$).

Set $I_{k+1} := \lim_{\varepsilon \to 0} J_{k+1}(\varepsilon)$. Hence the accumulation points mentioned above belong to the interval I_{k+1}. One has $l_k \leq l_{k+1} \leq r_{k+1} \leq r_k$ (this is shown in the same way as for $k = 0$ above).

Let us show that when $k \to \infty$ the lengths of the intervals I_k tend to 0 as fast as a geometric progression with a common ratio less than 1. Therefore their common intersection is a single point. In other words, there is only one accumulation point of the sequence of solutions to the initial equation (10) as $i \to \infty$.

To do this present equation (10) in the form $\Phi(x) = U(x)$ and let $i \to \infty$. The limits as $i \to \infty$ of the modified equations (10) are the ones of equation (13) with respectively $Y = Z(1/r_k)$ and $Y = Z(1/l_k)$, i.e.

$$\Phi(x) = \varphi(r_k, x) \quad \text{and} \quad \Phi(x) = \varphi(l_k, x), \tag{14}$$

where

$$\varphi(r, x) := \sum_{j=1}^{\infty} \frac{1}{x + (1/r)^j}.$$

The left (respectively right) equation in (14) has $-l_{k+1}$ (respectively $-r_{k+1}$) as its solution on $(0, 1)$. The series φ converges uniformly on $[l_0, r_0] \times [-1, 0]$.

Each of the functions $\varphi(r_k, x)$ and $\varphi(l_k, x)$ is decreasing on $(0, 1)$. The inequality $\varphi(r_k, x) > \varphi(l_k, x)$ holds for each fixed $x \in (0, 1)$. Therefore the intersection points of the graph of $\Phi(x)$ with that of $\varphi(r_k, x)$ and $\varphi(l_k, x)$ belong to the rectangle $[-r_k, -l_k] \times [\varphi(l_k, -l_k), \varphi(r_k, -r_k)]$.

For $x \in [-r_0, -l_0]$ one has $1/x^2 \geq 9$ and $2/(x+1)^2 > 3$, hence $|\Phi'(x)| > 12$. Therefore

$$|r_{k+1} - l_{k+1}| < \frac{1}{12}|\varphi(r_k, -r_k) - \varphi(l_k, -l_k)| \ .$$

To simplify the notation we write r instead of r_k and l instead of l_k. Set

$$M := \varphi(r, -r) - \varphi(l, -l) = \sum_{j=1}^{\infty} \frac{r^j - l^j + l^j r^j (r-l)}{(1-r^{j+1})(1-l^{j+1})} \ .$$

For each j there exists $\theta_j \in (l, r)$ such that $r^j - l^j = j\theta_j^{j-1}(r-l) < jr^{j-1}(r-l)$. As $l \leq r \leq 1/3$, one has

$$(1-r^{j+1})(1-l^{j+1}) > \left(\frac{2}{3}\right)^2 = \frac{4}{9} \quad \text{and} \quad r^j l^j \leq \frac{r^{j-1}}{9} \ .$$

Thus $0 \leq M \leq \frac{9}{4}(r-l)\sum_{j=1}^{\infty}\left(j + \frac{1}{9}\right) r^{j-1}$. Recall that

$$\sum_{j=1}^{\infty} jr^{j-1} = \frac{1}{(1-r)^2} \leq \frac{9}{4} \text{ and } \sum_{j=1}^{\infty} r^{j-1} = \frac{1}{1-r} \leq \frac{3}{2} \ .$$

Therefore $|r_{k+1} - l_{k+1}| < \frac{1}{12}M \leq \frac{1}{12}\cdot\frac{9}{4}\left(\frac{9}{4} + \frac{1}{9}\cdot\frac{3}{2}\right)(r_k - l_k) < \frac{r_k - l_k}{2}$, which proves part (i) of the theorem.

To settle part (ii) one has to observe that both the solution $-l_{k+1}$ of (14) and the parameter $-r_k$ tend to $-\lambda$, while

$$\varphi(r, -r) = \sum_{j=1}^{\infty} \frac{r^j}{1-r^{j+1}} = \psi(r) \ .$$

Therefore λ solves (12). □

The number λ is defined as

$$\lambda = \lim_{i\to\infty} \frac{\xi_{i+1}}{\xi_i},$$

where ξ_{i+1} is the rightmost critical point of the function $S_i(x) = xP_i(x)$. Recall that

$$P_i(x) = xP_{i-1}(x) - \xi_i P_{i-1}(\xi_i). \tag{15}$$

We now move further along in the proof of Theorem 4. Recall that the sequence $\{\widetilde{P}_i\}$ of scaled reciprocal polynomials is defined by

$$\widetilde{P}_i(x) = P_i(-\xi_i x)/P_i(0).$$

Each $\widetilde{P}_i(x)$ satisfies the conditions

$$\widetilde{P}_i(-1) = \widetilde{P}_i'(-1) = 0 \quad \text{and} \quad \widetilde{P}_i(0) = 1.$$

Dividing both sides of (15) by $-\xi_i P_{i-1}(\xi_i)$ and substituting $x \mapsto -\xi_i x$ we get:

$$(16)\qquad \widetilde{P}_i(x) := \frac{P_i(-\xi_i x)}{-\xi_i P_{i-1}(\xi_i)} = 1 + x\frac{P_{i-1}(-\xi_i x)}{P_{i-1}(\xi_i)} = 1 + x\frac{\widetilde{P}_{i-1}(\lambda_{i-1}x)}{\widetilde{P}_{i-1}(-\lambda_{i-1})},$$

where $\lambda_{i-1} = \xi_i/\xi_{i-1}$.

