# Deformed graphical zonotopal algebras

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**Abstract**

We study certain filtered deformations of the external zonotopal algebra $\mathcal{C}_G^f$ of a graph parametrized by a univariate polynomial $f$. We establish some general properties of these algebras and compute Hilbert function of $\mathcal{C}_G^f$ for a large number of examples. We also formulate several conjectures.

## 1 Introduction

Let $G$ be a finite undirected graph. A. Postnikov and the first author [7] introduced a commutative graded algebra $\mathcal{C}_G$ whose dimension is equal to the number of spanning forests of $G$. They also showed that the Hilbert series of $\mathcal{C}_G$ is a specialization of the Tutte polynomial of $G$ which enumerates the spanning forests of $G$ according to their external activity. These algebras can be defined for arbitrary representable matroids. Independently, D. Wagner [11] associated to a graph $G$ another algebra of the same dimension as $\mathcal{C}_G$ which is isomorphic to the Postnikov-Shapiro algebra of the cographical matroid of $G$.

Wagner's initial goal was to construct new algebraic invariants of graphs. Postnikov and Shapiro were motivated by the earlier work [9, 8], where it was shown that for the complete graph $G$, the algebra $\mathcal{C}_G$ is isomorphic to the algebra generated by the curvature forms of tautological Hermitian line bundles on the complete flag manifold. Soon it turned out that these algebras are connected to several other areas, such as the theory of power ideals, box splines, enumeration of lattice points, chip firing, etc. They have been studied under various names: circulation algebras, Postnikov-Shapiro algebras, forest-counting algebras, and (external) zonotopal algebras. We will use the latter term reflecting their connection with enumeration of lattice points in zonotopes (see e.g. [3]).

Wagner [10] and Nenashev [4] proved that the algebra $\mathcal{C}_G$ determines the graphical matroid of $G$. However, non-isomorphic graphs can have isomorphic algebras. For example, this is the case for all trees with the same number of vertices. In [6], Nenashev and the first author introduced a filtered algebra $\mathcal{K}_G$ which they called a K-theoretic analog of $\mathcal{C}_G$. They showed that the algebras $\mathcal{K}_G$ and $\mathcal{C}_G$ are isomorphic as (non-filtered) algebras but, unlike $\mathcal{C}_G$, the isomorphism type of $\mathcal{K}_G$ as a filtered algebra is a complete invariant of $G$.

The algebra $\mathcal{K}_G$ is a deformation of $\mathcal{C}_G$ in the class of filtered algebras. It belongs to a larger family $\mathcal{C}_G^f$ of filtered deformations of $\mathcal{C}_G$ parametrized by formal power series $f \in \mathbf{k}[[u]]$. In the current work, we begin to study this family of algebras, their relationship to each other and to the graph $G$.

We find this family of algebras interesting for several reasons. Compared to the graded algebra $\mathcal{C}_G$, which corresponds to $f = u$, the filtered algebras $\mathcal{C}_G^f$ are more sensitive to graph structure (for a generic $f$ they give a complete graph invariant) and their Hilbert functions, therefore, should contain useful graphical information. Often, by varying only a few coefficients in the function $f$, one can distinguish all graphs in a certain family by comparing the Hilbert functions of $\mathcal{C}_G^f$. Frequently, Hilbert functions for special families of graphs (like paths, cycles, stars, etc) exhibit interesting combinatorial patterns. In an earlier version of this work, we conjectured that the collection of the Hilbert functions of $\mathcal{C}_G^f$ for all $f$ completely determines $G$. However, we found several pairs of non-isomorphic connected graphs for which these Hilbert functions coincide for all $f$ we tested (see Question 2 in Section 7). Thus this conjecture seems to be false, and so it would be interesting to study the equivalence relation it imposes on graphs.

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We begin Section 2 with the definition and properties of the graded zonotopal algebra $\mathcal{C}_G$. We show how its functoriality with respect to graph morphisms leads to a deletion-contraction exact sequence connecting $\mathcal{C}_G$ and the zonotopal algebras of graphs $G-e$ and $G/e$ obtained from $G$ be deleting and contracting an edge respectively. Using this sequence we give a simple proof of the formulas for the dimension and the Hilbert series of $\mathcal{C}_G$, which were originally found in [10, 8].

In Section 3 we introduce the deformed algebras $\mathcal{C}_G^f$, our main object of study, which are not graded but endowed with an increasing filtration. We establish some general facts about these algebra. In particular, we prove that under mild nondegeneracy condition on $f$, the algebras $\mathcal{C}_G^f$ and the graded algebra $\mathcal{C}_G$ are isomorphic as unfiltered algebras.

In Section 4 we discuss algebraic properties of deformed zonotopal algebras. Unlike the graded case, we cannot explicitly find the Hilbert function of these algebras in general. However, we prove that under certain conditions the above deletion-contraction exact sequence preserves the filtration on $\mathcal{C}_G^f$ and can be used to compute its Hilbert series. We also find defining relations for the algebra $\mathcal{C}_G^f$ for the system of generators given by the elements of the first term of the filtration.

In Section 5 for a given graph $G$, we study the parameter space of its deformed zonotopal algebras and show that it is stratified by semi-algebraic subsets consisting of algebras with the same Hilbert series.

In Section 6 we collect computations of the Hilbert functions for several families of graphs and various choices of function $f$. Finally, in Section 7 we present a number of conjectures and questions for further study.

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# 2 External zonotopal algebras

## 2.1 Preliminaries on graphs

By a *graph* we understand a finite undirected multigraph $G=(V,E)$, possibly with loops, with a vertex set $V$ and a (multi)set of edges $E$.

Graphs form a category $\mathcal{G}r$ with morphisms

$$f : G=(V,E)\to G'=(V',E')$$

defined as maps of pairs

$$(\gamma,\lambda)\colon (V,E)\to (V',E'), \tag{2.1}$$

*injective on edges* and preserving incidences between vertices and edges (that is, if $e\in E$ is an edge in $G$ connecting $u$ and $v$, then $\lambda(e)$ is an edge in $G'$ connecting $\gamma(u)$ and $\gamma(v)$).

For an edge $e\in E$ we define two special graph morphisms,

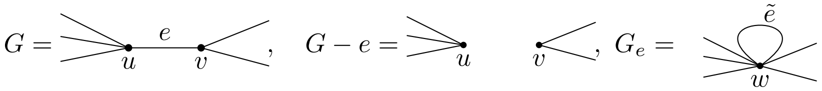
*the deletion of* $e$:

$$j_e\colon G-e\to G$$

and *the contraction of* $e$ *(assuming that* $e$ *is not a loop)*:

$$\pi_e\colon G\to G_e,$$

where $G-e$ is the subgraph of $G$ obtained by removing the edge $e$ from $E$, and $G_e$ is the graph obtained from $G$ by identifying the endpoints $u$ and $v$ of $e$ and replacing them with a new vertex $w$ and replacing each edge connecting $u$ and $v$ (including $e$) by a loop at $w$:



(2.2)

## 2.2 Graph algebras

By an algebra in this paper we understand a commutative unital algebras over a fixed field $\mathbf{k}$ of characteristic 0.

**Definition 2.1.** Given a graph $G = (V, E)$, its *edge algebra* $\Phi_G$ is the quotient of the polynomial algebra in *edge variables* $\phi_e, e \in E$, by their squares,

$$\Phi_G := \mathbf{k}[\phi_e : e \in E]/(\phi_e^2). \tag{2.3}$$

The edge algebra is a local algebra of dimension $2^{|E|}$ isomorphic to the tensor product of $|E|$ copies of the algebra of dual numbers $\mathbf{k}[\varepsilon]/(\varepsilon^2)$, namely

$$\Phi_G \simeq \bigotimes_{e\in E} \big(\mathbf{k}[\phi_e]/(\phi_e^2)\big).$$

It inherits the standard grading from the polynomial algebra. A basis of the $k$th graded component is given by square-free monomials of length $k$ in the edge variables $\phi_e$. Each such monomial corresponds to a unique subset of $E$ which we can identify with a spanning subgraph of $G$.

This construction is functorial. Every graph morphism (2.1)

$$(\gamma, \lambda) : G \to G'$$

induces a natural homomorphism of algebras

$$\lambda^* : \Phi_{G'} \to \Phi_G,$$

defined on generators $\phi_{e'} \in \Phi_{G'}$ by

$$\lambda^*(\phi_{e'}) := \sum_{e\in\lambda^{-1}(e')} \phi_e. \tag{2.4}$$

Thus, $\lambda^*(\phi_{\lambda(e)}) = \phi_e$ and $\lambda^*(\phi_{e'}) = 0$, if $e' \in E'$ is not in the image of $\lambda$.

**Definition 2.2.** Let $G = (V, E)$ be a graph with a linear order $<$ on its set of vertices $V$. The *zonotopal algebra* of $G$ is the subalgebra $\mathcal{C}_G$ of the edge algebra $\Phi_G$ generated by the elements

$$X_v = \sum_{e\in G} c_{v,e}\phi_e, \quad v \in V, \tag{2.5}$$

where

$$c_{v,e} = \begin{cases} 1, & \text{if } e = \{v, u\},\ v < u, \\ -1, & \text{if } e = \{v, u\},\ v > u, \\ 0 & \text{otherwise (in particular, if } e \text{ is a loop).} \end{cases} \tag{2.6}$$

Remark 2.3. Even though the coefficients $c_{v,e}$ in (2.6), and thus the elements (2.5), depend on the chosen ordering $<$ of the vertex set $V$, the subalgebras obtained from two different orderings will be identified under a graded automorphism of the algebra $\Phi_G$ given by changing signs of some generators $\phi_e$. It is clear that loop edges of $G$ do not contribute to the generators $X_v$. For this reason, in earlier papers [7, 4, 6] on this topic, graphs with loops were not considered. In this work we allow loops because they naturally appear when we consider graph homomorphisms involving edge contractions.

One of the important properties of the algebra $\mathcal{C}_G$ is that it is functorial with respect to graph morphisms (2.1).

**Proposition 2.4.** *For every graph morphism*

$$\phi = (\gamma, \lambda)\colon G \to G',$$

*the homomorphism $\lambda^*$ (2.4) sends the subalgebra $\mathcal{C}_{G'} \subset \Phi_{G'}$ to $\mathcal{C}_G \subset \Phi_G$. Thus,*

$$\mathcal{C} : G \mapsto \mathcal{C}_G$$

*is a contravariant functor from the category of graphs $\mathcal{G}r$ to the category of graded* **k***-algebras.*

*Proof.* All we need to show is that for every vertex $v' \in V'$, the image of the generator $X_{v'}$ of $\mathcal{C}_{G'}$ under the homomorphism $\lambda^*$ belongs to $\mathcal{C}_G \subset \Phi_G$. Indeed, if $v' \notin \gamma(V)$ then $\lambda(E)$ contains no edges incident to $v'$, and so $\lambda^*(X_{v'}) = 0$. If $v' \in \gamma(V)$, then, because of the injectivity of $\lambda$, we have

$$\lambda^*(X_{v'}) = \sum_{v \in \gamma^{-1}(v')} X_v \in \mathcal{C}_G,$$

as claimed.

□

Remark 2.5. The definition of $\mathcal{C}_G$ involves a choice of a linear order $<$ on $V$. In the above proposition we assume that the orders $<$ and $<'$ on $V$ and $V'$ are compatible, that is, they are chosen in such a way that the map $\gamma\colon V \to V'$ is monotone.

Since the loop edges of $G$ do not contribute to the generators $X_v$ (2.5) of the algebra $\mathcal{C}_G$, we may remove loops from $G$ without affecting $\mathcal{C}_G$. In particular, for a non-loop edge $e$, let $G/e$ be the graph obtained from $G$ by contracting $e$, that is removing $e$ and identifying the endpoints $u$ and $v$ of $e$. In other words, graph $G/e$ is obtained from the graph $G_e$ (2.2) by removing the loop $\tilde{e}$.

Thus $\mathcal{C}_{G/e}$ can be canonically identified with $\mathcal{C}_{G_e}$ and the contracting morphism $\pi_e\colon G \to G_e$ induces an injective homomorphism

$$\pi_e^*\colon \mathcal{C}_{G/e} = \mathcal{C}_{G_e} \to \mathcal{C}_G \tag{2.7}$$

which sends the generator $X_w$ corresponding to the new vertex $w = \{u, v\}$ to $X_v + X_u \in \mathcal{C}_G$.

## 2.3 Deletion-contraction exact sequence

The functoriality of $\mathcal{C}_G$ with respect to $G$ leads to an important relation between the zonotopal algebras of the three graphs $G$, $G/e$, and $G - e$.

Let $e \in E$ be a non-loop edge of a graph $G$ with the endpoints $u$ and $v$. Consider two algebra homomorphisms, the projection

$$j_e^*\colon \mathcal{C}_G \to \mathcal{C}_{G-e},$$

produced by sending $\phi_e \in \Phi_G$ to 0, and the embedding

$$\pi_e^*\colon \mathcal{C}_{G/e} = \mathcal{C}_{G_e} \to \mathcal{C}_G,$$

which maps $X_w$ to $X_u + X_v$.

Denote by

$$\partial_e := \frac{\partial}{\partial \phi_e}\colon \Phi_G \to \Phi_G/(\phi_e), \tag{2.8}$$

the "partial derivative" with respect to the edge variable $\phi_e \in \Phi_G$. Modulo the ideal $(\phi_e)$, the map $\partial_e$ is well-defined.

The identification

$$\Phi_G/(\phi_e) \simeq \Phi_{G-e}$$

endows $\Phi_{G-e}$ with a structure of a $\Phi_G$-algebra with respect to which $\partial_e$ becomes a $\Phi_{G-e}$-valued derivation

$$\delta_e : \Phi_G \to \Phi_{G-e}\ .$$

**Theorem 2.6.**

*(i) The derivation $\delta_e$ sends $\mathcal{C}_G$ into $\mathcal{C}_{G-e}$ lowering grading degree by* 1.

*(ii) The sequence*

$$0 \to \mathcal{C}_{G/e} \overset{\pi_e^*}{\to} \mathcal{C}_G \overset{\delta_e}{\to} \mathcal{C}_{G-e}[-1] \to 0, \tag{2.9}$$

*is exact.*

*Proof.* Assume that the edge $e$ is oriented from $u$ to $v$. The action of $\partial_e$ on generators $X_p$ (2.5), for $p \in V$, is given by

$$\partial_e(X_p) = \begin{cases} 1, & \text{if } p = u, \\ -1, & \text{if } p = v, \\ 0, & \text{if } p \notin \{u, v\} \ . \end{cases} \tag{2.10}$$

Together with the fact that $\delta_e$ is a derivation, this implies (i).

Since $\pi_e^*$ is an embedding, to prove (ii) we only need to check exactness at the second and third terms.

If we replace the element $X_v$ in the generating set $\{X_p : p \in V\}$ of the algebra $\mathcal{C}_G$ with

$$Y = X_u + X_v \in \mathcal{C}_G^{(1)},$$

then the new collection $\{Y,\ X_p : p \neq v\}$ will still generate $\mathcal{C}_G^{(1)}$. Therefore, every element $Z \in \mathcal{C}_G$ can be represented as

$$Z = \sum_{r \geq 0}^{\deg u} P_r \cdot X_u^r, \tag{2.11}$$

where $P_r$ is a polynomial in $Y$ and $X_p$, with $p \neq u, v$. Since, by (2.10), $\partial_e X_u = 1$, $\partial_e Y = 0$, and thus, $\partial_e P_r = 0$, we have

$$\partial_e Z = \sum_{r \geq 0}^{\deg u} r P_r \cdot X_u^{r-1}.$$

Any $\overline{Z} \in \mathcal{C}_{G-e}$ can be represented in a form similar to (2.11) as

$$\overline{Z} = \sum_{r \geq 0}^{\deg u - 1} P_r \cdot \overline{X}_u^r,$$

where $\overline{X}_u = X_u \mod (\phi_e)$. Therefore,

$$\overline{Z} = \partial_e \left( \sum_{r=1}^{\deg u} \frac{1}{r} P_r \cdot X_u^r \right),$$

which shows that $\delta_e$ is surjective.

To prove exactness of (2.9) in the $\mathcal{C}_G$-term, observe that $X_u$ is the only element in our generating set on which $\partial_e$ does not vanish. Therefore, the polynomials $P_r$ are unique and thus $\ker \delta_e$ consists of those $Z \in \mathcal{C}_G$ with $P_r = 0$ for $r > 0$. That is,

$$\ker \delta_e = \{P(Y, X_p) \,|\, P \in \mathbf{k}[y, x_p : p \in V - \{u, v\}]\},$$

which coincides with the image of $\pi_e^*$, because $\pi_e^*$ sends the generator $X_w$ of $\mathcal{C}_{G/e}$ to $X_u + X_v = Y$. □

For a graded $\mathbf{k}$-algebra $A = \bigoplus_{k \geq 0} A^{(k)}$, denote by

$$H_A(t) := \sum_{k \geq 0} \dim A^{(k)} t^k \tag{2.12}$$

its *Hilbert series*. The exact sequence (2.9) implies the following relation between the Hilbert series of the algebras $\mathcal{C}_G$, $\mathcal{C}_{G/e}$, and $\mathcal{C}_{G-e}$.

**Corollary 2.7.** *Let $e$ be a non-loop edge of $G$. Then*

$$H_{\mathcal{C}_G}(t) = H_{\mathcal{C}_{G/e}}(t) + tH_{\mathcal{C}_{G-e}}(t). \tag{2.13}$$

The above relation allows to easily derive the formulas for the dimension and the Hilbert series for any graph $G$ first proved in [10, 8].

In order to state the formula for the Hilbert function of the algebra $\mathcal{C}_G$, we will first recall the notion of external activity in graphs. A *spanning* forest of a graph $G = (V, E)$ is an acyclic subgraph of $G$ with the same set of vertices. In other words, a spanning forest can be viewed as an acyclic collection of edges $F \subset E$.

**Definition 2.8.** Let $G = (V, E)$ be a graph with some linear order on the set of edges $E$. Let $F \subset E$ be a spanning forest of $G$. An edge $e \in E - F$ is called *externally active* for $F$, if the subgraph $F \cup \{e\}$ contains a cycle and the edge $e$ is the smallest element of this cycle.

The number $\alpha(F)$ of externally active edges for $F$ is called the *external activity* of $F$ .