Theorem 15. (i) There exists a limiting function $V = \lim_{i\to\infty} \widetilde{P}_i$. (This limit is understood as a formal series.)
(ii) The limit $\lim_{i\to\infty} m_i = \lim_{i\to\infty} \frac{A_i^2}{A_{i-1}A_{i+1}}$ exists and equals $1/\lambda$.

Proof. Set $\widetilde{P}_i := \sum_{j=0}^{i} c_{j,i}x^j$. Equation (16) implies that $c_{0,i} = 1$ and for $j \geq 1$ one has $c_{j+1,i} = \lambda_{i-1}^j c_{j,i-1}/\widetilde{P}_{i-1}(-\lambda_{i-1})$. Hence for $j \geq 1$ the equality

$$\tilde{c}_{j,i} := c_{j+1,i}^2/c_{j,i}c_{j+2,i} = c_{j,i-1}^2/c_{j-1,i-1}c_{j+1,i-1}$$

holds.

Recall that by Theorem 2 which is already proven, for i fixed, $\tilde{c}_{0,i}$ is minimal among the quantities $\tilde{c}_{j,i}$. Denoting the first three coefficients of the polynomial $\widetilde{P}_{i-1}$ by a, b and c resp. (these are the coefficients of 1, x and x^2), we obtain that the first three coefficients of $\widetilde{P}_i$ are 1, $a/\widetilde{P}_{i-1}(-\lambda_{i-1})$ and $b\lambda_{i-1}/\widetilde{P}_{i-1}(-\lambda_{i-1})$. Further, the first three coefficients of $\widetilde{P}_{i+1}$ equal $1, 1/\widetilde{P}_i(-\lambda_i)$ and $a\lambda_i/\widetilde{P}_{i-1}(-\lambda_{i-1})\widetilde{P}_i(-\lambda_i)$. Finally, the first three coefficients of $\widetilde{P}_{i+2}$ are equal to $1, 1/\widetilde{P}_{i+1}(-\lambda_{i+1})$ and $\lambda_{i+1}/\widetilde{P}_i(-\lambda_i)\widetilde{P}_{i+1}(-\lambda_{i+1})$. Hence

$$\tilde{c}_{0,i+2} = \widetilde{P}_i(-\lambda_i)/\widetilde{P}_{i+1}(-\lambda_{i+1})\lambda_{i+1}.$$

This quantity tends to a finite positive limit as $i \to \infty$; see Theorem 2. As this is the case for λ_{i+1} as well, then the ratio $\widetilde{P}_i(-\lambda_i)/\widetilde{P}_{i+1}(-\lambda_{i+1})$ also has a finite positive limit which we temporarily denote by τ. In fact, $\tau = 1$. To show this observe that if $\tau < 1$, then $\lim_{i\to\infty} \widetilde{P}_i(-\lambda_i) = \infty$ which is impossible since $\widetilde{P}_i(x) \in [0,1]$ for $x \in [-1,0]$. If $\tau > 1$, then $\lim_{i\to\infty} \widetilde{P}_i(-\lambda_i) = 0$. This again is impossible. Indeed, one has $\widetilde{P}_i = (x+1)^2 \prod_{j=1}^{i-2}(1 + x/\alpha_{j,i})$, where the positive constants $\alpha_{j,i}$ are bounded from below (for all i simultaneously) by a geometric progression with common ratio > 2; see Lemmas 12 and 13. The latter presentation of the function $\widetilde{P}_i$ implies that:

(1) There exists $c \in (0,1)$ such that $\widetilde{P}_i(-\lambda_i) \geq c$ for all i.
(2) For each $r > 0$ there exists $C(r) > 0$ such that $|\widetilde{P}_i(x)| \leq C(r)$ for $x \in [-r, r]$ and for all i.
(3) For each $r > 0$ there exists $g(r) > 0$ such that $|\widetilde{P}_i'(x)| \leq g(r)$ for $x \in [-r, r]$ and for all i.