While the external activity of an individual forest $F$ depends on the ordering of the edges, the number of spanning forests $F$ with the given number of edges $|F|$ with the external activity $\alpha(F) = k$ is the same for all orderings. Notice that any loop edge is externally active for any forest, so $\alpha(F) \geq \ell$, where $\ell$ is the number of loops in $G$.

**Theorem 2.9** ([8, 10])**.** *Let $G = (V, E)$ be a graph.*

*(i) The dimension of the algebra $\mathcal{C}_G$ is equal to the number of spanning forests of $G$.*

*(ii) The dimension of the $k$th graded component $\mathcal{C}_G^{(k)}$ of $\mathcal{C}_G$ is equal to the number of spanning forests $F \subseteq E$ with the external activity*

$$\alpha(F) = |E| - |F| - k.$$

*Proof.* Since $\dim \mathcal{C}_G = \sum_{k \geq 0} \dim \mathcal{C}_G^{(k)}$, part (i) follows from (ii).

Let

$$a_k(G) = \big|\{F \subset E \,|\, \alpha(F) = |E| - |F| - k\}\big|$$

be the number of spanning forests of a graph $G$ with the external activity $|E|-|F|-k$. Then (ii) is equivalent to the equality

$$H_{\mathcal{C}_G}(t) = F_G(t)$$

where

$$F_G(t) = \sum_{k \geq 0} a_k(G) t^k$$

is the generating function of the sequence $a_k(G),\ k \geq 0$.

If $G$ is a graph without non-loop edges, then it has only one spanning forest $F = \emptyset$ for which all edges (loops) are externally active. Thus $\alpha(F) = |E| = |E| - |F|$ and $F_G(t) = 1$. Therefore, to prove (ii) it is sufficient to show that $F_G$ satisfies the deletion-contraction relation similar to (2.13):

$$F_G(t) = F_{G/e}(t) + tH_{G-e}(t)$$

or, equivalently, that for every $G$ we have

$$a_k(G) = a_k(G/e) + a_{k-1}(G - e),\ k \geq 0. \tag{2.14}$$

Denote by $\mathcal{F}(G)$ the set of forests in a graph $G$. If $e \in E$ is a non-loop edge, then $\mathcal{F}(G) = \mathcal{F}_+(G) \sqcup \mathcal{F}_-(G)$, where

$$\mathcal{F}_+(G) = \{F \in \mathcal{F}(G) \,|\, e \in F\}$$

and

$$\mathcal{F}_-(G) = \{F \in \mathcal{F}(G) \,|\, e \notin F\}.$$

These subsets can be identified with the sets of forests $\mathcal{F}(G/e)$ and $\mathcal{F}(G-e)$ by the bijections

$$f_+ : \mathcal{F}(G/e) \to \mathcal{F}_+(G),\ F \mapsto F \sqcup \{e\}$$

and

$$f_- : \mathcal{F}(G-e) \to \mathcal{F}_-(G),\ F \mapsto F.$$

To verify relation (2.14), we need to keep track of the numbers $a_k$. Let us order the set of edges $E$ so that the edge $e \in E$ is the largest. Then $\alpha(f_\pm(F)) = \alpha(F)$ both for $F \in \mathcal{F}(G/e)$ and $F \in \mathcal{F}(G-e)$. In the first case, the term $|E| - |F|$ stays the same, so forests from $\mathcal{F}(G/e)$ and $\mathcal{F}_+(G)$ contribute to the same coefficient $a_k$. In the second case the difference $|E| - |F|$ changes by 1, therefore forests from $\mathcal{F}(G-e)$ contribute to $a_{k-1}$, as required.

□

In [10, 8], it was proved the algebra $\mathcal{C}_G$ is a power algebra, that is it is isomorphic to a quotient of the polynomial algebra by an ideal generated by powers of linear forms. Namely, for a subset $I \subset V$ of vertices, denote by $D_I$ the number of edges $e \in E$ connecting a vertex from $I$ with one in the complementary subset $I^c = V - I$.

**Theorem 2.10** ([10, 8])**.** *The zonotopal algebra $\mathcal{C}_G$ is isomorphic to the quotient of the polynomial algebra $\mathbf{k}[x_v : v \in V]$ by the ideal generated by the power sums*

$$p_I = \Big(\sum_{v\in I} x_v\Big)^{D_I+1}, \quad \textit{for all} \ \ I \subseteq V. \tag{2.15}$$

This theorem can also be proved using the deletion and contraction morphisms by constructing an analogous to (2.9) exact sequence

$$0 \to \mathcal{B}_{G/e} \to \mathcal{B}_G \to \mathcal{B}_{G-e}[-1] \to 0,$$

for the quotient algebras

$$\mathcal{B}_G := \mathbf{k}[x_v : v \in V]/(p_I : I \subseteq V),$$

where the polynomial $p_I$ is given by (2.15), and noticing that, since the generators $X_v$ of $\mathcal{C}_G$ satisfy the relation

$$\Big(\sum_{v\in I} X_v\Big)^{D_I+1} = 0,$$

there is a natural projection $\mathcal{B}_G \to \mathcal{C}_G$.

Nenashev proved in [5] that the algebra $\mathcal{C}_G$ contains exactly the same information about $G$ as does its graphic matroid.

**Proposition 2.11** ([5])**.** *Given two connected graphs $G_1$ and $G_2$, the algebras $\mathcal{C}_{G_1}$ and $\mathcal{C}_{G_2}$ (viewed either as graded or non-graded algebras) are isomorphic if and only if the graphical matroids of $G_1$ and $G_2$ are isomorphic.*

Thus the algebra $\mathcal{C}_G$ contains exactly as much information about the graph $G$ as its Tutte polynomial. In the next section we will begin to study a filtered deformation $\mathcal{C}_G^f$ of $\mathcal{C}_G$ which, as it turns out (see Theorem 3.8), is a complete invariant of $G$.

# 3 Deformed zonotopal algebras

The main object of our study in this paper is a family of filtered algebras $\mathcal{C}_G^f$ parametrized by power series $f \in \mathbf{k}[[u]]$ which are deformations of the external zonotopal algebra $\mathcal{C}_G$ of a graph $G$.

**Definition 3.1.** A formal power series

$$f = a_0 + a_1 u + a_2 u^2 + \ldots \in \mathbf{k}[[u]]$$

is called *nondegenerate* if $a_1 \neq 0$, i.e. when $f'(0) \neq 0$.

**Definition 3.2.** Let $G = (V, E)$ be a graph and let $f \in \mathbf{k}[[u]]$ be a nondegenerate power series. The *deformed zonotopal algebra* of $G$ associated to $f$ is the subalgebra of the edge algebra $\Phi_G$ (2.3) generated by the elements

$$Y_v := f(X_v) = f\big(\sum_{e \in E} c_{v,e}\phi_e\big), \; v \in V, \tag{3.1}$$

where $c_{v,e}$ are given by (2.6).

If $f = u$, then $\mathcal{C}_G^f$ is the usual zonotopal algebra $\mathcal{C}_G$ of $G$, and if $f = e^u$, the deformed algebra $\mathcal{C}_G^f$ is the $K$-theoretic analog $\mathcal{K}_G$ of $\mathcal{C}_G$ studied in [6].

Remark 3.3. Since $\phi_e^2 = 0$, the element $X_v$ is nilpotent with $X_v^n = 0$ for $n > \deg v$, and so $f(X_v)$ into a power series is well-defined. This also shows that the terms of $f$ of degree higher than $\Delta(G)$, the maximum vertex degree of $G$, do not affect any of the generators $Y_v$ of $\mathcal{C}_G^f$. Therefore, we may restrict our attention to those $f \in \mathbf{k}[[u]]$ which are polynomials in $u$ of degree at most $\Delta(G)$.

Remark 3.4. Zonotopal algebras can be defined not only for graphs, but for arbitrary finite vectors configurations in $\mathbf{k}^n$ that correspond to representable matroids (see [8]). The same can also be done for the deformed algebras. However, unlike the case of graphs, the resulting algebra will depend not on the matroid, but on the specific representation. For this reason in this paper we restrict our attention to graphs, even though some of our results can be extended to arbitrary vector configurations.

The algebra $\mathcal{C}_G^f$ is endowed with an increasing filtration

$$\mathbf{k} = \mathcal{C}_G^{f,0} \subset \mathcal{C}_G^{f,1} \subset \mathcal{C}_G^{f,2} \subset \ldots, \tag{3.2}$$

where the subspace $\mathcal{C}_G^{f,i}$ is spanned by the monomials of degree at most $i$ in the generators $Y_v = f(X_v)$.

Remark 3.5. Notice that neither the algebra $\mathcal{C}_G^f$ nor the filtration (3.2) depend on the constant term $a_0 = f(0)$ of $f$, since replacing $f$ by $f + c$ for $c \in \mathbf{k} = \mathcal{C}_G^{f,0}$ simply changes each generator $Y_v$ to $Y_v + c$.

Similarly, is also clear that multiplying $f$ by a nonzero constant does not change the filtration.

For this reason, from now on, we may assume that $f$ has no constant term and that its coefficient at $u$ is equal to 1, i.e. that $f \equiv u \pmod{u^2}$.

When $f = u$, the filtration (3.2) coincides with the filtration induced on $\mathcal{C}_G$ by its standard grading.

The reason we limit our attention to nondegenerate $f$ is explained by the following basic property of $\mathcal{C}_G^f$ proved in [6].

**Proposition 3.6** ([6, Proposition 2])**.** *If $f \in \mathbf{k}[[u]]$ is a nondegenerate series, then $\mathcal{C}_G^f$ coincides, as a subalgebra of $\Phi_G$, with the usual zonotopal algebra $\mathcal{C}_G$. In other words, for nondegenerate $f$, the algebra $\mathcal{C}_G^f$ differs from $\mathcal{C}_G$ only in its filtration.*

*Proof.* By Remark 3.3 we may assume that $f$ is a polynomial. Since $X_v \in \mathcal{C}_G$ for $v \in V$, this implies that the all generators $Y_v = f(X_v)$ of $\mathcal{C}_G^f$ belong to $\mathcal{C}_G$. Thus we have that $\mathcal{C}_G^f \subseteq \mathcal{C}_G$.

To prove the opposite inclusion, it is enough to show that $X_v \in \mathcal{C}_G^f$ for $v \in V$. In view of Remark 3.3, this follows from the fact that for a non-degenerate series $f(u) = c_1 u + c_2 u^2 + \ldots,\; c_1 \neq 0$, there exists a polynomial $g \in \mathbf{k}[u]$ such that $z = g(f(Z))$ for every nilpotent element $\phi$ of the edge algebra $\Phi_G$. Indeed, first we find a formal power series $\tilde{g}(u) = \sum_{i \geq 1} a_i u^i$ satisfying

$$\begin{aligned} u =& \tilde{g}(f(u)) = \sum_{i \geq 1} a_i (f(u))^i \\ &= a_1 c_1 u + (a_1 c_2 + a_2 c_1^2) u^2 + (a_1 c_3 + 2 a_2 c_1 c_2 + a_3 c_1^3) u^3 + \ldots \end{aligned}$$

by solving for the coefficients $a_i$ inductively: $a_1 = 1/c_1,\ a_2 = -a_1 c_2/c_1^2, \dots$ . Now a required polynomial $g(u)$ can be obtained by truncating the resulting power series $\tilde{g}(u)$ after the term of degree $|E|$.

$\square$

Proposition 3.6 shows that varying a nondegenerate series $f \in \mathbf{k}[[u]]$ is equivalent to changing a filtration of the fixed algebra $\mathcal{C}_G$. The original grading can be recovered by taking the associated graded algebra with respect to the distinguished ideal $(Y_v \mid v \in V)$.

**Corollary 3.7.** *Let $f \in u\mathbf{k}[[u]]$ be a nondegenerate series and let $J$ be the ideal of the algebra $\mathcal{C}_G^f$ generated by the elements $Y_v = f(X_v)$, $v \in V$. Then the associated graded algebra $\mathrm{Gr}_J(\mathcal{C}_G^f)$ for the $J$-adic filtration*

$$\mathcal{C}_G^f \supset J \supset J^2 \supset \dots 0$$

*is isomorphic to the graded algebra $\mathcal{C}_G$. In particular, the Hilbert functions of $\mathcal{C}_G$ and of $\mathcal{C}_G^f$ with respect to the $J$-adic filtration coincide.*

*Proof.* This is simply because the algebra $\mathcal{C}_G$ is local and, for nondegenerate $f$, its maximal ideal is generated by elements $Y_v,\ v \in V$.

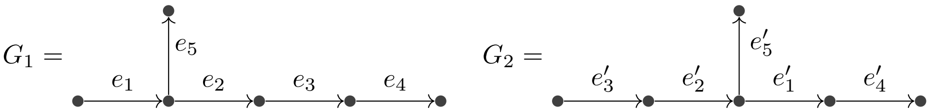
$\square$

The following result shows that the deformed algebra $\mathcal{C}_G^f$ is, in general, a much stronger invariant of $G$ than the usual external zonotopal algebra $\mathcal{C}_G$.

**Theorem 3.8** ([6, Theorem 6])**.** *Let $f$ be a polynomial with non-vanishing linear and quadratic terms and let $G_1$ and $G_2$ be two simple graphs without isolated vertices. Then $\mathcal{C}_{G_1}^f$ and $\mathcal{C}_{G_2}^f$ are isomorphic as filtered algebras if and only if the graphs $G_1$ and $G_2$ are isomorphic.*

If the quadratic term of the polynomial $f(u)$ vanishes, the corresponding algebras for non-isomorphic graphs can be isomorphic even as filtered algebras, as the following example shows.

**Proposition 3.9.** *Consider the trees $G_1 = D_6$ and $G_2 = E_6$ shown below and let $f(u) = u + u^3$. Then the deformed zonotopal algebras $\mathcal{C}_{G_1}^f$ and $\mathcal{C}_{G_2}^f$ are isomorphic as filtered algebras.*



*Proof.* Label the edges of the two graphs $e_1, \dots, e_5, e'_1, \dots, e'_5$ as shown on the figure (notice the difference in placing the labels $e_1, e_3$ and $e'_1, e'_3$) and denote by $\phi_1, \dots, \phi_5, \phi'_1, \dots, \phi'_5$ the corresponding generators of the edge algebras $\Phi_{G_1}$ and $\Phi_{G_2}$ (2.3). The algebras $\mathcal{C}_{G_1^f}$ and $\mathcal{C}_{G_2^f}$ are the subalgebras of $\Phi_{G_1}$ and $\Phi_{G_2}$ generated, respectively, by $\{f(X_v)\}$ and $\{f(X_w)\}$, where $v \in V(G_1)$, $w \in V(G_2)$ and elements $X_v$ are given by (2.5).

For $f = u + u^3$, since $\phi_i^2 = \phi'^2 = 0$, we see that these sets of generators for $\mathcal{C}_{G_1}^f$ are and $\mathcal{C}_{G_2}^f$ are, respectively,

$$S_1 = \{\phi_1, -\phi_1 + \phi_2 + \phi_5 - \phi_1\phi_2\phi_5, \phi_3 - \phi_2, \phi_4 - \phi_3, -\phi_4, -\phi_5\}$$

and

$$S_2 = \{\phi'_3, \phi'_2 - \phi'_3, \phi'_1 - \phi'_2 + \phi'_5 - \phi'_1\phi'_2\phi'_5, -\phi_1 + \phi_4, -\phi'_4, -\phi'_5\}.$$

The linear subspaces generated by these sets in $\Phi_{G_1}$ and $\Phi_{G_2}$ are, respectively, $L_1 = \langle S_1 \rangle = \langle \phi_i, \phi_1\phi_2\phi_5 \rangle$ and $L_2 = \langle S_2 \rangle = \langle \phi'_i, \phi'_1\phi'_2\phi'_5 \rangle$, $i = 1, \dots, 5$. Therefore, the isomorphism of algebras $\Phi_{G_1} \to \Phi_{G_2}$ sending $\phi_i$ to $\phi'_i$ sends $V_1$ to $V_2$ and thus gives an isomorphism of the filtered algebras $\mathcal{C}_{G_1}^f$ and $\mathcal{C}_{G_2}^f$.

$\square$

# 4 Algebraic properties

## 4.1 Hilbert series

Recall that Hilbert function and Hilbert series for a filtered algebra $R$ are defined as the corresponding notions for the corresponding associated graded algebra

$$\mathrm{Gr}(R) = \bigoplus_{i \in \mathbb{Z}} R^i / R^{i-1}. \tag{4.1}$$

**Definition 4.1.** Let $R$ be a filtered $\mathbf{k}$ algebra $R^0 \subset R^1 \subset \ldots \subset R$ with finite dimensional quotients $R^i/R^{i-1},\ i = 0, 1, 2, \ldots$.

The sequence of integers

$$h_i(R) = \dim_{\mathbf{k}} R^i/R^{i-1},\ i \geq 0,$$

is called the *Hilbert function* of $R$, and its generating function

$$\mathcal{H}_R(t) = \sum_{i \geq 0} h_i(R) t^i \tag{4.2}$$

is called the *Hilbert series* of $R$. (For convenience, we set $R^{-1} = 0$).

The Hilbert series of the deformed zonotopal algebras have some simple properties.

**Proposition 4.2.** *Let $\mathcal{H}^f_G(t) = \sum_{i \geq 0} h_i t^i$ be the Hilbert series of the algebra $\mathcal{C}^f_G$ of a graph $G = (G, V)$. Then*

*(i) $\mathcal{H}^f_G(t)$ is a polynomial in $t$ of degree $d \leq |E|$.*

*(ii) $h_i \leq \binom{|V|}{i}$ for $i = 0, \ldots, |V|$; in particular $h_0 = 1$.*

*(iii) If $f \in \mathbf{k}[u]$ is nondegenerate, then $h_1 = |V|$.*

*(iv) If $G = G_1 \sqcup G_2$ is a disjoint union of two graphs $G_1$ and $G_2$, then*

$$\mathcal{H}_G(t) = \mathcal{H}_{G_1}(t)\mathcal{H}_{G_2}(t). \tag{4.3}$$

□

In general, finding Hilbert functions of the deformed algebras $\mathcal{C}^f_G$ is more difficult task then in the graded case. However, for polynomials $f$ which are linear up to terms of sufficiently high order, the deletion-contraction relation (2.13) respects the filtration and therefore allows to express the Hilbert function of the algebra $\mathcal{C}^f_G$ through those of $\mathcal{C}^f_{G/e}$ and $\mathcal{C}^f_{G-e}$.