For brevity we write C and g instead of $C(r)$ and $g(r)$. Our final task is to show that the sequence $\{\widetilde{P}_i(x)\}$ is a Cauchy sequence on any closed interval of $\mathbb{R}$. To do this, we need to estimate the differences $\widetilde{P}_l(x) - \widetilde{P}_i(x)$ on such intervals. Consider the difference $W_1 := \widetilde{P}_l(x) - \widetilde{P}_i(x) = F_1(x) + B_1(x) + H_1(x)$, $l \geq 1$, where

$$\begin{aligned} F_1(x) &= \tfrac{x}{\tilde{P}_{l-1}(-\lambda_{l-1})}(\tilde{P}_{l-1}(\lambda_{l-1}x) - \tilde{P}_{i-1}(\lambda_{l-1}x)) \ , \\ B_1(x) &= \tfrac{x}{\tilde{P}_{l-1}(-\lambda_{l-1})}(\tilde{P}_{i-1}(\lambda_{l-1}x) - \tilde{P}_{i-1}(\lambda_{i-1}x)) \ , \\ H_1(x) &= x\tilde{P}_{i-1}(\lambda_{i-1}x)\left(\tfrac{1}{\tilde{P}_{l-1}(-\lambda_{l-1})} - \tfrac{1}{\tilde{P}_{i-1}(-\lambda_{i-1})}\right) \\ &= \tfrac{x\tilde{P}_{i-1}(\lambda_{i-1}x)}{\tilde{P}_{l-1}(-\lambda_{l-1})}\left(1 - \tfrac{\tilde{P}_{l-1}(-\lambda_{l-1})}{\tilde{P}_{i-1}(-\lambda_{i-1})}\right) \ . \end{aligned}$$

Present the difference $W_2 := \tilde{P}_{l-1}(\lambda_{l-1}x) - \tilde{P}_{i-1}(\lambda_{l-1}x)$ in the form $F_2(\lambda_{l-1}x) + B_2(\lambda_{l-1}x) + H_2(\lambda_{l-1}x)$, where the functions F_2, B_2 and H_2 are defined with respect to W_2 in the same way as F_1, B_1 and H_1 are defined with respect to W_1. That is,

$$\begin{aligned} F_2(\lambda_{l-1}x) &= \tfrac{\lambda_{l-1}x}{\tilde{P}_{l-2}(-\lambda_{l-2})}(\tilde{P}_{l-2}(\lambda_{l-2}\lambda_{l-1}x) - \tilde{P}_{i-2}(\lambda_{l-2}\lambda_{l-1}x)) \ , \\ B_2(\lambda_{l-1}x) &= \tfrac{\lambda_{l-1}x}{\tilde{P}_{l-2}(-\lambda_{l-2})}(\tilde{P}_{i-2}(\lambda_{l-2}\lambda_{l-1}x) - \tilde{P}_{i-2}(\lambda_{i-2}\lambda_{l-1}x)) \ , \\ H_2(\lambda_{l-1}x) &= \tfrac{\lambda_{l-1}x\tilde{P}_{i-2}(\lambda_{i-2}\lambda_{l-1}x)}{\tilde{P}_{l-2}(-\lambda_{l-2})}\left(1 - \tfrac{\tilde{P}_{l-2}(-\lambda_{l-2})}{\tilde{P}_{i-2}(-\lambda_{i-2})}\right) \ . \end{aligned}$$

Continue in the same way with $W_3 := \tilde{P}_{l-2}(\lambda_{l-2}\lambda_{l-1}x) - \tilde{P}_{i-2}(\lambda_{l-2}\lambda_{l-1}x)$ etc. To simplify the notation we denote by γ any of the quantities λ_k and by D any of the ratios $x/\tilde{P}_j(-\lambda_j)$ or $x\tilde{P}_k(\gamma^s x)/\tilde{P}_j(-\lambda_j)$. Thus

$$W_s = \tilde{P}_{l-s+1}(\gamma^{s-1}x) - \tilde{P}_{i-s+1}(\gamma^{s-1}x) \ , \quad F_s = D\gamma^{s-1}W_{s+1} \ ,$$

$$B_s = D\gamma^{s-1}(\tilde{P}_{i-s}(\gamma^s x) - \tilde{P}_{i-s}(\gamma^s x)) \ , \qquad H_s = D\gamma^{s-1}(1 - \tilde{P}_{l-s}(-\gamma)/\tilde{P}_{i-s}(-\gamma)) \ .$$

It is clear that $|W_s| \leq 2C$ for any s. Set

$$U_s := \tilde{P}_{i-s}(\gamma^s x) - \tilde{P}_{i-s}(\gamma^s x) \ , \ R_s := 1 - \tilde{P}_{l-s}(-\gamma)/\tilde{P}_{i-s}(-\gamma) \ .$$

As γ denotes different quantities λ_k, U_s is not identically 0. So for any $\nu < \min(i, l)$ the difference W_1 is of the form

$$W_1 = \sum_{s=1}^{\nu} D^s\gamma^{\binom{s}{2}}U_s + \sum_{s=1}^{\nu} D^s\gamma^{\binom{s}{2}}R_s + D^\nu\gamma^{\binom{\nu}{2}}W_\nu \ . \tag{17}$$

Recall that $0 < \gamma \leq 1/3$ (see Lemma 10). Denote by d some upper bound of all quantities D (valid for all $x \in [-r, r]$ and for all indices k, j and s; by properties (1) and (2) such a $d > 0$ exists). Denote by σ the sum of the series $\sum_{m=0}^{\infty} d^m/3^{\binom{m}{2}}$. As $|W_\nu|$ and $|D|$ remain bounded, the last summand in (17) tends to 0 as $\nu \to \infty$.

For any $\varepsilon > 0$ there exists ν so large that for $i - s > \nu$, $l - s > \nu$ one has $|R_s| < \varepsilon/4\sigma$ (this follows from $\tau = 1$). Hence the absolute value of the second summand in (17) is bounded from above by $\varepsilon/4$. The absolute value of the first one is

$$\leq \sigma \max_{s \leq \nu, x \in [-r,r]} U_s(x) \leq \sigma g r \max_{s \leq \nu} |\gamma^s - \gamma^s| \ , \ \text{see property (3).}$$

This absolute value is $< \varepsilon/4$ if i and l are large enough. Indeed, the difference $\gamma^s - \gamma^s$ is close to 0 for s sufficiently large because $\gamma \in (0, 1/3]$. For the remaining finitely many values of s it can be made arbitrarily close to 0 by choosing the indices i and l large enough because each factor γ is a quantity λ_k, the indices k are close to l or i and the limit $\lim_{k\to\infty} \lambda_k$ exists.

Thus if ν (hence i and l as well) is large enough, then $\max_{s\leq\nu} |\gamma^s - \gamma^s| < \varepsilon/(4\sigma g r)$, $|\sum_{s=1}^{\nu} D^s \gamma^{\binom{s}{2}} U_s| < \varepsilon/4$ and $|W_1| < 3\varepsilon/4 < \varepsilon$. Hence the sequence $\{\widetilde{P}_i(x)\}$ of polynomials is a Cauchy sequence on any interval $[-r, r]$ and has a limit. This proves part (i) of the theorem.

To prove part (ii) recall that $\tilde{c}_{0,i+2} = \widetilde{P}_i(-\lambda_i)/\widetilde{P}_{i+1}(-\lambda_{i+1})\lambda_{i+1}$, $\lim_{i\to\infty} \lambda_{i+1} = \lambda$ and $\lim_{i\to\infty} \widetilde{P}_i(-\lambda_i)/\widetilde{P}_{i+1}(-\lambda_{i+1}) = 1$. Hence $\lim_{i\to\infty} \tilde{c}_{0,i+2} = 1/\lambda$. On the other hand, $\tilde{c}_{0,i+2} = A_{i+1}^2/(A_i A_{i+2})$. □

Proposition 16. *The function V defined in part (i) of Theorem 15 enjoys the following properties:*

(i) *V belongs to the Laguerre-Pólya class $\mathcal{L} - \mathcal{P}I$, i.e. the limit in Theorem 15 can be understood as the uniform convergence on compact sets in $\mathbb{C}$;*

(ii) *V satisfies the functional relation: $V(x) = 1 + xV(\lambda x)/V(-\lambda)$.*

Notice that the latter relation implies that for any choice of λ one has $V(0) = 1$ and $V(-1) = 0$. On the other hand, $V'(-1) = 0$ is an additional condition which together with (ii) determines λ.

Proof of Proposition 16. To prove part (i) notice that the function V is a limit of a sequence of hyperbolic polynomials $\widetilde{P}_i$ with negative roots. If the roots are numbered in the order of increasing absolute values, then for every fixed j the root $-\alpha_{j,i}$ has a finite limit $-\alpha_j$ when $i \to \infty$.

Indeed, for i fixed the absolute values of the roots $-\alpha_{j,i}$ increase faster than a geometric progression with ratio 2.6 (see Lemmas 10, 12 and 13). Hence there exist $a > 0$ and $b > 0$ such that $|\widetilde{P}_i'| \geq b$ for $x \in [-\alpha_{1,i} - a, -\alpha_{1,i} + a]$ and for all i ($-\alpha_{1,i}$ is the first root of $\widetilde{P}_i$ to the left of -1). As $V := \lim_{i\to\infty} \widetilde{P}_i$ exists, there exists also the lower bound of the values of $h < 0$ such that $V > 0$ for $x \in (h, -1]$. It is clear that $V(h) = 0$ (V is continuous), hence $\lim_{i\to\infty}(-\alpha_{1,i}) = h$ and one can set $\alpha_1 = -h$. It is also clear that $-\alpha_1$ is a simple root of V and that $V'(-\alpha_1) \geq b$.

Having proved the existence of the limits $-\alpha_j$ for $j < j_0$ one proves the existence of $-\alpha_{j_0} := \lim_{i\to\infty}(-\alpha_{j_0,i})$ by analogy.

The absolute values of the roots $-\alpha_j$ increase at least as fast as a geometric progression with ratio 2.6. Hence the sequence of polynomials $\widetilde{P}_i$ is uniformly convergent on any compact set Ω. Indeed, consider the product $\prod_{j=N}^{\infty}(1 + x/\alpha_j)$. The absolute value of its logarithm is bounded from above by $C\sum_{j=N}^{\infty} |x/\alpha_j|$, where $C > 0$ depends only on the set Ω. Obviously, $C\sum_{j=N}^{\infty} |x/\alpha_j|$ can be made arbitrarily small uniformly on Ω if one chooses N sufficiently large. To prove this notice that

$$|\log(1 + y)| < |y| + |y|^2 + |y|^3 + \ldots = \frac{|y|}{1 - |y|},$$

where $y = x/\alpha_j$. For j large enough the expression $|y|/(1 - |y|)$ is smaller than $2|y|$. Thus we have proven uniform convergence of the above sequence of polynomials on compact sets in $\mathbb{C}$ and the limit of this sequence belongs to the class $\mathcal{L}-\mathcal{P}I$ by definition.