**Theorem 4.3.** *Let $f \in \mathbf{k}[u]$ be a polynomial without terms of degrees $2, 3, \ldots, d-1$, i.e. $f \equiv cu \pmod{u^d}$.*

*If $e \in E$ is a non-loop edge connecting vertices $v$ and $u$ of a graph $G = (V, E)$, such that*

$$\deg v + \deg u < d + 2, \tag{4.4}$$

*then the Hilbert series of the deformed zonotopal algebras of the three graphs $G,\ G - e$, and $G/e$ satisfy*

$$\mathcal{H}^f_G(t) = \mathcal{H}^f_{G/e}(t) + t\mathcal{H}^f_{G-e}(t). \tag{4.5}$$

*Proof.* Under assumptions (4.4), it is clear that both the embedding $\pi^*_e$ and the derivation $\delta_e$ (2.8) preserve the filtration (3.2) of $\mathcal{C}^f_G$. Therefore, in this case the exact sequence (2.9) holds for individual terms of the filtration, just as in the graded case. This, in turn, implies the corresponding relation between the Hilbert series. □

**Corollary 4.4.** *If $e \in E$ is an edge connecting vertices of degrees $1$ and $d$, and $f \in \mathbf{k}[u]$ is such that $f \simeq cu \pmod{u^{d+1}}$, then*

$$\mathcal{H}_{\mathcal{C}^f_G}(t) = (1+t)\mathcal{H}_{\mathcal{C}^f_{G/e}}(t). \tag{4.6}$$

*Proof.* Indeed, in this case the graph $G - e$ is the disjoint union of $G/e$ and the one-vertex graph $G_0 = \bullet$. Thus the result follows from the deletion-contraction relation (4.5) and Proposition 4.2.(iv). $\square$

## 4.2 Two families of examples

In this subsection we compute the Hilbert series (4.2) of the algebra $\mathcal{C}^f_G$ for various non-degenerate polynomials

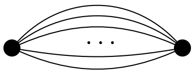
$$f = u + \sum_{i\geq 2} \alpha_i u^i \tag{4.7}$$

for two families of graphs: two-vertex multiedge graphs and star graphs.

### 4.2.1 Two-vertex multi-edge

Consider the graph

$$E_m = \tag{4.8}$$



consisting of two vertices connected by $m$ edges.

If $f = u$, the algebra $\mathcal{C}^f_{E_m}$ is graded and is generated by a single element

$$Y_1 = \sum_{i=1}^m \phi_i = -Y_2.$$

Therefore, in this case $\mathcal{C}^f_{E_m}$ is isomorphic to the truncated polynomial algebra $\mathbf{k}[x]/(x^{m+1})$ with the Hilbert series

$$1 + t + t^2 + \cdots + t^m.$$

By Theorem 3.8, this implies that for all nondegenerate $f$ we have

$$\dim \mathcal{C}^f_{E_m} = m + 1.$$

Notice that if the polynomial (4.7) has no nonzero monomials of even degree $\leq m$ (i.e. $\alpha_{2k} = 0$ for $k \leq m/2$), then the generators (3.1) of $\mathcal{C}^f_G$,

$$Y_1 = f\left(\sum_{i=1}^m \phi_i\right) \quad \text{and} \quad Y_2 = f\left(-\sum_{i=1}^m \phi_i\right),$$

satisfy $Y_1 + Y_2 = 0$. Therefore all the quotients $\mathcal{C}^{f,k}_G/\mathcal{C}^{f,k-1}_G$, for $k \leq m$, are one-dimensional. Thus, in this case the Hilbert series of $\mathcal{C}^f_G$ is the same as for $f = u$.

The above remark helps to find $\mathcal{H}^f_{E_m}$ for *all* non-degenerate $f$ for small values of $m$.

**Proposition 4.5.** *Let $G = E_m$ and let $f$ be a nondegenerate polynomial (4.7) with at least one nonzero coefficient $\alpha_{2\ell}$, $2\ell \leq m$. Then the Hilbert series of the algebra $\mathcal{C}^f_G$ is as follows.*

*(1) If $m = 2$ then $\mathcal{H}^f_G = 1 + 2t$,*

*(2) If $m = 3$ then $\mathcal{H}^f_G = 1 + 2t + t^2$,*

*(3) If $m = 4$ then* $\mathcal{H}^f_G = \begin{cases} 1 + 2t + 2t^2, & \text{if } \alpha_2 \neq 0 \\ 1 + 2t + t^2 + t^3, & \text{if } \alpha_2 = 0, \alpha_4 \neq 0 \end{cases}$

*(4) If* $m = 5$ *then* $\mathcal{H}_G^f = 1 + 2t + 2t^2 + t^3$.

*Proof.* Denote by $s = \sum_{i=1}^{m} \phi_i$ the sum of all edge variables (2.3) of the graph $G = E_m$. Clearly, $s^i \neq 0$ for $i \leq m$, and $s^i = 0$ for $i > m$. The algebra $\mathcal{C}_G^f$ is generated by

$$Y_1 = f(s) = s + \sum_{i=2}^{m} \alpha_i s^i \quad \text{and} \quad Y_2 = f(-s) = -s + \sum_{i=2}^{m} \alpha_i (-s)^i,$$

or equivalently by

$$a = \frac{1}{2}(Y_1 - Y_2) = s + \alpha_3 s^3 + \alpha_5 s^5 + \dots$$

and

$$b = \frac{1}{2}(Y_1 + Y_2) = \alpha_2 s^2 + \alpha_4 s^4 + \dots .$$

When $m = 2$, we have $f = u + \alpha_2 u^2$ with $\alpha_2 \neq 0$. In this case the generators

$$a = s \quad \text{and} \quad b = \alpha_2 s^2$$

of $\mathcal{C}_G^f$ are linearly independent and, since $\dim \mathcal{C}_G^f = 3$, the filtration (3.2) stabilizes at the second term. Thus in this case we have $\mathcal{H}_{E_2}^f(t) = 1 + 2t$.

For $m = 3$, we have $f = u + \alpha_2 u^2 + \alpha_3 u^3$ with $\alpha_2 \neq 0$. In this case the generators

$$a = s + \alpha_3 s^3 \quad \text{and} \quad b = \alpha_2 s^2$$

of $\mathcal{C}_G^f$ are linearly independent, and therefore,

$$\dim \mathcal{C}_G^{f,1} / \mathcal{C}_G^{f,0} = 2.$$

Since $\dim \mathcal{C}_G^f = 4$ and

$$s^3 = \frac{1}{\alpha_2} ab \in \mathcal{C}_G^{f,2},$$

we have the Hilbert series $1 + 2t + t^2$.

For $m = 4$, we have $f = u + \alpha_2 u^2 + \alpha_3 u^3 + \alpha_4 u^4$ with $\alpha_2 \neq 0$ or $\alpha_4 \neq 0$. The generators

$$a = s + \alpha_3 s^3 \quad \text{and} \quad b = \alpha_2 s^2 + \alpha_4 s^4$$

are linearly independent, and so

$$\dim \mathcal{C}_G^{f,1} / \mathcal{C}_G^{f,0} = 2.$$

If $\alpha_2 \neq 0$, then the elements of $\mathcal{C}_G^{f,2} / \mathcal{C}_G^{f,1}$ represented by $ab = \alpha_2 s^3$ and $b^2 = \alpha_2^2 s^4$, whereas $a^2 \in \langle b, b^2 \rangle$. Thus $\dim \mathcal{C}_G^{f,2} / \mathcal{C}_G^{f,1} = 2$ and, since $\dim \mathcal{C}_G^f = 5$, we have that the Hilbert series is $\mathcal{H}_G^f = 1 + 2t + 2t^2$.

If $\alpha_2 = 0$ and $\alpha_4 \neq 0$, then the elements $a = s + \alpha_3 s^3$ and $b = \alpha_4 s^4$ are still linearly independent in $\mathcal{C}_G^{f,1} / \mathcal{C}_G^{f,0}$. Since $ab = b^2 = 0$, the only nonzero monomials in $a, b$ of degree higher than 1 are $a^2$ and $a^3$. Thus $\dim \mathcal{C}_G^{f,2} / \mathcal{C}_G^{f,1} = \mathcal{C}_G^{f,3} / \mathcal{C}_G^{f,2} = 1$ and the Hilbert series in this case is $\mathcal{H}_G^f = 1 + 2t + t^2 + t^3$.

For $m = 5$, the generators

$$a = s + \alpha_3 s^3 + \alpha_5 s^5 \text{ and } b = \alpha_2 s^2 + \alpha_4 s^4,$$

where $\alpha_2 \neq 0$ or $\alpha_4 \neq 0$, are linearly independent, so $\dim \mathcal{C}_G^{f,1} / \mathcal{C}_G^{f,0} = 2$.

The space $\mathcal{C}_G^{f,2} / \mathcal{C}_G^{f,1}$ is spanned by the classes of three monomials $a^2$, $ab$ and $b^2$.

If $\alpha_2 \neq 0$ and $\alpha_4 \neq 2\alpha_2\alpha_3$, then $a^2 = s^2 + 2\alpha_3 s^4$ and $ab = \alpha_2 s^3 + (\alpha_4 + \alpha_2\alpha_3)s^5$ are linearly independent modulo $\mathcal{C}_G^{f,1}$ and the element $b^2 = \alpha_2^2 s^4$ is a linear combination of $b$ and $a^2$. Thus $\dim \mathcal{C}_G^{f,2}/\mathcal{C}_G^{f,1} = 2$. If $\alpha_4 = 2\alpha_2\alpha_3$, then still $\dim \mathcal{C}_G^{f,2}/\mathcal{C}_G^{f,1} = 2$, since $b^2$ does not belong to the span of $a, b, a^2$ and $ab$.

Since $\dim \mathcal{C}_G^{f} = 6$ and the element $ab^2 = \alpha_2 s^5$ does not belong to $\mathcal{C}_G^{f,2}$, we see that $\dim \mathcal{C}_G^{f,3}/\mathcal{C}_G^{f,2} = 1$ and so the Hilbert series in this case is $\mathcal{H}_G^f = 1 + 2t + 2t^2 + t^3$.

If $\alpha_2 = 0$ and $\alpha_4 \neq 0$, then $b = \alpha_4 s^4$. Therefore, the only nonzero monomials of degree $\geq 2$ are $a^2 = s^2 + 2\alpha_3 s^4$, $a^3 = s^3 + 3\alpha_3 s^5$, $a^4 = s^4$ and $ab = \alpha_4 s^5$. Since $a^2$ and $ab$ are linearly independent modulo $\mathcal{C}_G^{f,1}$ we obtain that $\dim \mathcal{C}_G^{f,2}/\mathcal{C}_G^{f,1} = 2$. Finally, since $\dim \mathcal{C}_G^{f} = 6$ and $a^3 = s^3\alpha_2 s^5 \notin \mathcal{C}_G^{f,2}$, we see that $\dim \mathcal{C}_G^{f,3}/\mathcal{C}_G^{f,2} = 1$. Thus we get the same Hilbert series as in the previous case $\mathcal{H}_G^f = 1 + 2t + 2t^2 + t^3$.

□

If the polynomial (4.7) is of the type $f = \sum_{i=1}^{\ell} u^i$, then we can find the Hilbert series $\mathcal{H}_{E_m}^f$ for all $m$ and $\ell = 2, 3, 4$.

**Proposition 4.6.** *Let*

$$f_\ell = u + u^2 + u^3 + \ldots + u^\ell,$$

*then we have*

*(1)* $$\mathcal{H}_{E_m}^{f_2} = \begin{cases} 1 + 2t + 2t^2 + \cdots + 2t^{m/2}, & \text{if } m \text{ is even;} \\ 1 + 2t + 2t^2 + \cdots + 2t^{(m-1)/2} + t^{(m+1)/2}, & \text{if } m \text{ is odd,} \end{cases}$$

*for* $m \geq 2$*,*

*(2)* $$\mathcal{H}_{E_m}^{f_3} = \begin{cases} 1 + 2t + 3t^2 + \cdots + 3t^{(m+1)/3}, & \text{if } m \equiv 2 \pmod 3; \\ 1 + 2t + 3t^2 + \cdots + 3t^{m/3} + t^{(m+3)/3}, & \text{if } m \equiv 0 \pmod 3; \\ 1 + 2t + 3t^2 + \cdots + 3t^{(m-1)/3} + 2t^{(m+2)/3}, & \text{if } m \equiv 1 \pmod 3, \end{cases}$$

*for* $m \geq 6$*,*

*(3)* $$\mathcal{H}_{E_m}^{f_4} = \begin{cases} 1 + 2t + 3t^2 + 4t^3 \cdots + 4t^{m/4} + 3t^{(m+4)/4}, & \text{if } m \equiv 0 \pmod 4; \\ 1 + 2t + 3t^2 + 4t^3 \cdots + 4t^{(m-1)/4} + 3t^{(m+3)/4} + t^{(m+7)/4}, & \text{if } m \equiv 1 \pmod 4; \\ 1 + 2t + 3t^2 + 4t^3 \cdots + 4t^{(m+2)/4} + t^{(m+6)/4}, & \text{if } m \equiv 2 \pmod 4; \\ 1 + 2t + 3t^2 + 4t^3 \cdots + 4t^{(m+1)/4} + 2t^{(m+5)/4}, & \text{if } m \equiv 3 \pmod 4, \end{cases}$$

*for* $m \geq 10$.

*Proof.* In the notation of the proof of Proposition 4.6, the algebra $\mathcal{C}_{E_m}^{f_\ell}$ is the truncated polynomial algebra

$$\mathbf{k}[s] = \mathbf{k}[x]/(x^{m+1}),$$

with the filtration generated by the two elements

$$a = \frac{1}{2}(Y_1 - Y_2) = s + s^3 + s^5 + \ldots + s^{2\lfloor \ell/2 \rfloor - 1}$$

and

$$b = \frac{1}{2}(Y_1 + Y_2) = s^2 + s^4 + \ldots s^{2\lfloor \ell/2 \rfloor},$$

where the term $\mathcal{C}_{E_m}^{f_\ell,n}$ is spanned by the polynomials in $a, b$ of degree $\leq n$.

Thus, the components of the associated graded algebra (4.1) are spanned by the $n+1$ degree-$n$ monomials in $a$ and $b$

$$Q_n := \mathcal{C}_{E_m}^{f_\ell,n}/\mathcal{C}_{E_m}^{f_\ell,n-1} = \langle a^n, a^{n-1}b, \ldots, ab^{n-1}, b^n \rangle. \tag{4.9}$$

Here $a$ and $b$ are viewed as elements of the quotient $Q_1 = \mathcal{C}_{E_m}^{f_\ell,1}/\mathcal{C}_{E_m}^{f_\ell,0}$, and our task is to study linear dependencies between them.

Notice that $\dim Q_1 = 2$ for all $m, \ell \geq 2$, since $a$ and $b$ are linearly independent in $Q_1$ for all such $m$ and $\ell$.

(1) In the case $\ell = 2$, we have $f = u + u^2$ and so $a = s$ and $b = s^2$. In this case we see from (4.9) that

$$\mathcal{C}_{E_m}^{f_\ell, n} = \langle 1, s, s^2, \dots, s^{2n} \rangle$$

and therefore

$$Q_n = \langle s^{2n-1}, s^{2n} \rangle.$$

Since $s^i$ and $s^{i-1}$ are linearly independent for $i \leq m$, we obtain that

$$\dim Q_n = \begin{cases} 2 & \text{if } 1 \leq n \leq m/2 \\ 1 & \text{if } n = 0 \text{ or } m \text{ is even and } n = (m+1)/2, \end{cases}$$

as required.

(2) If $\ell = 3$, i.e., $f = u + u^2 + u^3$, then $a = s + s^3 = s(1+s^2)$ and $b = s^2$. In this case we have the following description of the terms of the filtration.

**Lemma 4.7.** *The space* $\mathcal{C}_{E_m}^{f_3, n}$ *is spanned by the elements*

$$s^2, s^4, \dots, s^{2\lfloor 3n/2 \rfloor} \text{ and } s + s^3, s^3 + s^5, \dots, s^{2\lfloor (3n-1)/2 \rfloor - 1} + s^{2\lfloor (3n-1)/2 \rfloor + 1}.$$

*Proof.* By induction in $n$, we can check that

$$a^{n-2}b^2 = s^{n+2}(1+s^2)^{n-2}, a^{n-1}b = s^{n+1}(1+s^2)^{n-1}, \text{ and } a^n = s^n(1+s^2)^n$$

are the only monomials $a^i b^{n-i} \in \mathcal{C}_{E_m}^{f_3, n}$ which do not belong to $\mathcal{C}_{E_m}^{f_3, n-1}$. □

This implies that

$$Q_n = \begin{cases} \langle s, 1+s^2, s^3 \rangle \cdot s^{3n-3}, & \text{if } n \geq 2 \text{ is even} \\ \langle s+s^2, s^3, s^2+s^4 \rangle \cdot s^{3n-4}, & \text{if } n \geq 3 \text{ is odd,} \end{cases}$$

and, since the collection of these spanning elements of $Q_n$, for $n \leq m/3$, is linearly independent, the claim in part (2) follows.

(3) When $\ell = 4$, we have $f = u + u^2 + u^3 + u^4$, so the elements $a = s + s^3 = s(1+s^2)$ and $b = s^2 + s^4 = s^2(1+s^2)$ span $Q_1$.

Similarly to the previous case, we can prove by induction in $n$ the following description of the component $Q_n = \mathcal{C}_{E_m}^{f_4, n} / \mathcal{C}_{E_m}^{f_4, n-1}$.

**Lemma 4.8.** *We have*

$$Q_n = \begin{cases} 1, & \text{if } n = 0, \\ \langle 1, s \rangle \cdot s(1+s^2), & \text{if } n = 1, \\ \langle s, 1+s^2, s^3 \rangle \cdot s^3(1+s^2), & \text{if } n = 2, \\ \langle 1+s^2, s^3, s^2+s^4, s^5 \rangle \cdot s^{4n-7}(1+s^2), & \text{if } n \geq 3. \end{cases}$$

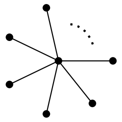
Since $s^{4n} \neq 0$ for $n \leq m/4$ and all these spanning elements have different degrees in $s$, they are linearly independent. This implies (3). □

Using a similar argument, we can show that in general, for $m \geq \ell(\ell+1)/2$, the Hilbert series of the algebra $\mathcal{C}_{E_m}^{f_\ell}$ has the form

$$\mathcal{H}_{E_m}^{f_\ell} = 1 + 2t + 3t^2 + \ldots + \ell t^{\ell-1} + \ell t^\ell + \ldots + \ell t^{\lfloor m/\ell \rfloor} + \ldots,$$

where the tail has a decreasing sequence of coefficients for which we currently do not have an explicit description.

### 4.2.2 Star graphs

Consider the $n$-vertex star graph $S_n =$  , i.e. the complete bipartite graph $K_{1,n-1}$.

Let $f = u + \sum_{i=2}^{n-1} \alpha_i u^i$ be a nondegenerate polynomial with not all $\alpha_i$ equal to 0. Label the vertices $0, 1, \ldots, n-1$ with 0 being the central vertex and orient the edges away from the center. As in the previous example, let

$$s = \sum_{i=1}^{n-1} \phi_i$$

be the sum of all edge variables. Then the generators $Y_i$ of the algebra $\mathcal{C}^f_{S_n}$ are

$$Y_0 = f(s) = s + \sum_{i=2}^{n-1} \alpha_i s^i \quad \text{and} \quad Y_i = -\phi_i, \text{ for } i = 1, \ldots, n-1.$$

Our goal is to find dimensions of the quotients

$$Q_p := \mathcal{C}^{n,p}_{S_n} / \mathcal{C}^{n,p-1}_{S_n}$$

spanned by the monomials of degree $p$ in these generators or, equivalently, in $\phi_i,\ i = 1, \ldots, n-1$, and

$$z := Y_0 - \sum_{i=0}^{n-1} Y_i = \sum_{i=2}^{n-1} \alpha_i s^i \in \mathcal{C}^{f,1}_{S_n}. \tag{4.10}$$

Therefore

$$Q_1 = \langle Y_0, Y_1, \ldots \rangle = \langle \phi_1, \ldots, \phi_{n-1}, z \rangle$$

and so $\dim Q_1 = n$, because $Y_0 \notin \langle Y_1, \ldots, Y_{n-1} \rangle$, when $f(u) \neq u$.

**Proposition 4.9.** *For $n = 4$ and $n = 5$ and a nondegenerate polynomial $f$, the Hilbert series of the algebra $\mathcal{C}^f_G$ of the star graph $G = S_n$ are as follows.*

- *If $n = 4$, then*

$$\mathcal{H}^f_G = \begin{cases} 1 + 3t + 3t^2 + t^3, & \text{if } f = u, \\ 1 + 4t + 3t^2, & \text{if } f \neq u. \end{cases} \tag{4.11}$$

- *If $n = 5$, then*

$$\mathcal{H}^f_G = \begin{cases} 1 + 4t + 6t^2 + 4^3 + t^4, & \text{if } f = u, \text{ i.e. } \alpha_2 = \alpha_3 = \alpha_4 = 0, \\ 1 + 5t + 6t^2 + 4t^3, & \text{if } \alpha_2 = \alpha_3 = 0, \alpha_4 \neq 0, \\ 1 + 5t + 7t^2 + 3t^3, & \text{if } \alpha_2 = 0, \alpha_3 \neq 0, \\ 1 + 5t + 10t^2, & \text{if } \alpha_2 \neq 0. \end{cases}$$

*Proof.* When $n = 4$, we have $\dim \mathcal{C}^f_{S_n} = 2^3 = 8$ and $\dim Q_1 = 4$. Therefore $\dim Q_2 \leq 3$. If $\alpha_3 \neq 0$, then the elements $\phi_i \phi_j,\ 1 \leq i < j \leq 3$ of $Q_2$ are linearly independent. If $\alpha_3 = 0$, then $\alpha_2 \neq 0$ and we have that the element $s^3 = \dfrac{1}{\alpha_2} \phi_1 Y_0 \in Q_2$ is not a linear combination of the degree-two elements $\phi_i \phi_j$. In both cases we see that $\dim Q_2 = 3$ and therefore the Hilbert polynomial of $\mathcal{C}^f_{S_4}$ is equal to $1 + 4t + 3t^2$ for all nondegenerate $f \neq u$.

Let $n = 5$ and $Q_1 = \langle \phi_1, \dots, \phi_4, z \rangle$. If $\alpha_2 \neq 0$, then among the 11 elements spanning $Q_2$ given by

$$\phi_i \phi_j,\ 1 \leq i < j \leq 4,\quad \phi_i z,\ 1 \leq i \leq 4 \ \text{ and } z^2,$$

there are 10 linearly independent, because the only dependence between them is given by

$$Y_0 = s + 2\alpha_2 \sum_{i<j} \phi_i \phi_j + \frac{\alpha_3}{6} \sum_i \phi_i z + \alpha_4 z^2 \in Q_1.$$

Therefore, in this case we have $\mathcal{H}^f_{S_5} = 1 + 5t + 10t^2$.

If $\alpha_2 = 0$ and $\alpha_3 \neq 0$, then $z = \alpha_3 s^3 + \alpha_4 s^4$ and therefore

$$s^4 = \frac{1}{\alpha_3} \phi_1 z = \ldots = \frac{1}{\alpha_3} \phi_4 z \in Q_2.$$

Thus, $Q_2 = \langle s^4, \phi_i \phi_j,\ 1 \leq i < j \leq 4 \rangle$ and $\dim Q_2 = 7$. Since $s^4$ belongs to $\mathcal{C}^{f,2}_{S_4}$, we see that $s^3 = \dfrac{1}{\alpha_3}(z - \alpha_4 s^4)$ is also in $\mathcal{C}^{f,2}_{S_4}$. Therefore, $Q_3$ is spanned by the four elements, $\phi_i \phi_j \phi_k,\ 1 \leq i < j < k \leq 4$, with

$$s^3 = \frac{1}{6}(\phi_1\phi_2\phi_3 + \phi_1\phi_2\phi_4 + \phi_1\phi_3\phi_4 + \phi_2\phi_3\phi_4) = 0$$

being the only linear relation between them. This shows that in this case $\dim Q_3 = 3$ and $\mathcal{H}^f_{S_5} = 1 + 5t + 7t^2 + 3t^3$.

Finally, if $\alpha_2 = \alpha_3 = 0$ and $\alpha_4 \neq 0$, then $z = \alpha_4 s^4$ and we have

$$\dim Q_2 = \dim \langle \phi_i \phi_j,\ 1 \leq i < j \leq 4 \rangle = 6$$

and

$$\dim Q_3 = \dim \langle \phi_i \phi_j \phi_k,\ 1 \leq i < j < k \leq 4 \rangle = 4.$$

Thus in this case, $\mathcal{H}^f_{S_5} = 1 + 5t + 6t^2 + 4t^3$.

□

Using the symmetry of the star graph $S_n$, we can compute the Hilbert function of the algebra $\mathcal{C}^f_{S_n}$ for polynomials $f = u + u^p$ for all $n$.

Consider the edge algebra of the graph $S_n$,

$$\Phi = \bigoplus_{k=0}^{n-1} \Phi_k,$$

where

$$\Phi_k := \langle \phi_{i_1} \phi_{i_2} \ldots \phi_{i_k} \ :\ 1 \leq i_i < i_2 < \ldots < i_k \leq n-1 \rangle$$

is the span of all square-free monomials of degree $k$. The action of the symmetric group $\Sigma_{n-1}$ on $S_n$ makes the algebra $\Phi$ and each of the components $\Phi_k$ an $\Sigma_{n-1}$-module.

The following proposition describes the structure of these representations.

**Proposition 4.10.**

*(i) If $k > (n-1)/2$, then $\Phi_k \simeq \Phi_{n-1-k}$ as $\Sigma_{n-1}$-modules.*

*(ii) If $k \le (n-1)/2$, then*

$$\Phi_k \simeq \bigoplus_{p=0}^{k} W_p, \tag{4.12}$$

*where $W_p$ is a simple $\Sigma_{n-1}$-module of dimension*

$$\dim W_p = \binom{n-1}{p} - \binom{n-1}{p-1} = \binom{n-1}{p} \cdot \frac{n-2p}{n-p}. \tag{4.13}$$

*Proof.* Let us first recall some facts about permutation representations. For a finite group $\Gamma$, let $R(\Gamma)$ be its character ring (the Grothendieck ring of complex finite dimensional representations). For a finite $\Gamma$-set $X$, denote by $V_X \in R(\Gamma)$ the corresponding permutation representation. The paring

$$\langle V, W\rangle := \dim \mathrm{Hom}_\Gamma(V, W) = (\mathrm{Hom}_{\mathbb{C}}(V, W))^\Gamma$$

defines a non-degenerate inner product on $R(\Gamma)$ for which the irreducible representations form an orthonormal basis. Every permutation representation $V_X$ possesses a $\Gamma$-invariant inner product and, therefore, is self-dual, $V_X \simeq V_X^*$. Since $\dim_{\mathbb{C}}(V_X)^\Gamma = |X/\Gamma|$, i.e. the number of $\Gamma$-orbits in $X$, we have that, for any two $\Gamma$-sets $X$ and $Y$,

$$\begin{aligned}\langle V_X, V_Y\rangle &= \dim \mathrm{Hom}_{\mathbb{C}}(V_X, V_Y)^\Gamma = \dim(V_X^* \otimes_{\mathbb{C}} V_Y)^\Gamma = \dim(V_X \otimes_{\mathbb{C}} V_Y)^\Gamma \\ &= \dim(V_{X\times Y})^\Gamma = |(X\times Y)/\Gamma|.\end{aligned}$$

Returning to the $\Sigma_{n-1}$-representation on the edge algebra $\Phi$, we observe that the $\Phi_k$ is the permutation representation corresponding to the action of $\Sigma_{n-1}$ on the set of $k$-element subsets of $[n-1] = \{1, 2, \ldots, n-1\}$. Passing from $A \in X_k$ to the complementary subset $A^c = [n-1] - A \in X_{n-1-k}$ establishes an isomorphism between $\Sigma_{n-1}$-actions on $X_k$ and $X_{n-1-k}$. This gives part (i).

The $\Sigma_{n-1}$-orbits on the set of pairs of subsets $(A, B) \in X_\ell \times X_k$, for $0 \le \ell \le k \le \lfloor (n-1)/2 \rfloor$, are uniquely determined by the cardinality of the intersection $|A \cap B| \in \{0, 1, 2, \ldots, \ell\}$. Therefore, in this case we have

$$\langle \Phi_\ell, \Phi_k\rangle = \langle V_{X_\ell}, V_{X_k}\rangle = |X_\ell \times X_k|/\Sigma_{n-1} = \ell + 1, \tag{4.14}$$

which allows to prove part (ii) of the Proposition by induction in $k$.

Indeed, if $k = 0$, then $\Phi_0 = W_0$ is a one-dimensional trivial module.

Assume that the decomposition (4.12) holds for all $p < k \le \lfloor (n-1)/2 \rfloor$, i.e. that

$$W_p = \Phi_p - \Phi_{p-1} \in R(\Sigma_{n-1})$$

is a genuine (i.e. not virtual) irreducible representation for $p \le k-1$. Then, using (4.14), we have

$$\begin{aligned}\langle W_p, \Phi_k\rangle &= \langle \Phi_p - \Phi_{p-1}, \Phi_p\rangle = \langle \Phi_p, \Phi_k\rangle - \langle \Phi_{p-1}, \Phi_k\rangle \\ &= (p+1) - (p-1+1) = 1.\end{aligned}$$

Therefore, for each $p = 1, ..., k-1$, the irreducible module $W_p$, appears in $\Phi_k$ with multiplicity 1. This implies that $\Phi_{k-1} \simeq \bigoplus_{p=0}^{k-1} W_p$ is isomorphic to a submodule of $\Phi_k$. Thus we have a decomposition

$$\Phi_k \simeq \Phi_{k-1} \oplus W_k$$

for some representation $W_k$ of $\Sigma_{n-1}$. Since

$$\dim W_k = \dim \Phi_k - \dim \Phi_{k-1} = \binom{n-1}{k} - \binom{n-1}{k-1},$$

this module has the required dimension. To show that it is irreducible, we will use (4.14) to check that $\langle W_k, W_k\rangle = 1$. Indeed,

$$\begin{aligned}\langle W_k, W_k\rangle &= \langle \Phi_k - \Phi_{k-1}, \Phi_k - \Phi_{k-1}\rangle = \langle \Phi_k, \Phi_k\rangle - 2\langle \Phi_{k-1}, \Phi_k\rangle + \langle \Phi_{k-1}, \Phi_{k-1}\rangle \\ &= (k+1) - 2k + k = 1.\end{aligned}$$

□

In addition to an $\Sigma_{n-1}$-module structure, the space $\Phi$ has a less obvious action of the Lie algebra $\mathfrak{sl}_2$.

**Proposition 4.11.** *Consider the degree* 1 *element*

$$s = \sum_{i=1}^{n-1} \phi_i \in \Phi_1$$

*and let*

$$E : \Phi \to \Phi,\ x \mapsto sx \tag{4.15}$$

*be the operator of multiplication by* $s$.

*Let* $\partial_i : \Phi \to \Phi$ *be the partial derivative with respect to* $\phi_i$ *and consider the degree* $-1$ *map*

$$F := \sum_{i=1}^{n-1} \partial_i : \Phi \to \Phi.$$

*(i) The restriction of the commutator* $H = [E, F]$ *to the subspace* $\Phi_k$ *is the scalar operator*

$$[E, F]_{|\Phi_k} = (2k - n + 1)\mathrm{Id}_{\Phi_k}. \tag{4.16}$$

*(ii) We have*

$$[H, E] = 2E \text{ and } [H, F] = -2F,$$

*and therefore the triple* $(E, H, F)$ *defines a representation of the Lie algebra* $\mathfrak{sl}_2$ *on* $\Phi$.

*(iii) As an* $\mathfrak{sl}_2$*-module,* $\Phi$ *decomposes as*

$$\Phi = \sum_{p=0}^{\lfloor (n-1)/2 \rfloor} V_{n-1-2p}^{\oplus m_p}, \tag{4.17}$$

*where* $V_k$ *is the* $(k+1)$*-dimensional irreducible* $\mathfrak{sl}_2$*-representation and*

$$m_k = \binom{n-1}{k} \cdot \frac{n-2k}{n-k} = \dim W_k.$$

*(iv) We have the following decomposition of* $\Phi$ *as a* $\Sigma_{n-1} \times \mathfrak{sl}_2$*-module:*

$$\Phi = \bigoplus_{p=0}^{\lfloor (n-1)/2 \rfloor} W_p \otimes V_{n-1-2p}.$$

*Proof.* The edge algebra $\Phi$ has a basis consisting of the monomials

$$\phi_A := \phi_{i_1}\phi_{i_2}\dots\phi_{i_k} \in \Phi_k$$

indexed by subsets $A = \{i_1, \dots, i_k\}$ of $[n-1]$.

Let $e_i : \Phi \to \Phi$ be the operator of multiplication by the element $\phi_i \in \Phi_1$. Computing the action of $e_i$ and $\partial_i$ on monomials $\phi_A$,

$$e_i\phi_A = \begin{cases} \phi_{A \sqcup \{i\}}, & \text{if } i \notin A, \\ 0, & \text{if } i \in A, \end{cases} \quad \text{and} \quad \partial_i \phi_A = \begin{cases} \phi_{A - \{i\}}, & \text{if } i \in A, \\ 0, & \text{if } i \notin A, \end{cases}$$

we find that the commutator of $e_i$ and $\partial_j$ is

$$[e_i, \partial_j]\phi_A = \begin{cases} 0, & \text{if } i \neq j, \\ \phi_A, & \text{if } i = j \text{ and } i \in A, \\ -\phi_A, & \text{if } i = j \text{ and } i \notin A. \end{cases}$$

Since $E = \sum_{i=1}^{n-1} e_i$ and $F = \sum_{i=1}^{n-1} \partial_i$, this gives the relation (4.16) in part (i) from which (ii) immediately follows.

Thus $\Phi$ is a $\mathfrak{sl}_2$-module, and the component $\Phi_k$ consists of all vectors of weight $2k-n+1$. For $0 \leq p \leq \min(k, n-1-k)$, denote by $W_p^{(k)} \subset \Phi_k$ the $\Sigma_{n-1}$-submodule isomorphic to the irreducible summand $W_p$ in the decomposition (4.12).