To prove part (ii) recall the equality $S_{m+1}(x) = x(S_m(x) - S_m(\xi_{m+1}))$ or, equivalently, $P_{m+1}(x) = xP_m(x) - \xi_{m+1}P_m(\xi_{m+1})$. The function V is the limit when $m \to \infty$ of the polynomials $\widetilde{P}_m(x) = P_m(-\xi_m x)/P_m(0)$. $\square$

In order to finish the proof of Theorem 4 we determine which entire functions satisfy conditions (i) and (ii) of Proposition 16. The following definition is crucial for our further considerations. Recall that $\Psi(q,u) = \sum_{j=0}^{\infty} q^{\binom{j+1}{2}} u^j$.

Definition. We say that a pair $(\hat{q},\hat{u})$ is *critical* for $\Psi(q,u)$ if $|\hat{q}| < 1$ and $\Psi(\hat{q},u)$ considered as a function of u has a double root at $\hat{u}$.

Theorem 17. *There exists a function $V(x)$ analytic in a disk $|x| \le r,\ r > 1$ and satisfying for some $\hat{u} \in \mathbb{C}^*$ and $|\hat{q}| < 1$ the relation:*

$$V(x) = -\hat{u}\hat{q}(xV(\hat{q}x) + V(-\hat{q})) \tag{18}$$

together with the boundary condition $V'(-1) = 0$ if and only if the pair $(\hat{q},\hat{u})$ is critical.

Notice that (18) is exactly the relation (ii) of Proposition 16 with an undefined scalar factor $\hat{u}$.

Proof of Theorem 17. Assume that $V(x)$ is analytic in some disk $|x| \le r,\ r > 1$ and satisfies the relation $V(x) = -uq(xV(qx) + V(-q))$ for some fixed $u \ne 0$ and $q \ne 0$. W.l.o.g. we can assume $V(0) = 1$ which is equivalent to $\beta_0 = 1$. Substitution of $V(x) = \sum_{j=0}^{\infty} \beta_j x^j$ in the latter relation gives

$$\sum_{j=0}^{\infty} \beta_j x^j = -quV(-q) - qux \sum_{j=0}^{\infty} \beta_j q^j x^j\ .$$

Comparing the coefficients at equal powers in the latter relation we get the system of equalities

$$-quV(-q) = 1, \quad \text{and} \tag{19}$$

$$\beta_1 = -qu,\ \beta_2 = -q^2u\beta_1,\ \beta_3 = -q^3u\beta_2,\ \dots\ ,\beta_k = -q^k u\beta_{k-1}, \dots$$

implying $\beta_j = q^{\binom{j+1}{2}}(-u)^j$ $(j = 1,2,3,\dots)$. Substituting these coefficients in $V(x)$ we get $V(x) = \sum_{j=0}^{\infty} q^{\binom{j+1}{2}}(-ux)^j$. In terms of the partial theta function $\Psi(q,u)$ given by (4) one gets $V(x) = \Psi(q,-ux)$ and condition (19) takes the form $\Psi(q,qu) = -\frac{1}{qu}$.

Let us show that it is equivalent to $\Psi(q,u) = 0$. Indeed, expanding $\Psi(q,qu) = -\frac{1}{qu}$ we get $qu\sum_{j=0}^{\infty} q^{\binom{j+1}{2}}(qu)^j = -1 \Leftrightarrow 1 + \sum_{j=0} q^{\binom{j+1}{2}}(qu)^{j+1} = 0 \Leftrightarrow 1 + \sum_{j=0} q^{\binom{j+2}{2}}u^{j+1} = 0 \Leftrightarrow \Psi(q,u) = 0$. Notice that for the power series expressing $V(x)$ to have a radius of convergence exceeding 1 it is necessary and sufficient to have $|q| < 1$ in which case $V(x)$ is entire. Now we use the last boundary condition $V'(-1) = 0$. With $V(x) = \Psi(q,-ux)$ we get $V'(x) = -u\partial\Psi(q,-ux)$, where ∂ stands for the partial derivative w.r.t. second argument. Finally, $V'(-1) = -u\Psi'_u(q,u)$. Thus one gets the system

$$\begin{cases}\Psi(q,u)=0\\ -u\Psi'_u(q,u)=0\end{cases} \Leftrightarrow \begin{cases}\Psi(q,u)=0\\ \Psi'_u(q,u)=0\end{cases}$$

since $u=0$ is never a solution of $\Psi(q,u)=0$.

An arbitrary solution $(\hat{q},\hat{u})$ of the latter system is exactly a critical pair in the above definition, i.e. $\hat{q}$ is such that the function $\Psi(\hat{q},u)$ as a function of u has a double root at $\hat{u}$. □

Finally to finish the proof of Theorem 4 notice that by Proposition 16 and Theorem 17 the function $V(x)$ satisfies the functional relation (ii), $V'(-1)=0$ and belongs to $\mathcal{L}-\mathcal{P}I$ which implies that $\lambda=\widetilde{q}$ and the function V equals $\Psi(\widetilde{q},-\widetilde{u}x)$ where $\widetilde{u}$ is the double root of the function $\Psi(\widetilde{q},x)$. □

Remark 6. A somewhat mysterious equation (12) describes the set of all critical points of the limiting function $\Psi(\widetilde{q},-\tilde{u}x)$ of which $\tilde{u}$ is the only critical point belonging to the interval $(0,1)$.

4. Proving miscellaneous results

Denote by $g_q(x)$ the series and the function $\sum_{k=0}^{\infty} q^{k^2}x^k = \Phi(q^2, x/q)$, $0<q<1$ and by $S_n(q,x)=\sum_{k=0}^{n} q^{k^2}x^k$ its n-th section.

Lemma 18. *For every $q\in(0,1)$ there exists a number $m\in\mathbb{N}$ such that for all $n\geq 2m+2$ the number of non-real zeros of $S_n(q,x)$ is not greater than $2m+2$.*

Proof. We will use the following well-known identity

$$\prod_{k=1}^{\infty}\frac{1-q^k}{1+q^k} = 1+2\sum_{k=1}^{\infty}(-1)^k q^{k^2}$$

valid for $|q|<1$, (see e.g. [23], Chapter 1, Problem 56). By this identity we have

$$1+2\sum_{k=1}^{\infty}(-1)^k q^{k^2} > 0,\ q\in(0,1).$$

Thus for every $q\in(0,1)$ there exists $m=2s+1\in\mathbb{N}$ such that the inequality

$$1+2\sum_{k=1}^{m}(-1)^k q^{k^2} > 0 \tag{20}$$

holds. Then for every $n\geq 2m+2$ and k satisfying the condition $m+1\leq k\leq n-m-1$ we get:

$$\begin{aligned}(-1)^k q^{k^2} S_n(q,-q^{-2k}) &= \textstyle\sum_{j=0}^{n}(-1)^{j-k}q^{(j-k)^2}\\ &= \left(\textstyle\sum_{j=0}^{k-m-1}+\sum_{j=k-m}^{k+m}+\sum_{j=k+m+1}^{n}\right)(-1)^{j-k}q^{(j-k)^2}\\ &=: \Sigma_1+\Sigma_2+\Sigma_3\ .\end{aligned}$$

We see that the summands in Σ_1 are sign-alternating and their absolute values are increasing. Therefore the sign of Σ_1 coincides with that of the $(k-m-1)$-th summand which means that it is equal to $(-1)^{k-m-1-k}=(-1)^{-m-1}=(-1)^{-2s-2}=1$.

Analogously the summands in Σ_3 are sign-alternating and their absolute values are decreasing. So the sign of Σ_3 coincides with that of the $(k+m+1)$-st summand. In other words, $\operatorname{sgn}\Sigma_3 = (-1)^{k+m+1-k} = (-1)^{m+1} = (-1)^{2s+2} = 1$. Therefore $\Sigma_1 \geq 0$ and $\Sigma_3 \geq 0$. By (20) we have

$$\Sigma_2 = q^{-k^2} \sum_{j=k-m}^{k+m} (-1)^{j-k} q^{(j-k)^2} = q^{-k^2} \left(1 + 2\sum_{k=1}^{m} (-1)^k q^{k^2}\right) > 0 \ .$$

Therefore $(-1)^k S_n(q, -\frac{1}{q^{2k}}) > 0$ for every k such that $m+1 \leq k \leq n-m-1$. Thus $S_n(q,x)$ has not less than $n-2m-2$ real zeros for $n \geq 2m+2$ (and the number of non-real zeros of $S_n(q,x)$ is not greater than $2m+2$). □

Corollary 2. *For every real $q \in (0,1)$ the functions $g_q(x)$ and $\Psi(q,x)$ have a finite number of non-real zeros. Moreover, the number of non-real zeros is a non-decreasing function of parameter q.*

Proof. Since $\Psi(q,x)$ is obtained from $g_q(x)$ by rescaling of x it suffices to consider $g_q(x)$ only. To prove the first statement fix an arbitrary $q \in (0,1)$. By Lemma 18 we know that there exists m such that the number of non-real roots of any section $S_n(q,x)$ for n sufficiently large does not exceed m. Assume that the function $g_q(x)$ has $l > m$ non-real roots. Take small circles surrounding these roots and not intersecting the real axis. By the Hurwitz theorem all sections with large n should have exactly l roots in the union of disks bounded by these l circles. This is a contradiction.