The operators $E = \sum e_i$ and $F = \sum \partial_i$ commute with the $\Sigma_{n-1}$-action on $\Phi$. Therefore, they send the subspace $W_p^{(k)}$ to $W_p^{(k+1)}$ and $W_p^{(k-1)}$, respectively. Thus we obtain a decomposition of $\Phi$ into $\mathfrak{sl}_2$-submodules

$$\Phi = \bigoplus_{p=0}^{\lfloor (n-1)/2 \rfloor} U_p, \text{ where } U_p := \bigoplus_{k=p}^{n-1-p} W_p^{(k)}.$$

Since there is no $\Sigma_{n-1}$-submodule isomorphic to $W_p$ in $\Phi_k$ for $k \geq n-p$, we see that $E$ vanishes on the subspace $W_p^{(n-1-p)} \subset \Phi_{n-1-p}$. Therefore every non-zero element of $v \in W_p^{(n-1-p)}$ is a highest weight vector of weight $n-1-2p$ for $\mathfrak{sl}_2$. Thus each of them generates an irreducible $\mathfrak{sl}_2$-module

$$\mathfrak{sl}_2 \cdot v = \langle v, Fv, F^2 v, \ldots, F^{n-1-2p} v \rangle \simeq V_{n-1-2p}$$

of dimension $n-2p$. This implies parts (iii) and (iv) of the Proposition. $\square$

The following diagram shows the structure of the edge algebra $\Phi$ as a $\Sigma_{n-1} \times \mathfrak{sl}_2$-module with the vertical arrows describing the action of $E \in \mathfrak{sl}_2$ given by multiplication by $s$.

$$\begin{tikzcd}
\mathbf{\Phi} & = & \mathbf{U_0} & \oplus & \mathbf{U_1} & \oplus & \mathbf{U_2} & \oplus & \ldots & \oplus & \mathbf{U}_{\lfloor n/2 \rfloor} \\
\mathbf{\Phi_0} & = & W_0^{(0)} \arrow[d, "s"'] & & & & & & & & \\
\mathbf{\Phi_1} & = & W_0^{(1)} \arrow[d, "s"'] & \oplus & W_1^{(0)} \arrow[d, "s"] & & & & & & \\
\mathbf{\Phi_2} & = & W_0^{(2)} \arrow[d, "s"'] & \oplus & W_1^{(1)} \arrow[d, "s"] & \oplus & W_2^{(0)} \arrow[d, "s"] & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\mathbf{\Phi}_{\lfloor \frac{\mathbf{n-1}}{\mathbf{2}} \rfloor} & = & W_0^{(\lfloor \frac{n-1}{2} \rfloor)} \arrow[d, "s"'] & \oplus & W_1^{(\lfloor \frac{n-3}{2} \rfloor)} \arrow[d, "s"] & \oplus & W_2^{(\lfloor \frac{n-5}{2} \rfloor)} \arrow[d, "s"] & \oplus & \cdots \arrow[d, "s"'] & \oplus & W_{\lfloor \frac{n-1}{2} \rfloor}^{(0)} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\mathbf{\Phi_{n-3}} & = & W_0^{(n-3)} \arrow[d, "s"'] & \oplus & W_1^{(n-2)} \arrow[d, "s"] & \oplus & W_2^{(n-5)} & & & & \\
\mathbf{\Phi_{n-2}} & = & W_0^{(n-2)} \arrow[d, "s"'] & \oplus & W_1^{(n-3)} & & & & & & \\
\mathbf{\Phi_{n-1}} & = & W_0^{(n-1)} & & & & & & & &
\end{tikzcd}$$

The above description of the $\Sigma_{n-1} \times \mathfrak{sl}_2$-module structure on $\Phi$ leads to the following way of computing the Hilbert functions of the algebra $\mathcal{C}_{S_n}^{f}$ for all $f$.

**Proposition 4.12.** *Let $f \in \mathbf{k}[u]$ be a nondegenerate polynomial. Consider the algebra $P_m := \mathbf{k}[x]/(x^{m+1})$ of truncated polynomials with the increasing filtration*

$$\mathbf{k} = P_m^{(0)} \subset P_m^{(1)} \subset \dots$$

*generated by $x$ and $f(x)$, i.e.*

$$P_m^{(r)} := \langle x^i f^j(x) : i + j \leq r \rangle$$

*and let*

$$h_m^f(t) = \sum_{i \geq 0} (\dim P_m^{(r)}/P_m^{(r-1)}) t^i$$

*be the Hilbert series of this filtered algebra.*

*Then the Hilbert series of the algebra $\mathcal{C}_{S_n}^f$ for the star graph $S_n$ is*

$$\mathcal{H}_{S_n}^f(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (\dim W_i) t^i h_{n-1-2i}^f(t). \tag{4.18}$$

*Proof.* Since the operator $E$ of multiplication by $s$ commutes with the $\Sigma_{n-1}$-action on the edge algebra $\Phi_{S_n}$, the Hilbert series of the algebra $\mathcal{C}_{S_n}^f$ is the sum of the Hilbert series of the isotypical components $U_m$ with the filtration starting in degree $m$ and given by the action of $s$ and $f(s)$. By choosing a lowest weight vector in the irreducible $\mathfrak{sl}_2$-module $V_m$, we can identify $V_m$ with $P_m$ so that the action of $E \in \mathfrak{sl}_2$ becomes multiplication by $x \in P_m$. This identifies $U_m$ and $W_m \otimes V_m$ with the filtration starting at the level $m$, and the result follows. □

For some polynomials $f$, such as $f = u + u^p$, the Hilbert series $h_m^f$ can be computed explicitly. As a concrete example, we consider the case of $f = u + u^2$.

**Lemma 4.13.** *For $f = u + u^2$ the Hilbert series of the truncated polynomial algebra $P_m$ is equal to*

$$h_m^f(t) = \begin{cases} 1 + 2t + 2t^2 + \ldots + 2t^{m/2} & \textit{if } m \textit{ is even;} \\ 1 + 2t + 2t^2 + \ldots + 2t^{(m-1)/2} + t^{(m+1)/2} & \textit{if } m \textit{ is odd.} \end{cases}$$

*Proof.* Indeed, for $f = u + u^2$, the term $P_m^{(r)}$ of the filtration is spanned by monomials of degree $\leq r$ in $x$ and $x^2 = f(x) - x$. Thus $P_m^{(r)} = \langle 1, x, \ldots, x^{2r} \rangle$ and $P_m^{(r)}/P_m^{(r-1)} = \langle x^{2r-1}, x^{2r} \rangle$. Since $P_m = \langle 1, x, x^2, \ldots, x^m \rangle$, we have

$$\dim P_m^{(r)}/P_m^{(r-1)} = \begin{cases} 2 & \text{if } r \leq m/2, \\ 1 & \text{if } m \text{ is odd and } r = (m-1)/2, \\ 0, & \text{if } r > m/2. \end{cases}$$

□

Now Proposition 4.12 together with the above lemma and the dimension formula (4.13) give the following result.

**Proposition 4.14.** *For $f = u + u^2$, the Hilbert series of the algebra $\mathcal{C}_{S_n}^f$ is*

$$\mathcal{H}_{S_n}^{u+u^2} = \begin{cases} \displaystyle\sum_{k=0}^{(n-1)/2} \binom{n}{k} t^k, & \textit{if } n \textit{ is odd,} \\ \displaystyle\sum_{k=0}^{n/2-1} \binom{n}{k} t^k + \binom{n-1}{(n-1)/2} t^{n/2}, & \textit{if } n \textit{ is even.} \end{cases}$$

□

## 4.3 Generators and relations

**Proposition 4.15.** *Let $A$ be a finite dimensional local algebra over $\mathbf{k}$ with maximal ideal $\mathfrak{m}$ and with a set of algebra generators $x_1, \dots, x_n \in \mathfrak{m}$. Let $f \in \mathbf{k}[[u]]$ be a nondegenerate series with $f(0) = 0$. Then*

*(i) the map $f\colon A \to A,\ a \mapsto f(a)$ is well-defined and invertible;*

*(ii) the elements $y_1 = f(x_1), \dots, y_n = f(x_n)$ generate $A$;*

*(iii) for $L \in \mathbf{k}[t_1, \dots, t_n]$, the relation $L(x_1, \dots, x_n) = 0$ holds in $A$ if and only if the relation $L(f^{-1}(y_1), \dots, f^{-1}(y_n)) = 0$ holds in $A$ with respect to the generators $y_1, \dots, y_n$.*

*Proof.* To prove (i), notice that, since $A$ is an Artinian algebra, any $a \in A$ is nilpotent, which implies that $f(a)$ is a finite sum. Therefore $f(a)$ is well-defined for any $a$. The inverse of $f$ has been already constructed in the proof of Proposition 3.6.

Now, (ii) is immediate, since $f$ is invertible as a map of $A$ and therefore $x_i = f^{-1}(y_i)$ for every $i$.

The first part of (iii), claiming that the relation

$$L(f^{-1}(y_1), \dots, f^{-1}(y_N)) = 0$$

holds in $A$, is obvious. Conversely, assume that a relation $R(y_1, \dots, y_n) = 0$ holds in $A$. Then the equation $R(f(x_1), \dots, f(x_n)) = 0$ gives a relation in $A$ in terms of the original generating set $x_1, \dots, x_n$, which then satisfies the claim because of invertibility of $f$.

□

**Corollary 4.16.** *For a graph $G = (V, E)$ and a non-degenerate $f \in \mathbf{k}[[u]]$, consider the ideal*

$$I_G^f := \left( y_v^{\deg v + 1}, \Big( \sum_{v \in I} f^{-1}(y_v) \Big)^{D_I + 1} \right)_{v \in V,\ I \subset V,\ |I| \geq 2} \tag{4.19}$$

*in $\mathbf{k}[y_v : v \in V]$, where $D_I$, as before, is the number of edges having one end in $I$ and the other in the complementary subset $V - I$.*

*Then $I_G^f$ the ideal of relations for $\mathcal{C}_G^f$, i.e. it is the kernel of the epimorphism $\mathbf{k}[y_v \colon v \in V] \to \mathcal{C}_G^f$ . Therefore*

$$\mathcal{C}_G^f \simeq \mathbf{k}[y_v \colon v \in V] / I_G^f \ .$$

*Proof.* As we have already seen, $X_v^{\deg v + 1} = 0$ which implies that $Y_v^{\deg v + 1} = (f(X_v))^{\deg v + 1} = 0$ by plugging directly into the formula for $f(u)$. By Proposition 4.15 and Theorem 2.10, $\big(\sum_{v \in I} f^{-1}(Y_v)\big)^{D_I + 1}$ are still relations in $\mathcal{C}_G^f$. Thus by mapping $y_i \mapsto Y_i$ we see that $\mathcal{C}_G^f$ is a homomorphic image of $\mathbf{k}[y_v : v \in V]/I_G^f$. Due to invertibility of $f(u)$ and Theorem 2.10, there are no additional relations (i.e., the kernel of this homomorphism is zero).

□

Several concrete examples of relations and their applications can be found in § 5.3.

Remark 4.17. The relation $\sum\limits_{v \in V} f^{-1}(Y_v) = 0$ corresponding to $I = V$ in Corollary 4.16 can be used to remove half of the generators of the ideal $I_G^f$. Namely, for a subset $I \subset V$ with $1 < |I| < |V|$, consider the complementary subset $I^c = V - I$. If $|I| < |I^c|$, then keep the generator $\big(\sum_{v \in I} f^{-1}(Y_v)\big)^{D_I + 1}$ corresponding to $I$ and remove the one corresponding to $I^c$. If $|I^c| > |I|$, then keep the generator corresponding to $I^c$ and remove the one corresponding to $I$. Finally, if $|I| = |I^c|$ (which can happen only if $|V|$ is even), then keep any of these two generators and remove the other one. The generator corresponding to $I = V$ should not be removed.

# 5 Parameter space and its stratification

## 5.1 Parameter space

From Remark 3.3 we know that every deformed zonotopal algebra of a graph $G$ is isomorphic as filtered algebra to an algebra $\mathcal{C}_G^f$, where $f \in \mathbf{k}[u]$ is a polynomial of degree at most $\Delta(G)$ such that $f(0) = 0$. In the nondegenerate case, we can further restrict $f$ by requiring $f'(0) = 1$.

**Definition 5.1.** Given a graph $G$, we will call the affine space

$$\mathcal{A}_G = \{f \in \mathbf{k}[u] : f(0) = 0, f'(0) = 1, \deg f \leq \Delta(G)\} \tag{5.1}$$

*the space of parameters of deformed zonotopal algebras* of $G$.

**Proposition 5.2** ([6, Proposition 3]; see also Theorem 5.7.)**.** *There exists a non-empty Zariski open subset $U \subset \mathcal{A}_G$ such that all Hilbert functions $\mathcal{H}_G^f$ with $f \in U$ are equal to each other and are greater, in lexicographic order on sequences, than any Hilbert function $\mathcal{H}_G^{f'}$ for $f' \in \mathcal{A}_G - U$.*

Let $\mathcal{A}_G$ be the affine space (5.1) of parameters of algebras $\mathcal{C}_G^f$ for a graph $G$. Our goal is to study the stratification of the space $\mathcal{A}_G$ according to the Hilbert functions $\mathcal{H}_G^f$ of $\mathcal{C}_G^f$.

Since the generators $Y_v,\ v \in V$, of the algebra $\mathcal{C}_G^f$ are nilpotent, the corresponding Hilbert function $\mathcal{H}_G^f$ has finitely many non-zero terms. We denote by $\overline{\mathcal{H}}_G^f$ the finite sequence obtained by removing all zero terms of $\mathcal{H}_G^f$.

**Definition 5.3.** Given a graph $G$ and a sequence $M = (m_0, m_1, m_2, \ldots, m_n)$ of positive integers, the *associated Hilbert stratum* is the subset $S_G^M$ of the parameter space $\mathcal{A}_G$ consisting of all $f$ such that $\overline{\mathcal{H}}_G^f = M$.

We will show that each $S_G^M$ is a constructible algebraic set (see Section 5.2). (Observe that $S_G^M$ might be empty.) First let us consider some examples.

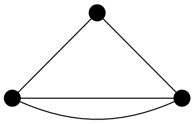
Example 5.4. Let us consider the stratification of the parameter space $\mathcal{A}_G$ for some graphs.

1. If $G$ is a two-vertex graph with two edges then the parameter space is an affine line $\mathcal{A}_G = \{u + \alpha u^2 : \alpha \in \mathbf{k}\}$ and there are only two strata
   - $\alpha = 0$ with the Hilbert series $1 + t + t^2$,
   - $\alpha \neq 0$ with the Hilbert series $1 + 2q$ (all algebras with $\alpha \neq 0$ have the same Hilbert series due to the action (5.3)).
2. For a two-vertex graph with three edges, the parameter space is an affine plane $\mathcal{A}_G = \{u + \alpha u^2 + \beta u^3 : \alpha, \beta \in \mathbf{k}\}$. There are two strata:
   - $\alpha = 0, \beta \in \mathbf{k}$ with the Hilbert series $1 + t + t^2 + t^3$,
   - $\alpha \neq 0, \beta \in \mathbf{k}$ with the Hilbert series $1 + 2t + t^2$.
3. If $G$ is a two-vertex graph with four edges, then the parameter space $\mathcal{A}_G = \{u + \alpha u^2 + \beta u^3 + \gamma u^4 : \alpha, \beta, \gamma \in \mathbf{k}\}$ is three-dimensional with the three strata
   - $\alpha = \gamma = 0, \beta \in \mathbf{k}$ with the Hilbert series $1 + t + t^2 + t^3 + t^4$,
   - $\alpha = 0, \gamma \neq 0, \beta \in \mathbf{k}$ with the Hilbert series $1 + 2t + t^2 + t^3$,
   - $\alpha \neq 0, \beta, \gamma \in \mathbf{k}$ with the Hilbert series $1 + 2t + 2t^2$.
4. If $G = D_4 = S_4$ is the 4-vertex star, then the parameter space is the affine plane $\mathcal{A}_G = \{u + \alpha u^2 + \beta u^3 : \alpha, \beta \in \mathbf{k}\}$ with two strata
   - $\alpha = \beta = 0$ with the Hilbert series $1 + 3t + 3t^2 + t^3$,

- $\alpha \neq 0$ or $\beta \neq 0$ with the Hilbert series $1+4t+3t^2$.

5. If $G = S_5$ is the 5-vertex star, then by Proposition 4.9, we have a three-dimensional parameter space $\mathcal{A}_G = \{u + \alpha u^2 + \beta u^3 + \gamma u^4 : \alpha, \beta, \gamma \in \mathbf{k}\}$ with four strata
   - $\alpha = \beta = \gamma = 0$, with the Hilbert series $1+4t+6t^2+4t^3+t^4$,
   - $\alpha = \beta = 0, \gamma \neq 0$, with the Hilbert series $1+5t+6t^2+4t^3$,
   - $\alpha = 0, \beta \neq 0, \gamma \in \mathbf{k}$, with the Hilbert series $1+5t+7t^2+3t^3$,
   - $\alpha \neq 0, \beta, \gamma \in \mathbf{k}$ with the Hilbert series $1+5t+10t^2$.

6. Let $G$ be the three-cycle graph with one extra edge

$G =$  . (5.2)

Then $\mathcal{A}_G$ is the affine plane $\mathcal{A}_G = \{u + \alpha u^2 + \beta u^3 : \alpha, \beta \in \mathbf{k}\}$ with the strata

- a point $\{u\}$, with the Hilbert series $1+2t+3t^2+3t^3+t^4$,
- the union of two lines and a parabola

$$\{u + \alpha u^2 : \alpha \neq 0\} \cup \{u + \beta u^3 : \beta \neq 0\} \cup \{u + \alpha u^2 + \frac{4}{3}\alpha^2 u^3 : \alpha \neq 0\},$$

  with the Hilbert series $1+3t+4t^2+2t^3$,
- a two-dimensional stratum

$$\{u + \alpha u^2 + \beta u^3 : 3\beta \neq 4\alpha^2, \alpha \neq 0, \beta \neq 0\},$$

  with the Hilbert series $1+3t+5t^2+t^3$.

For the proofs for the first two graphs, see Section 4.2.1, for $D_4$ and $S_5$, see Section 4.2.2, and for three-cycle with an extra edge, see Proposition 5.12.

## 5.2 Semicontinuity

First recall the following notion.

**Definition 5.5.** Let $X$ be a topological space and let $(\Lambda, \prec)$ be a partially ordered set. We say that a function $g\colon X \to \Lambda$ is *upper semicontinuous* if for every $\lambda \in \Lambda$, the set

$$g^{-1}(\prec \lambda) := \{x \in X \mid g(x) \prec \lambda\}$$

is open. *Lower semicontinuity* is defined in the opposite way.

**Lemma 5.6.** *Let $A$ be a commutative ring and $M$ be a matrix with entries in $A$. For a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, let $M(\mathfrak{p})$ be the matrix obtained from $M$ by replacing the entries by their images in $k(\mathfrak{p})$. Then the real-valued function $\mathfrak{p} \mapsto \operatorname{rk} M(\mathfrak{p})$ (respectively, $\mathfrak{p} \mapsto \dim\ker M(\mathfrak{p})$) is constructive and lower (resp., upper) semicontinuous on* $\operatorname{Spec} A$.

*Proof.* We use the fact that non-vanishing of a minor is an open condition. Namely, if $B$ is a square matrix then $\det B(\mathfrak{p}) \neq 0$ if and only if $\det B \notin \mathfrak{p}$. Hence, for any $\mathfrak{p}$ such that $\operatorname{rk} M(\mathfrak{p}) \geq r$, there is an open neighborhood where the same condition holds.