To prove the second statement consider the sequence of sections. We prove that the number of real roots of any section $S_n(q,x)$ and $g_q(x)$ itself is a monotone non-increasing function of $q \in (0,1)$. Indeed, the sequence q^{n^2} is a complex zero decreasing sequence (CZDS) for any $q \in (0,1)$; see e.g. [7]. (We recall that the sequence $\{c_k\}_{k\geq 0}$ is a CZDS if for any polynomial $a_0+a_1x+\ldots+a_nx^n$ the inequality

$$Z_c\left(\sum_{k=0}^{n} c_k a_x x^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_x x^k\right),$$

is valid, where Z_c is the number of non-real zeros of a polynomial counting multiplicities.) For any $0 < q_1 < q_2 < 1$ one obtains the section with the value of parameter q_1 from that of q_2 multiplying the coefficients of the former by $(q_1/q_2)^{n^2}$. Since the latter sequence is CZDS the result for all finite sections follows. For $g_q(x)$ itself the same argument applies since it only has finitely many non-real zeros and is of genus 0. □

We finally prove the remaining Proposition 7. Denote by $\mathrm{Pol}_n^+ \subset \mathrm{Pol}_n$ the set of all monic degree n polynomials with all positive coefficients. It contains Σ_n, the set of degree n polynomials with all roots negative. Let $p(x) = a_nx^n + \ldots + a_0$ be a polynomial.

Lemma 19. *For any $k = 1, 2, \ldots, n-1$ there exists a polynomial $p \in \mathrm{Pol}_n^+$, $p \notin \Sigma_n$, for which one has $a_i^2 \geq 4a_{i-1}a_{i+1}$ for $i \neq k$ and $a_k^2 < 4a_{k-1}a_{k+1}$.*

Proof. Fix the triple of coefficients (a_{k-1}, a_k, a_{k+1}) such that $a_k^2 < 4a_{k-1}a_{k+1}$. Hence the polynomial $g := x^{k-1}(a_{k+1}x^2 + a_kx + a_{k-1})$ has a root at 0 of multiplicity $k-1$ and a complex conjugate pair.

For $i > k+1$ (resp. for $i < k-1$) set $a_i = b_i\varepsilon^{i-k-1}$ (resp. $a_i = b_i\varepsilon^{k-1-i}$), where $\varepsilon \in (0,1]$. Choose the coefficients $b_i > 0$ such that the inequalities $a_i^2 \geq 4a_{i-1}a_{i+1}$ hold for $i \neq k$ and $\varepsilon = 1$. One can do this choice consecutively. First we choose $b_{k+2} > 0$ sufficiently small to ensure $a_{k+1}^2 \geq 4a_k b_{k+2}$, then we choose b_{k+3} satisfying $b_{k+2}^2 \geq 4b_{k+3}a_{k+1}$ etc. Proceed analogously with b_{k-2}, b_{k-3} etc.

If the inequalities $a_i^2 \geq 4a_{i-1}a_{i+1}$ $(i \neq k)$ hold for $\varepsilon = 1$, then they hold for any $\varepsilon \in (0,1]$ (to be checked directly).

For ε small enough the polynomial p is a perturbation of the polynomial g. Fix two circles centered at the complex roots of g and not intersecting the real axis. For $\varepsilon > 0$ small enough one has $|p - g| < |g|$ on these circles. By the Hurwitz theorem p has a root inside each of them. Hence $p \notin \Sigma_n$. □

Proof of Proposition 7. To prove (i) notice that by Lemma 5 of [21] Hutchinson's cone is the minimal polyhedral cone containing the set of the so-called sign-independently hyperbolic polynomials and is, on the other hand, contained in $\Delta_n \subset \Sigma_n$. (A sign-independently hyperbolic polynomial is a hyperbolic polynomial with all positive coefficients such that all polynomials obtained by arbitrary sign changes of its coefficients are hyperbolic as well; see [21].) By Lemma 19 we see that an arbitrarily small parallel translation 'outward' of any of the hyperplanes defining the logarithmic image of Hutchinson's cone results into getting outside the logarithmic image of the largest set Σ_n. Therefore the logarithmic image of Hutchinson's cone is the largest polyhedral cone contained in $L\Delta_n$ and, analogously, in $L\Sigma_n$.

To prove (ii) observe that Theorem 2 can be interpreted as follows. Consider Hutchinson's cone in the space Pol_n^1 and its image under the logarithmic map. Then it has a unique apex, i.e. the vertex where all inequalities become equalities. Look for a parallel translation of the logarithmic image of Hutchinson's cone containing the whole $L\Delta_n$. Then if you take the parallel translation such that this apex is placed at the logarithmic image of P_n, then the whole $L\Delta_n$ is covered. Since the translated cone and $L\Delta_n$ still have a common point this position is minimal for containment. Exactly the same argument using the known properties of Newton's inequalities tells us that placing the apex at the logarithmic image of $(x + 1/n)^n$ does the job. □

5. Appendix. An interesting iteration scheme

The main ingredient in the proof of Theorem 4 is a construction of the sequence $\{S_j\}$ starting from a hyperbolic polynomial S_1 with all simple roots the rightmost of which is at the origin. The next polynomial is obtained from the previous one by subtracting its minimum of smallest absolute value followed by multiplication by x. We have shown that after appropriate scaling the limiting entire function is a specialization of a partial theta function and also that the critical point at which this minimum is located asymptotically stabilizes. In view of these results one can ask what happens if we consider a similar iterative procedure where the point at which we take a value to subtract is fixed from the beginning. A more detailed consideration leads to the following natural set-up.