$\square$

**Theorem 5.7.** *Let $A$ be a Noetherian commutative ring and let*

$$R = A[X_1, \dots, X_p]/I$$

*be an $A$-algebra finitely generated as an $A$-module. Denote by $x_i$ the image of $X_i$ in $R$. We endow $R$ with the filtration by $A$-submodules generated by monomials in $x_i$ of degree at most $m$:*

$$F_m = \langle x_1^{\alpha_1} \cdots x_p^{\alpha_p} \mid \alpha_1 + \cdots + \alpha_p \leq m \rangle, \ m \geq 0 \ .$$

*For a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, denote by $k(\mathfrak{p})$ its residue field and consider the filtration of*

$$R(\mathfrak{p}) = R \otimes_A k(\mathfrak{p})$$

*by the images $F_m(\mathfrak{p})$ of the projection $F_m \otimes_A k(\mathfrak{p}) \to R(\mathfrak{p})$, for $m \geq 0$.*

*Then*

*(i) the function $\mathfrak{p} \mapsto \dim_{k(\mathfrak{p})} R(\mathfrak{p})$ is upper semicontinuous on $\operatorname{Spec} A$;*

*(ii) the function*

$$H_m \colon \mathfrak{p} \mapsto \dim_{k(\mathfrak{p})} F_m(\mathfrak{p})$$

*is lower semicontinuous on $\operatorname{Spec} A$;*

*(iii) for $\mathfrak{p} \subset \mathfrak{q}$, the equality $H_m(\mathfrak{p}) = H_m(\mathfrak{q})$ holds for all $m$ if and only if $F_m \otimes_A (A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q})$ is a free $A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}$-module for all $m$.*

*Proof.* Since $R$ is a finite $A$-module and $A$ is Noetherian, we can find a finite collection of monomials $Y_1, \dots, Y_n \in A[X_1, \dots, X_N]$ whose images generate $R$ as an $A$-module and obtain a presentation of $R$ as an $A$-module

$$A^{\oplus \ell} \xrightarrow{M} A^{\oplus n} \to R \to 0,$$

where $M$ is a matrix with entries in $A$. Then it is clear that

$$\dim_{k(\mathfrak{p})} R \otimes_A k(\mathfrak{p}) = n - \operatorname{rk} M(\mathfrak{p}),$$

so the first claim follows from Lemma 5.6.

For the second claim, let

$$V_m = \bigoplus_k \langle Y_{i_k} \rangle$$

be the free submodule of

$$A^{\oplus n} = \bigoplus_{i=1}^n \langle Y_i \rangle$$

corresponding to the monomials $Y_{i_k}$ among $Y_1, \dots, Y_n$ with $\deg Y_{i_k} \leq m$. Then $F_m(\mathfrak{p})$ is the image of the induced projection $V_m(\mathfrak{p}) \to R(\mathfrak{p})$. and we have

$$H_m(\mathfrak{p}) = \dim_{k(\mathfrak{p})} F_m(\mathfrak{p}) = \operatorname{rk}_A V_m - \dim_{k(\mathfrak{p})}(\operatorname{Im} M(\mathfrak{p}) \cap V_m(\mathfrak{p})).$$

Let us show how to represent the intersection of linear subspaces $\operatorname{Im} M(\mathfrak{p})$ and $V_m(\mathfrak{p})$ of $k(\mathfrak{p})^{\oplus n}$ as the kernel of a suitable matrix. Let $X$ be a matrix whose columns are formed by coordinates of vectors from some basis of $V_m$, and let

$$N = M \oplus (-X) = [M| - X]$$

be the block matrix formed by placing $M$ and $-X$ next to each other. Then

$$\operatorname{Im} M(\mathfrak{p}) \cap V_m(\mathfrak{p})) = \ker N$$

and therefore

$$\dim_{k(\mathfrak{p})} \operatorname{Im} M(\mathfrak{p}) = \dim_{k(\mathfrak{p})} \ker N$$

is an upper semicontinuous function by Lemma 5.6.

Now we prove the third assertion. For the ease of notation let us replace $A$ by $A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}$ so that we will assume now that $A$ is a local domain with the maximal ideal $\mathfrak{m}$, $\mathfrak{p} = 0$, and $\mathfrak{q} = \mathfrak{m}$. Let $L$ and $k$ be the field of fractions and the residue field of $A$. By the assumption,

$$\dim_L(F_m \otimes_A L) = \dim_k(F_m \otimes_A k)$$

for all $m$. It remains to recall that for a finitely generated $A$-module $S$, the equality $\dim_k(S \otimes_A k) = \dim_L(S \otimes_A L)$ holds if and only if $S$ is free.

□

**Corollary 5.8.** *Let $A$ be a commutative ring and let $R$ be an $A$-algebra with a set of generators $x_1, \ldots, x_n \in R$ such that $R$ is generated as an $A$-module by monomials in $x_i$ of degree at most $D$.*

*In the notation of Theorem 5.7, define the function*

$$H\colon \operatorname{Spec} A \to \Lambda,\ \mathfrak{p} \mapsto (H_0(\mathfrak{p}), H_1(\mathfrak{p}), \ldots, H_D(\mathfrak{p}))$$

*where $\Lambda = \mathbb{Z}^{D+1}$ with the lexicographic order $\preceq_{lex}$.*

*Then $H$ is lower semicontinuous on $\operatorname{Spec} A$. Moreover, for any $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_D) \in \Lambda$, the set*

$$H^{-1}(\lambda) = \{\mathfrak{p} \in \operatorname{Spec} A \mid H(\mathfrak{p}) = \lambda\}$$

*is constructible and its closure is contained in the closed set $H^{-1}(\preceq_{lex} \lambda)$.*

*Proof.* Since functions $H_k$ are lower semicontinuous and integer-valued, the sets $H_k^{-1}(\geq \lambda_k) = H_k^{-1}(> \lambda_k - 1)$ are open and the sets $H_i^{-1}(< \lambda_i + 1)$ are closed. We also have

$$H^{-1}(\lambda) = \left(\bigcap_{i=0}^{D} H_i^{-1}(\geq \lambda_i)\right) \bigcap \left(\bigcap_{i=0}^{D} H_i^{-1}(< \lambda_i + 1)\right).$$

Thus, the set $H^{-1}(\lambda)$ is an intersection of an open and a closed sets, and so it is constructible.

For the second claim, it is sufficient to show that $H^{-1}(\preceq_{lex} \lambda)$ is closed. By definition of the order $\preceq_{lex}$ we have

$$\begin{aligned} H^{-1}(\preceq_{lex} \lambda) = & H_0^{-1}(< \lambda_0) \bigcup H_0^{-1}(\leq \lambda_0) \cap H_1^{-1}(< \lambda_1) \bigcup \cdots \\ & \bigcup H_0^{-1}(\leq \lambda_0) \cap \cdots \cap H_{D-1}^{-1}(\leq \lambda_{D-1}) \cap H_D^{-1}(< \lambda_D). \end{aligned}$$

So $H^{-1}(\preceq_{lex} \lambda)$ is a union of closed sets.

□

**Corollary 5.9.** *For a graph $G = (V, E)$ and a sequence $M = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, where $d = \Delta(G)$ is the maximum degree of a vertex in $G$, the Hilbert stratum $S_G^M$ of the parameter set $\mathcal{A}_G$ (5.1) is constructible and its closure is contained in the union $S_G^{\preceq_{lex} M} := \bigcup_{M' \preceq_{lex} M} S_G^{M'}$.*

*Proof.* Introduce formal parameters $t_2, \ldots, t_d$ and consider the polynomial $f = u + t_2 u^2 + \cdots + t_d u^d \in \mathbf{k}[u, t_2, \ldots, t_d]$. Let $R_G$ be the quotient algebra $R_G = \mathbf{k}[y_v, t_i : v \in V, i \in \{2, \ldots, d\}]/I_G$ where $I$ is the ideal generated by relations (4.19). Notice that since $y_v^{d+1} \in I$, an element $f^{-1}$ satisfying $f^{-1}(f(u)) = u$ exists.

Since $R$ is a finitely generated module over $A = \mathbf{k}[t_2, \ldots, t_d]$, Corollary 5.8 shows that the level sets of the Hilbert function $\mathcal{H}_G^k(\mathfrak{p}) = H_k(\mathfrak{p}) - H_{k-1}(\mathfrak{p})$ are constructible. Also, notice that for sequences $(m_0, \ldots, m_d)$ and $(m_0', \ldots, m_d') \in \mathbb{Z}^d$ the inequality $(m_0, \ldots, m_d) \preceq_{lex} (m_0', \ldots, m_d')$ is equivalent to $(m_0, m_1 - m_0, \ldots, m_d - m_{d-1}) \preceq_{lex} (m_0', m_1' - m_0', \ldots, m_d' - m_{d-1}')$. Therefore the inclusion $\overline{S_G^M} \subset S_G^{\preceq_{lex} M}$ follows from the second claim of Corollary 5.8.

□

## 5.3 $\mathbf{k}^*$-action on the parameter space

**Lemma 5.10.** *For a graph $G$, a polynomial*

$$f = u + \sum_{i=2}^{\Delta(G)} \alpha_i u^i \in \mathcal{A}_G,$$

*and a nonzero element $\varepsilon \in \mathbf{k}$, let $f_\varepsilon$ be the polynomial*

$$f_\varepsilon(u) = \frac{1}{\varepsilon} f(\varepsilon u) \in \mathcal{A}_G. \tag{5.3}$$

*This defines a $\mathbf{k}^*$-action on the parameter space $\mathcal{A}_G$ which preserves the Hilbert stratification. This action is free on $\mathcal{A}_G \setminus \{u\}$ and the point $u \in \mathcal{A}_G$ belongs to the closure of every Hilbert stratum.*

*In addition, this action is quasihomogeneous in the affine coordinates $\alpha_i$ on $\mathcal{A}_G$.*

*Proof.* We substitute $Y_v = X_v/\varepsilon$ to get an isomorphism of *filtered* algebras

$$\mathbf{k}[X_v : v \in V]/I_G^{f(u)} \cong \mathbf{k}[Y_v : v \in V]/(I_G^{f(\varepsilon u)}),$$

clearly this transformation does not affect the Hilbert function.

The second claim follows from the fact that if $f = u + \ldots \neq u$, then $f(u) \neq \frac{1}{\varepsilon} f(\varepsilon u)$ for $\varepsilon \neq 1$. But the closure of each stratum is a closed variety invariant under the $\mathbf{k}^\star$-action, so by Borel's fixed point theorem it should contain a fixed point.

From the explicit formula for the action

$$f_\varepsilon(u) = u + \sum_{i=2}^{\Delta(G)} \varepsilon^{i-1} \alpha_i u^i,$$

we see that it is quasihomogeneous with the weight of the coordinate $\alpha_i$ being equal to $i-1$. ☐

**Corollary 5.11.** *The Hilbert function of the graded algebra $\mathcal{C}_G$ is the smallest in the lexicographic order. The maximum length of the Hilbert function of $\mathcal{C}_G^f$ among all $f \in \mathcal{A}_G$ is attained for $f(u) = u$ and is equal to the number of edges $|E|$ of $G$.*

*Proof.* By Lemma 5.10 the polynomial $f = u$ is contained in the closure of every stratum, so by Corollary 5.9 it must be the minimum in the lexicographic order. ☐

Since the action (5.3) is quasihomogeneous, the quotient

$$\mathbf{P}\mathcal{A}_G = (\mathcal{A}_G \setminus \{u\}) / \mathbf{k}^\star$$

is a weighted projective space.

Since the action preserves Hilbert strata, we obtain an induced Hilbert stratification on $\mathbf{P}\mathcal{A}_G$. While the stratification of $\mathcal{A}_G$ has a minimum element given by the one-point stratum $\{u\}$, the stratification of the projectivization $\mathbf{P}\mathcal{A}_G$ might not have a minimum element. However, it always has a maximum element which corresponds to the stratum of the maximum dimension with the generic Hilbert function.

Let us consider two examples in which we can completely describe the Hilbert stratifications of $\mathbf{P}\mathcal{A}_G$.

We begin with the three-cycle graph with a double edge.

**Proposition 5.12.** *Let $G$ be the three-cycle with an extra edge (see (5.2)). The projectivized parameter space $\mathbf{P}\mathcal{A}_G$ is a weighted projective line. Its Hilbert stratification consists of three points corresponding to*

$$f(u) = u + u^2, \; f(u) = u + u^3, \; \text{and } f(u) = u + u^2 + \frac{4}{3}u^3$$

*with the Hilbert series*

$$\mathcal{H}_G^f = 1 + 3t + 4t^2 + 2t^3 \tag{5.4}$$

*and a one-dimensional stratum corresponding to*

$$\{f(u) = u + bu^2 + cu^3 : 3c = 4b^2 \neq 0\}$$

*with the generic Hilbert series $1 + 3t + 5t^2 + t^3$.*

*Proof.* By Corollary 4.16 the ideal of relations for $\mathcal{C}_G^f$ is generated by

$$\begin{cases} x_1^4 = x_2^4 = x_3^3 = 0, \\ f(x_1) + f(x_2) + f(x_3) = 0, \\ (f(x_1) + f(x_2))^3 = (f(x_1) + f(x_3))^4 = (f(x_2) + f(x_3))^4 = 0. \end{cases}$$

Note that, the first group of equations implies that $f(x_1)^4 = f(x_2)^4 = f(x_3)^3 = 0$, so the third group of equations is redundant.

When $b \neq 0$ we can substitute $u \mapsto u/b$ transforming $f(u)$ into the function of the form $f(u) = u + u^2 + \lambda u^3$. Using $\lambda$ as a variable, we can verify using Macaulay2 [2] that $(x_1 + x_2 + x_3)^2$ is always a relation. Note that since the sums of consecutive entries in any Hilbert function are increasing, the function $f(u)$ is uniquely determined if there is no further relation in degree 2. We want to show that $\lambda = 0, 4/3$ are the only two exceptional cases where we get an additional quadratic relation which will explain the two rows of the table. It can be verified by hand or using Macaulay2, that Hilbert functions of these exceptional values of $\lambda$ coincide.

Using symmetry we must have a quadratic relation of the form

$$a_1(x_1^2 + x_2^2) + a_2 x_1 x_2 + a_3(x_1 + x_2) + b x_3(x_1 + x_2) + c_2 x_3^2 + c_1 x_3 = 0.$$

By subtracting the existing relation $(x_1 + x_2 + x_3)^2 = 0$ we may assume that $b = 0$. Since $f(x_3)^2 = x_3^2$ we can rewrite the relation as

$$a_1(x_1^2 + x_2^2) + a_2 x_1 x_2 + a_3(x_1 + x_2) = Q(-f(x_3)),$$

where $Q$ is a quadratic polynomial satisfying $Q(0) = 0$. Since, $Q(-f(x_3)) = Q(f(x_1) + f(x_2))$ we need to find out under which conditions there exists a $Q$ such that $\deg Q(f(x_1) + f(x_2)) \leq 2$. We may assume that $Q(T) = T^2 + aT$, since $Q(T) = T$ does not satisfy the required condition.

One can check that modulo the existing relations $x_1^4 = x_2^4 = (f(x_1) + f(x_2))^3 = 0$, the polynomial $Q(f(x_1) + f(x_2))$ has degree at most 3 and its cubic term is

$$(-a\lambda + 4/3a)(x_1 + x_2)^3 + (\lambda + 4/3a)(x_1^3 + x_2^3).$$

Both coefficients of the latter expression vanish if and only if either $\lambda = 4/3, a = -1$ or $a = \lambda = 0$. ☐

As a second example let us consider the complete graph $K_4$ on 4 vertices.

**Proposition 5.13.** *For $G = K_4$, the projectivized parameter space $P\mathcal{A}_G$ is a weighted projective line. Its Hilbert stratification consists of two points corresponding to $f(u) = u + u^2$, with the Hilbert series*

$$\mathcal{H}_{K_4}^{u+u^2} = 1 + 4t + 9t^2 + 15t^3 + 5t^4 + 3t^5 + t^6,$$

*and $f(u) = u + u^3$, with the Hilbert series*

$$\mathcal{H}_{K_4}^{u+u^3} = 1 + 4t + 9t^2 + 12t^3 + 8t^4 + 3t^5 + t^6,$$

*and a generic one-dimensional stratum corresponding to $f = u + bu^2 + cu^3$ with $b \neq 0$, $c \neq 0$, and the Hilbert series*

$$\mathcal{H}_{K_4}^{f} = 1 + 4t + 10t^2 + 14t^3 + 5t^4 + 3t^5 + t^6.$$

*Proof.* Since $b \neq 0$, using rescaling we can make $b = 1$ and therefore assume that $f = u + u^2 + cu^3$. Consider the presentation of the algebra $\mathcal{C}_G^f$ as a quotient of $\mathbf{k}[x_1, x_2, x_3, x_4]$.

By Corollary 4.16 the ideal of relations is generated by

$$\begin{cases} x_i^4 = 0 & \text{for } 1 \leq i \leq 4, \\ (x_i + x_j + x_i^2 + x_j^2 + c(x_i^3 + x_j^3))^5 = 0 & \text{for } i \neq j, \\ (x_i + x_j + x_k + x_i^2 + x_j^2 + x_k^2 + c(x_i^3 + x_j^3 + x_k^3))^4 = 0 & \text{for } i \neq j \neq k \neq i, \\ x_1 + x_2 + x_3 + x_4 + (x_1^2 + x_2^2 + x_3^2 + x_4^2) + c(x_1^3 + x_2^3 + x_3^3 & +x_4^3) = 0. \end{cases}$$

It is then easy to see that

$$(x_i + x_j + x_i^2 + x_j^2 + c(x_i^3 + x_j^3))^5 \equiv (x_i + x_j + x_i^2 + x_j^2)^5 \mod (x_1^4, x_2^4, x_3^4, x_4^4).$$

In fact, this relation reduces to $40x_i^3x_j^3 + 10x_i^3x_j^2 + 10x_i^2x_j^3 \equiv 0$. After multiplication by $x_i$ we get that $x_i^3x_j^3 \equiv 0 \mod (x_1^4, x_2^4, x_3^4, x_4^4)$. So the ideal generated by the first two classes of relations is generated by $x_i^4, x_i^3x_j^2 + x_i^2x_j^3$ for $i \neq j$. It is now easy to verify that

$$(x_i + x_j + x_k + x_i^2 + x_j^2 + x_k^2 + c(x_i^3 + x_j^3 + x_k^3))^4 \equiv (x_i + x_j + x_k + x_i^2 + x_j^2 + x_k^2)^4$$

modulo the ideal generated by the previous relations. We conclude that our relations can be rewritten as:

$$\begin{cases} x_i^4 = 0 & \text{for } 1 \leq i \leq 4, \\ (x_i + x_j + x_i^2 + x_j^2)^5 = 0 & \text{for } i \neq j, \\ (x_i + x_j + x_k + x_i^2 + x_j^2 + x_k^2)^4 = 0 & \text{for } i \neq j \neq k \neq i, \\ x_1 + x_2 + x_3 + x_4 + (x_1^2 + x_2^2 + x_3^2 + x_4^2) + c(x_1^3 + x_2^3 + x_3^3 & + x_4^3) = 0. \end{cases}$$

It is now immediate that $c = 0$ makes the last relation quadratic. It is also easy to see that no other value of $c$ can do that. Indeed, since the ideal $I$ generated by the first three groups of relations, which are independent of $c$, does not contain a polynomial of degree less than 4, for all $f \in I$, the polynomial

$$f + x_1 + x_2 + x_3 + x_4 + (x_1^2 + x_2^2 + x_3^2 + x_4^2) + c(x_1^3 + x_2^3 + x_3^3 + x_4^3)$$

has a nonzero cubic term.

In a similar way we can find the reduced set of relations in the case of $f(u) = u + cu^3$. The corresponding Hilbert series have been calculated using Macaulay2. □

# 6 Computations and experimental results

In this section, we present results of computations of the Hilbert functions $\mathcal{H}_G^f$ of the deformed zonotopal algebras $\mathcal{C}_G^f$ for a number of graphs $G$ and nondegenerate polynomials $f$ which were computed with the help of the Macaulay2 computer algebra system [2]. If $f = u$, i.e. when $\mathcal{C}_G^f$ is the usual graded zonotopal algebra the answer is given by Theorem 2.9. In particular, if $G$ is a tree with $n$ edges, then

$$\mathcal{H}_G^u = (1+t)^n.$$

Also we will assume that $\deg f \leq \Delta(G)$, since as explained in Remark 3.3, the terms in $f$ of degree higher than $\Delta(G)$ do not affect $\mathcal{C}_G^f$.

Under this assumption for any graph $G$ and a nondegenerate function

$$f = u + \alpha u^m + \dots, \ \alpha \neq 0,$$

we have

$$\mathcal{H}_G^f(t) = 1 + h_1 t + \dots,$$

where

$$h_1 = \begin{cases} |V| - 1, & \text{if } G \text{ is a two-vertex multigraph } E_m \text{ and } f \text{ is odd} \\ |V|, & \text{otherwise} \end{cases}$$

Thus the number of vertices of the graph can be recovered from the second coefficient of the Hilbert series.

## 6.1 Graphs with at most four vertices

In the tables below we present the Hilbert functions of the algebra $\mathcal{C}_G^f$ for all graphs with at most 5 vertices and for a selected collection of functions $f$. For all other functions $f$ for which we computed these Hilbert functions, the answer coincides with the one given by a generic polynomial in the same table of the same degree as $f$.

### 6.1.1 Graphs with 4 or 5 vertices and $\Delta = 3$

In the three tables below we present the Hilbert functions of the algebras $\mathcal{C}_G^f$ for all connected simple graphs with 4 or 5 vertices with the maximum vertex degree $\Delta = 3$, as well as for all 4-vertex multigraphs with $\Delta = 3$. For every non-degenerate polynomial $f \in \mathbf{k}[u]$, the corresponding Hilbert function is the same as for one of the following four polynomials: $u + u^2$, $u + u^3$, $u + \frac{1}{2}u^2 + \frac{1}{3}u^3$ and $u + u^2 + u^3$, with the last representing the generic stratum.

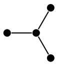
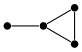
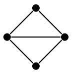
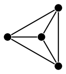
| $G =$ <br> $f(u) =$ |  |  |  |  |
|---|---|---|---|---|
| $u+u^2$ | 1,4,3 | 1,4,7 | 1,4,9,10 | 1,4,9,15,5,3,1 |
| $u+u^3$ | 1,4,3 | 1,4,5,2 | 1,4,9,8,2 | 1,4,9,12,8,3,1 |
| $u+u^2+u^3$ | 1,4,3 | 1,4,7 | 1,4,10,9 | 1,4,10,14,5,3,1 |

Table 1: Hilbert functions of $\mathcal{C}_G^f$ for all 4-vertex connected simple graphs with $\Delta = 3$

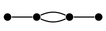
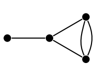
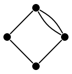
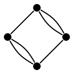
| $G =$ <br> $f(u) =$ |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| $u+u^2$ | 1,4,7 | 1,4,7 | 1,4,9,4 | 1,4,8,7 | 1,4,9,8 | 1,4,9,15,3 |
| $u+u^3$ | 1,4,5,2 | 1,4,5,2 | 1,4,7,5,1 | 1,4,8,6,1 | 1,4,8,7,2 | 1,4,10,11,5,1 |
| $u+\frac{u^2}{2}+\frac{u^3}{3}$ | 1,4,7 | 1,4,7 | 1,4,9,4 | 1,4,9,6 | 1,4,9,8 | 1,4,10,14,3 |
| $u+u^2+u^3$ | 1,4,7 | 1,4,7 | 1,4,9,4 | 1,4,9,6 | 1,4,10,7 | 1,4,10,15,2 |

Table 2: Hilbert functions of $\mathcal{C}_G^f$ for all 4-vertex connected multigraphs with $\Delta = 3$

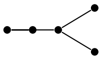
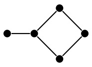
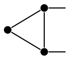
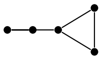
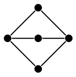
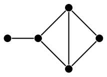
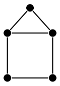
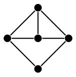
| $G =$ <br> $f(u) =$ |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|
| $u+u^2$ | 1,5,10 | 1,5,13,10,1 | 1,5,12,10 | 1,5,12,8,2 | 1,5,14,25,9 | 1,5,13,21,8 | 1,5,14,23,9 | 1,5,14,29,33,3,1 |
| $u+u^3$ | 1,5,7,3 | 1,5,11,10,3 | 1,5,12,9,1 | 1,5,10,9,3 | 1,5,15,22,11 | 1,5,13,19,10 | 1,5,14,20,11,1 | 1,5,15,29,26,9,1 |
| $u+\frac{u^2}{2}+\frac{u^3}{3}$ | 1,5,10 | 1,5,14,9,1 | 1,5,13,9 | 1,5,13,7,2 | 1,5,15,24,9 | 1,5,14,22,6 | 1,5,15,24,7 | 1,5,15,33,28,3,1 |
| $u+u^2+u^3$ | 1,5,10 | 1,5,14,9,1 | 1,5,13,9 | 1,5,13,8,1 | 1,5,15,26,7 | 1,5,14,23,5 | 1,5,15,26,5 | 1,5,15,33,28,3,1 |

Table 3: Hilbert functions of $\mathcal{C}_G^f$ for all 5-vertex connected simple graphs with $\Delta = 3$

### 6.1.2 Simple graphs with 5 vertices and $\Delta = 4$

In the following table we present the Hilbert functions of the algebra $\mathcal{C}_G^f$ for all 5-vertex simple graphs with the maximum vertex degree $\Delta = 4$. We have not analyzed the parametric Gröbner bases for these algebras and therefore, in principle, that for some non-degenerate $f \in \mathbf{k}[u]$ the corresponding Hilbert function is different from the ones given in the table.

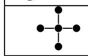
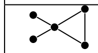
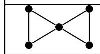
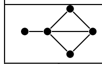
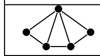
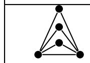
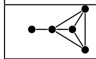
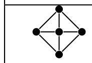
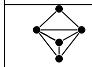
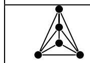
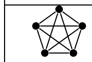
| $f(u) =$<br>$G =$ | $u+u^4$ | $u+u^2$ | $u+u^3$ | $u+u^3+u^4$ | $u+u^2+u^4$ | $u+\frac{u^2}{2}+\frac{u^3}{3}$ | $u+u^2+u^3$ | $u+\frac{u^2}{2}+\frac{u^3}{3}+\frac{u^4}{4}$ | $u+u^2+u^3+u^4$ |
|---|---|---|---|---|---|---|---|---|---|
|  | 1,5,6,4 | 1,5,10 | 1,5,7,3 | 1,5,7,3 | 1,5,10 | 1,5,10 | 1,5,10 | 1,5,10 | 1,5,10 |
|  | 1,5,9,9,4 | 1,5,12,9,1 | 1,5,10,9,3 | 1,5,10,9,3 | 1,5,13,8,1 | 1,5,13,8,1 | 1,5,13,8,1 | 1,5,13,8,1 | 1,5,13,8,1 |
|  | 1,5,12,15,12,4 | 1,5,14,20,9 | 1,5,13,16,11,3 | 1,5,13,16,11,3 | 1,5,15,19,9 | 1,5,15,21,7 | 1,5,15,21,7 | 1,5,15,21,7 | 1,5,15,21,7 |
|  | 1,5,11,15,12,4 | 1,5,13,23,6 | 1,5,14,19,9 | 1,5,14,20,8 | 1,5,14,23,5 | 1,5,14,23,5 | 1,5,14,25,3 | 1,5,14,23,5 | 1,5,14,25,3 |
|  | 1,5,13,21,21,12,3 | 1,5,14,28,28 | 1,5,15,28,22,5 | 1,5,15,29,4,2 | 1,5,15,32,23 | 1,5,15,32,23 | 1,5,15,32,23 | 1,5,15,32,23 | 1,5,15,33,22 |
|  | 1,5,14,23,23,14,1, | 1,5,14,27,32,2 | 1,5,15,26,25,9 | 1,5,15,27,25,8 | 1,5,15,30,28,2 | 1,5,15,31,29 | 1,5,15,31,29 | 1,5,15,32,28 | 1,5,15,32,28 |
|  | 1,5,11,19,21,15,4 | 1,5,13,25,24,6,2 | 1,5,14,24,22,9,1 | 1,5,14,25,23,7,1 | 1,5,14,27,23,5,1 | 1,5,14,28,22,5,1 | 1,5,14,28,22,5,1 | 1,5,14,28,20,6,2 | 1,5,14,29,21,5,1 |
|  | 1,5,13,26,34,33,18,4 | 1,5,14,30,51,26,7 | 1,5,15,33,45,29,6 | 1,5,15,34,47,28,4 | 1,5,15,34,56,19,4 | 1,5,15,34,56,19,4 | 1,5,15,34,56,19,4 | 1,5,15,35,51,20,7 | 1,5,15,35,56,18,4 |
|  | 1,5,14,27,35,30,15,1 | 1,5,14,29,46,28,5 | 1,5,15,31,43,28,5 | 1,5,15,32,47,25,3 | 1,5,15,34,53,19,1 | 1,5,15,33,52,21,1 | 1,5,15,33,52,21,1 | 1,5,15,34,49,20,4 | 1,5,15,34,53,19,1 |
|  | 1,5,14,30,48,50,37,13 | 1,5,14,30,53,62,33 | 1,5,15,33,55,59,28,2 | 1,5,15,34,58,64,20,1 | 1,5,15,35,65,64,13 | 1,5,15,34,62,68,13 | 1,5,15,34,62,68,13 | 1,5,15,35,60,59,23 | 1,5,15,35,65,64,13 |
|  | 1,5,14,30,53,73,60,<br>41,9,4,1 | 1,5,14,30,55,80,77,<br>15,9,4,1 | 1,5,15,33,60,76,60,<br>27,9,4,1 | 1,5,15,34,63,82,56,<br>21,9,4,1 | 1,5,15,35,67,91,48,<br>15,9,4,1 | 1,5,15,34,64,90,53,<br>15,9,4,1 | 1,5,15,34,64,90,53,<br>15,9,4,1 | 1,5,15,35,63,84,59,<br>15,9,4,1 | 1,5,15,35,67,91,48,<br>15,9,4,1 |

Table 4: Hilbert functions of $\mathcal{C}_G^f$ for all 5-vertex connected simple graphs with a 4-valent vertex

## 6.2 Families of graphs with small maximum degree $\Delta_G \leq 3$

In this subsection, for several families of graphs $G$ with the maximum vertex degree $\Delta_G \leq 3$ and specific functions $f$, we present the Hilbert function of the algebra $\mathcal{C}_G^f$ computed using the Macaulay2 system [2]. We also formulate some conjectures based on the analysis of these computations.

We begin with the chain graphs $A_n$ and the cycles $C_n$, the only connected graphs with $\Delta_G = 2$. In this case, the projectivized parameter space $P\mathcal{A}_G$ consists of a single point, so the only polynomial we need to consider, besides the graded case $f = u$, is $f = u + u^2$.

We then consider some families of graphs with $\Delta_G = 3$ obtained from $A_n$ or $C_n$ by adding one or two extra edges. In this case, the space $P\mathcal{A}_G$ is a weighted projective line, and the only functions $f$ we need to consider (besides the graded case $f = u$) are $f = u + u^3$ and $f = u + u^2 + au^3$ with $a \in \mathbf{k}$. We computed the Hilbert function of the algebra $\mathcal{C}_G^{u+u^2+au^3}$ for various values of $a$. For each of the graphs we considered, this Hilbert function coincided with the one for $a = 0$, except when $f = u + u^2 + 2u^3$, where the result differs in many cases. In the tables below, we use the following notation for these functions

$$f_1 = u + u^2, \;\; f_2 = u + u^2 + 2u^3, \quad \text{and} \quad f_3 = u + u^3. \tag{6.1}$$

### 6.2.1 Chain and cycle graphs

Example 6.1. The Hilbert function of the algebra $\mathcal{C}_G^f$ of the $n$-vertex chain graph $A_n = \bullet\!\!-\!\!\bullet\!\!- \cdots -\!\!\bullet\!\!-\!\!\bullet$, with $3 \leq n \leq 17$ and $f_1 = u + u^2$, is as follows.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 3 | | | | | | | |
| 1 | 4 | 3 | | | | | | |
| 1 | 5 | 10 | | | | | | |
| 1 | 6 | 16 | 9 | | | | | |
| 1 | 7 | 23 | 33 | | | | | |
| 1 | 8 | 31 | 61 | 27 | | | | |
| 1 | 9 | 40 | 98 | 108 | | | | |
| 1 | 10 | 50 | 145 | 225 | 81 | | | |
| 1 | 11 | 61 | 203 | 397 | 351 | | | |
| 1 | 12 | 73 | 273 | 636 | 810 | 243 | | |
| 1 | 13 | 86 | 356 | 955 | 1551 | 1134 | | |
| 1 | 14 | 100 | 453 | 1368 | 2665 | 2862 | 729 | |
| 1 | 15 | 115 | 565 | 1890 | 4258 | 5895 | 3645 | |
| 1 | 16 | 131 | 693 | 2537 | 6452 | 10788 | 9963 | 2187 |
| 1 | 17 | 148 | 838 | 3326 | 9386 | 18232 | 21924 | 11664 |

Observations (conjectures):

(i) For $n = 2k$, the $(k+1)$st entry equals $3^{k-1}$.

(ii) For $n = 2k+1$, the $(k+1)$st entry equals $(k+8)3^{k-2}$.

(iii) For each $k \geq 1$, the $k$th entry in $n$th row, for $n \geq 2k-1$, is a polynomial of degree $k-1$ in $n$.

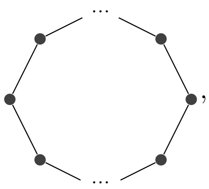
In particular,

- for $n \geq 5$, the 3rd entry is $\dfrac{n^2 + n - 10}{2}$;
- for $n \geq 7$, the 4th entry is $\dfrac{n^3 + 3n^2 - 46n + 30}{6}$;
- for $n \geq 9$, the 5th entry is $\dfrac{n^4 + 6n^3 - 121n^2 + 42n + 1080}{24}$;
- for $n \geq 11$, the 6th entry is $\dfrac{n^5 + 10n^4 - 245n^3 - 250n^2 + 9364n - 12000}{120}$;

- for $n \geq 13$, the 7th entry is

$$\frac{n^6 + 15n^5 + 3325n^4 + 1785n^3 + 11874n^2 - 201960n + 93600}{6!}.$$

Remark 6.2. The above formulas were obtained from the numerical information in the table using the main assumption (iii).

Example 6.3. The Hilbert function of the algebra $\mathcal{C}_G^f$ of the $n$-cycle graph $C_n =$ , with $3 \leq n \leq 13$ and $f_1 = u + u^2$, is as follows.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 2 | 1 | | | | | |
| 1 | 4 | 7 | 3 | | | | | |
| 1 | 5 | 14 | 10 | 1 | | | | |
| 1 | 6 | 20 | 31 | 5 | | | | |
| 1 | 7 | 27 | 63 | 28 | 1 | | | |
| 1 | 8 | 35 | 96 | 106 | 9 | | | |
| 1 | 9 | 44 | 138 | 243 | 75 | 1 | | |
| 1 | 10 | 54 | 190 | 405 | 346 | 17 | | |
| 1 | 11 | 65 | 253 | 627 | 891 | 198 | 1 | |
| 1 | 12 | 77 | 328 | 921 | 1620 | 1103 | 33 | |
| 1 | 13 | 90 | 416 | 1300 | 2691 | 3159 | 520 | 1 |

Observations (conjectures):

(i) The last entry in a row corresponding to an odd $n$ is equal to 1.

(ii) The last entry in the row with $n = 2k$ is $2^{k-1} + 1$.

(iii) The next to the last entry in $2k + 1$st row for $k \geq 2$ is $L(2k + 1) - 1$, where $L(n)$ is the $n$th Lucas number (a member of the Fibonacci-like sequence $1, 3, 4, 7, 11, 18, 29, 47, \ldots$).

(iv) For each $k \geq 1$, the $k$th entry in $n$th row, for $n \geq 2k - 1$, is a polynomial in $n$ of degree $k - 1$.

In particular,

- for $n \geq 5$, the 3rd entry equals $\frac{n^2+n-2}{2}$,
- for $n \geq 7$, the 4th entry equals $\frac{n^3+3n^2-16n}{6}$.