Given an initial analytic function $f_1(x)$ defined (at least) in a small open neighborhood of the interval $[-1,0]$ on the real line and a number $0 < q < 1$ define the

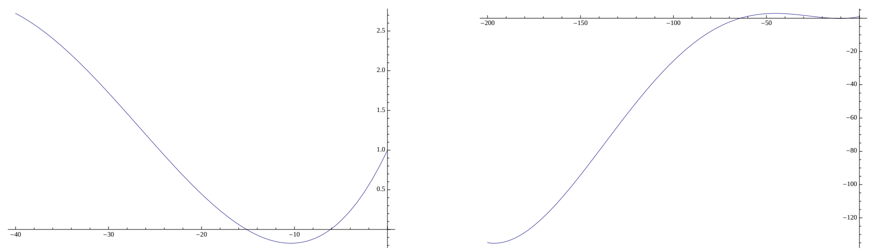

FIGURE 4. $\Psi(1/4,u)$ in the intervals $[-40,0]$ and $[-200,0]$.

sequence $\{f_i\}_{i\geq 1}$ given by

$$f_i(x) = \left(1 + \frac{x f_{i-1}(qx)}{f_{i-1}(-q)}\right),\ i \geq 2\ . \tag{21}$$

Obviously, $f_i(x)$ will be well-defined and analytic in the same neighborhood of $[-1,0]$ unless $-q$ is a root of $f_{i-1}(x)$. For generic choices of f_1 the latter circumstance never happens. One can easily check that if f_i is well-defined, then it satisfies the normalization conditions $f_i(0)=1$ and $f_i(-1)=0$.

Looking at the first part of the proof of Theorem 17 before we use the additional condition $V'(-1)=0$ one can see that the fixed points of (21), i.e. the analytic functions satisfying on $[0,1]$ the functional relation

$$F(x) = \left(1 + \frac{xF(qx)}{F(-q)}\right),$$

are exactly of the form $F(x)=\Psi(q,-\hat{u}x)$ where with $q\in(0,1)$ fixed and $\hat{u}$ is one of the real roots of the equation $\Psi(q,u)=0$.

Thus for any fixed $q\in(0,1)$ the iteration scheme (21) considered as a self-map of an appropriate space of functions analytic in a neighborhood of $[-1,0]$ has countably many fixed points. At the moment it is completely unclear what local properties these fixed points have. For example, which of the above fixed points are repelling and which are attracting? Under which additional assumptions on f_1 the sequence $\{f_j\}$ obtained via scheme (21) converges?

Our computer experiments and Theorem 4 suggest that the following statement should be true.

Conjecture 1. *For any positive $q \leq \widetilde{q}$ and any initial function of the form $f_1=(x+1)Q(x)$ where $Q(x)$ is a hyperbolic polynomial with all negative roots smaller than -1, the polynomial sequence $\{f_j(x)\}$ converges uniformly on $[-1,0]$ with any number of derivatives to the function $\Psi(q,-u(q)x)$, where $u(q)$ is the negative solution of the equation $\Psi(q,u)=0$ with the minimal absolute value.*

Remark 7. Seemingly, under the condition $0<q\leq\widetilde{q}$ the attraction domain of the latter fixed point is much larger than f_1 of the form given in the above conjecture. In particular, iterations starting with $f_1=\sin\pi x$ converge very quickly to the same limit. On the other hand, if $q>\widetilde{q}$ numerical experiments show that iterations typically diverge; see Fig 3. It might be that for $q>\widetilde{q}$ all the fixed points of (21) become repelling. There are superficial similarities of the scheme (21) and

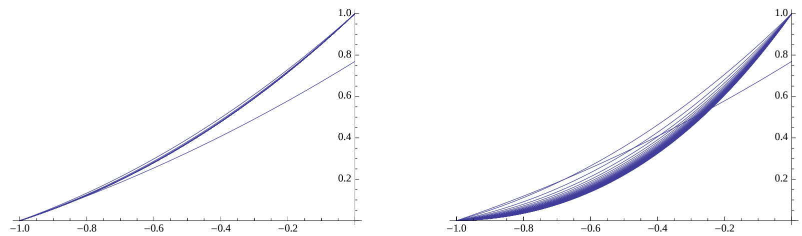

Figure 5. Convergence of iterations for $q = 1/4$ (left) and the critical $\widetilde{q}$ (right).

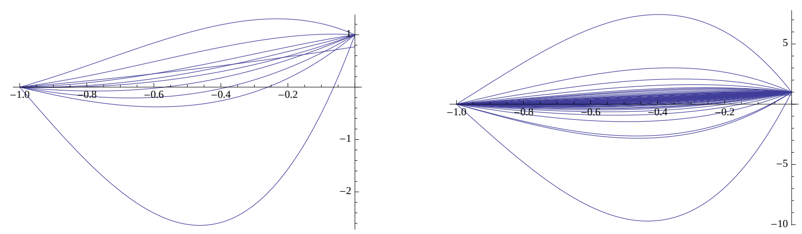

Figure 6. Divergence of iterations for $q = 1/2$. The number of iterations on the left is 10 and on the right is 50.

the famous logistic map in dynamical systems which also depends crucially on the value of the additional parameter q.

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