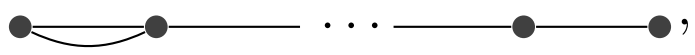
### 6.2.2 Chains and cycles with one double edge

In this subsection we present the results of computations of the Hilbert functions of the algebra $\mathcal{C}_G^f$, where the graph $G$ is an $n$-chain $A_n$ or an $n$-cycle $C_n$ with one double edge for the three polynomials $f_1, f_2$ and $f_3$ from (6.1).

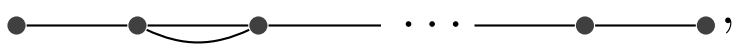
Each of these graphs has an edge connecting two vertices of degree two. Therefore, we can use the deletion-contraction relation (4.5) to compute the Hilbert series of the algebra $\mathcal{C}_G^f$ for $f_3 = u + u^3$ recursively. This leads to the following closed formula for chain graphs with a double edge (which also explains the Pascal-like behavior of the numbers in the column for $f_3$ in the table below).

**Proposition 6.4.** *Let $G$ be an $n$-vertex chain graph with one double edge at any place. The Hilbert series of the algebra $\mathcal{C}_G^{u+u^3}$ for $n \geq 3$ is equal to*

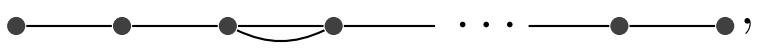
$$\mathcal{H}_G^{u+u^3}(t) = (1 + 3t + 2t^2)(1 + t)^{n-3}. \tag{6.2}$$

Example 6.5. For the graph $B_n =$  the chain $A_n$ with doubled first edge with $3 \leq n \leq 10$, we have the following Hilbert functions.

| $f_1 = u + u^2,\ f_2 = u + u^2 + 2u^3$ | $f_3 = u + u^3$ |
|---|---|
| 1, 3, 2 | 1, 3, 2 |
| 1, 4, 7 | 1, 4, 5, 2 |
| 1, 5, 12, 6 | 1, 5, 9, 7, 2 |
| 1, 6, 18, 23 | 1, 6, 14, 16, 9, 2 |
| 1, 7, 25, 45, 18 | 1, 7, 20, 30, 25, 11, 2 |
| 1, 8, 33, 75, 75 | 1, 8, 27, 50, 55, 36, 13, 2 |
| 1, 9, 42, 114, 164, 54 | 1, 9, 35, 77, 105, 91, 49, 15, 2 |
| 1, 10, 52, 163, 299, 243 | 1, 10, 44, 112, 182, 196, 140, 64, 17, 2 |

Example 6.6. For the graph $G =$  the chain $A_n$ with the doubled second edge, $4 \leq n \leq 10$, we have the following Hilbert functions.

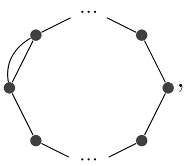
| $f_1 = u + u^2$ | $f_2 = u + u^2 + 2u^3$ |
|---|---|
| 1, 4, 7 | 1, 4, 7 |
| 1, 5, 12, 6 | 1, 5, 11, 7 |
| 1, 6, 18, 23 | 1, 6, 17, 24 |
| 1, 7, 25, 45, 18 | 1, 7, 24, 43, 21 |
| 1, 8, 33, 75, 75 | 1, 8, 32, 72, 79 |
| 1, 9, 42, 114, 164, 54 | 1, 9, 41, 110, 160, 63 |
| 1, 10, 52, 163, 299, 243 | 1, 10, 51, 158, 290, 258 |

Example 6.7. For the graph $G =$  the chain $A_n$ with the doubled third edge, $4 \leq n \leq 10$, we have the following Hilbert functions.

| $f_1 = u + u^2$ | $f_2 = u + u^2 + 2u^3$ |
|---|---|
| 1, 4, 7 | 1, 4, 7 |
| 1, 5, 12, 6 | 1, 5, 11, 7 |
| 1, 6, 17, 24 | 1, 6, 16, 25 |
| 1, 7, 24, 46, 18 | 1, 7, 23, 44, 21 |
| 1, 8, 32, 73, 78 | 1, 8, 31, 70, 82 |
| 1, 9, 41, 111, 168, 54 | 1, 9, 40, 107, 164, 63 |
| 1, 10, 51, 159, 295, 252 | 1, 10, 50, 154, 286, 267 |

Observations (conjectures):

1. in Example 6.5 $f_1$ and $f_2$ have the same Hilbert functions.
2. The Hilbert functions for $f_1$ coincide in Examples 6.5 and 6.6.
3. After finitely many exceptions in the beginning the $i$th row in the above 3 tables is given by a polynomial of degree $(i - 1)$ in the number of vertices.

EXAMPLE 6.8. For the graph $\widehat{C}_n =$ , the $n$-cycle $C_n$ with one doubled edge, for $3 \le n \le 9$, we have the following Hilbert functions.

| $f_1 = u + u^2$ | $f_2 = u + u^2 + 2u^3$ | $f_3 = u + u^3$ |
|---|---|---|
| 1, 3, 5, 1 | 1, 3, 5, 1 | 1, 3, 4, 2 |
| 1, 4, 10, 7 | 1, 4, 9, 8 | 1, 4, 8, 7, 2 |
| 1, 5, 15, 23, 2 | 1, 5, 14, 23, 3 | 1, 5, 13, 16, 9, 2 |
| 1, 6, 21, 46, 20 | 1, 6, 20, 43, 24 | 1, 6, 19, 30, 25, 11, 2 |
| 1, 7, 28, 72, 80, 2 | 1, 7, 27, 68, 82, 5 | 1, 7, 26, 50, 55, 36, 13, 2 |
| 1, 8, 36, 106, 181, 50 | 1, 8, 35, 101, 174, 63 | 1, 8, 34, 77, 105, 91, 49, 15, 2 |
| 1, 9, 45, 149, 306, 254, 2 | 1, 9, 44, 143, 293, 267, 9 | 1, 9, 43, 112, 182, 196, 140, 64, 17, 2 |

Similar to chain graphs with one double edge, the Hilbert series of $\mathcal{C}_G^f$ for $G = \widehat{C}_n$ and $f = f_3 = u+u^3$ can be found explicitly using the deletion-contraction formula (4.5). This also explains the Pascal-like behavior of the results in the third column of the above table.

**Proposition 6.9.** *The Hilbert series of the algebra* $\mathcal{C}^{u+u^3}_{\widehat{C}_n}$ *for* $n \ge 3$ *is equal to*

$$\mathcal{H}^{u+u^3}_{\widehat{C}_n}(t) = (1+2t)(1+t)^{n-1} - t - t^2. \tag{6.3}$$

*Proof.* We use induction on $n$. For $n = 3$, equation (6.3) agrees with the answer given in (5.4).

If $n \ge 4$, then the graph $G = \widehat{C}_n$ has an edge $e$ connecting two vertices of degree 2 and we can apply the deletion-contraction formula (4.5). In this case $G/e = \widehat{C}_{n-1}$ and $G - e$ is a chain $A_n$ with one doubled edge. By Proposition 6.4 we have $\mathcal{H}^{u+u^3}_{G-e} = (1+3t+2t^2)(1+t)^{n-3}$. Therefore by induction we have

$$\mathcal{H}^{u+u^3}_{\widehat{C}_n} = [(1+2t)(1+t)^{n-2} - t - t^2] + t[(1+3t+2t^2)(1+t)^{n-3}] = (1+2t)(1+t)^{n-1} - t - t^2$$

as required. □

### 6.2.3 Families of trees with a single trivalent vertex

In this subsection we present the results of computations of the Hilbert functions of the algebra $\mathcal{C}^f_G$, for several families of trees obtained from the chain $A_{n-1}$ by adding a new edge at a vertex of degree two.

For $f = u + u^3$, the deletion-contraction relation (4.5) allows to find a closed formula for $\mathcal{H}^f_G$.

**Proposition 6.10.** *Let $G$ be a tree with $n \ge 4$ vertices, with one trivalent vertex and all other vertices of degree $\le 2$. Then the Hilbert series of the algebra* $\mathcal{C}^{u+u^3}_G$ *is equal to*

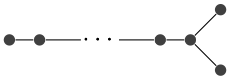
$$\mathcal{H}^{u+u^3}_G = (1+4t+3t^2)(1+t)^{n-4}. \tag{6.4}$$

*Proof.* If $n = 4$, then $G$ is the star graph $S_4$, for which the Hilbert series is given by (4.11).

For $n \ge 5$, the $G$ must contain an edge $e$ connecting a vertex of degree 1 with a vertex of degree 2. Since $G/e$ also satisfies the assumption of the proposition and $G - e = (G/e) \sqcup \bullet$, the relation (4.5) gives

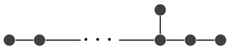
$$\mathcal{H}^{u+u^3}_G = \mathcal{H}^{u+u^3}_{G/e} + t\mathcal{H}^{u+u^3}_{G-e} = (1+t)\mathcal{H}^{u+u^3}_{G/e}$$

and the statement follows by induction. □

Example 6.11. Let $G$ be the graph $D_n =$  with $n \geq 4$ vertices.

Our computations show that for $f_1 = u + u^2$ and $f_2 = u + u^2 + 2u^3$, the Hilbert series $\mathcal{H}_G^f$ of the algebra $\mathcal{C}_G^f$ coincides with the one of the chain graph $A_n$ for $f_1 = u + u^2$, see Example 6.1).

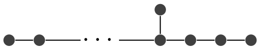
*Proof.* This follows by induction on $n$ from the deletion-contraction relation (4.5) applied to an edge connecting two vertices of degree 2. ☐

Example 6.12. Consider the chain $A_{n-1}$ with an extra edge on the third place, $G =$ , with $n \geq 5$ vertices.

For $f_3 = u + u^3$, the Hilbert series of the algebra $\mathcal{C}_G^f$ is the same as for the graph $D_n$ from Example 6.11 given by (6.4). This follows from the deletion-contraction relation (4.5) similar to the proof of Proposition 6.10.

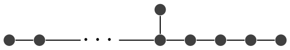
For $f_1 = u + u^2$ and $f_2 = u + u^2 + 2u^3$ the Hilbert functions of the algebra $\mathcal{C}_G^f$ are as follows:

| $f_1 = u + u^2$ | $f_2 = u + u^2 + 2u^3$ |
|---|---|
| 1, 5, 10 | 1, 5, 10 |
| 1, 6, 16, 9 | 1, 6, 15, 10 |
| 1, 7, 23, 33 | 1, 7, 22, 34 |
| 1, 8, 31, 61, 27 | 1, 8, 30, 59, 30 |
| 1, 9, 40, 98, 108 | 1, 9, 39, 95, 112 |
| 1, 10, 50, 145, 225, 81 | 1, 10, 49, 141, 221, 90 |
| 1, 11, 61, 203, 397, 351 | 1, 11, 60, 198, 388, 366 |

Example 6.13. Consider the chain $A_{n-1}$ with an extra edge on the 4th place, $G =$ , with $n \geq 5$ vertices.

For $f_1 = u + u^2$ and $f_2 = u + u^2 + 2u^3$ we have the following Hilbert functions.

| $f_1 = u + u^2$ | $f_2 = u + u^2 + 2u^3$ |
|---|---|
| 1, 5, 10 | 1, 5, 10 |
| 1, 6, 16, 9 | 1, 6, 16, 9 |
| 1, 7, 23, 33 | 1, 7, 22, 34 |
| 1, 8, 31, 61, 27 | 1, 8, 30, 62, 27 |
| 1, 9, 40, 98, 108 | 1, 9, 39, 96, 111 |
| 1, 10, 50, 145, 225, 81 | 1, 10, 49, 142, 229, 81 |
| 1, 11, 61, 203, 397, 351 | 1, 11, 60, 199, 393, 360 |

Example 6.14. Consider the chain $A_{n-1}$ with an extra edge on the 5th place, $G =$  with $n \geq 6$ vertices. For $f_1 = u + u^2$ and $f_2 = u + u^2 + 2u^3$, we have the following Hilbert functions.

| $f_1 = u + u^2$ | $f_2 = u + u^2 + 2u^3$ |
|---|---|
| 1, 6, 16, 9 | 1, 6, 16, 9 |
| 1, 7, 23, 33 | 1, 7, 23, 33 |
| 1, 8, 31, 61, 27 | 1, 8, 30, 59, 30 |
| 1, 9, 40, 98, 108 | 1, 9, 39, 96, 111 |
| 1, 10, 50, 145, 225, 81 | 1, 10, 49, 140, 222, 90 |
| 1, 11, 61, 203, 397, 351 | 1, 11, 60, 197, 392, 363 |

### 6.2.4 Families of trees with two trivalent vertices

In this subsection we present the results of computations of the Hilbert series of the algebra $\mathcal{C}_G^f$ for several families of trees obtained from the chain $A_{n-1}$ by adding two new edges at vertices of degree two.

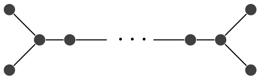
Our computations show that for all such graphs $G$ with $n$ vertices the Hilbert series $\mathcal{H}_G^f$ for $f = u+u^2+au^3$ with $a \neq 2$ are the same as for $f = u + u^2$. Moreover, if one of three-valent vertices of $G$ is connected to two leaves (one-valent vertices), then the Hilbert series $\mathcal{H}_G^{u+u^2}$ is the same as for the chain graph $A_n$ (see Example 6.1).

For the polynomial $f_3 = u+u^3$, the deletion-contraction relation (4.5) allows to compute $\mathcal{H}_G^{f_3}$ recursively.

**Proposition 6.15.** *Let $G$ be a tree with $n \geq 7$ vertices, with two trivalent vertices and all other vertices of degree $\leq 2$. Let $k$ denote the number of vertices in the unique path connecting the trivalent vertices. Then the Hilbert series of the algebra $\mathcal{C}_G^{u+u^3}$ is equal to*

$$\mathcal{H}_G^{u+u^3} = \begin{cases} (1+3t)^2(1+t)^{k-5} - t + t^3)(1+t)^{n-k-4} & \text{if } k \geq 3; \\ (1+6t+14t^2+10t^3+t^4)(1+t)^{n-6} & \text{if } k = 2. \end{cases} \tag{6.5}$$

*Proof.* Applying Corollary 4.4, the statement is reduced to the case when all leaves are connected to the tree-valent vertices, i.e. to the case of the graph

$G_0 =$ 

with $k+4$ vertices.

This case is treated in the following lemma. □

**Lemma 6.16.** *The Hilbert series $\mathcal{H}_n := \mathcal{H}_{G_0}^{u+u^3}$ for the above $n$-vertex graph $G_0$ is given by*

$$\mathcal{H}_n = \begin{cases} (1+3t)^2(1+t)^{n-5} - t + t^3 & \text{if } n \geq 7, \\ 1+6t+14t^2+10t^3+t^4 & \text{if } n = 6. \end{cases}$$

*Proof.* Applying the deletion-contraction formula (4.5) to an edge $e$ connecting two two-valent vertices and using the factorization formula (4.3), we have, for $n \geq 8$,

$$\mathcal{H}_n = \mathcal{H}_{n-1} + t\mathcal{H}_{D_4}^{u+u^3}\mathcal{H}_{D_{n-4}}^{u+u^3},$$

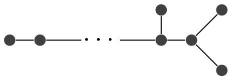
where $D_n$ is the graph from Example 6.11. Therefore, by the induction hypotheses and Proposition 6.10 we have

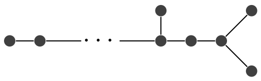
$$\begin{aligned} \mathcal{H}_n &= \big((1+3t)^2(1+t)^{n-6} - t + t^3\big) + t(1+4t+3t^2)^2(t+1)^{n-8} \\ &= (1+3t)^2(1+t)^{n-5} - t + t^3. \end{aligned}$$

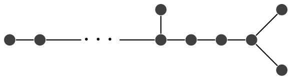
as required.

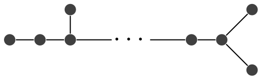
The Hilbert series for the base of induction case $n = 7$ and for the special case $n = 6$ have been found using Macaulay2. □

To illustrate Proposition 6.15 we present below the closed formulas for $\mathcal{H}_G^{u+u^3}$ for the following four families of such graphs:

$G_1(n) =$ 

$G_2(n) =$ 

$G_3(n) =$ 

$G_4(n) =$ 

(Here as above $n$ stands for the number of vertices of a graph).

**Corollary 6.17.** *The Hilbert series of the algebra* $\mathcal{C}_G^{u+u^3}$ *for the above graphs is equal to*

(a) $\mathcal{H}_{G_1(n)}^{u+u^3} = (1+6t+14t^2+10t^3+t^4)(1+t)^{n-6}$ *for* $n \geq 6$;

(b) $\mathcal{H}_{G_2(n)}^{u+u^3}(t) = (1+7t+22t^2+25t^3+9t^4)(1+t)^{n-7}$ *for* $n \geq 7$;

(c) $\mathcal{H}_{G_3(n)}^{u+u^3}(t) = (1+8t+30t^2+47t^3+33t^4+9t^5)(1+t)^{n-8}$ *for* $n \geq 8$;

(d) $\mathcal{H}_{G_4(n)}^{u+u^3} = (1+3t)^2(1+t)^{n-5} - t - t^2 + t^3 + t^4$ *for* $n \geq 8$.

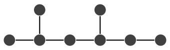
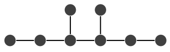
Our computations show that for $G$ from one of the above four families of graphs and $f = u + u^2 + au^3$, with $a \neq 2$, the Hilbert function of the algebra $\mathcal{C}_G^f$ is the same as for the $n$-chain graph $A_n$ and $f = u + u^2$. The results for the exceptional case $f = u + u^2 + 2u^3$ and $6 \leq n \leq 11$ are given in the table below.

| $G_1$ | $G_2$ | $G_3$ | $G_4$ |
|---|---|---|---|
| 1, 6, 16, 9 | 1, 6, 16, 9 | 1, 6, 15, 10 | 1, 6, 16, 9 |
| 1, 7, 22, 34 | 1, 7, 23, 33 | 1, 7, 23, 33 | 1, 7, 23, 33 |
| 1, 8, 30, 62, 27 | 1, 8, 30, 59, 30 | 1, 8, 31, 61, 27 | 1, 8, 31, 61, 27 |
| 1, 9, 39, 96, 111 | 1, 9, 39, 96, 111 | 1, 9, 39, 95, 112 | 1, 9, 40, 98, 108 |
| 1, 10, 49, 142, 229, 81 | 1, 10, 49, 140, 222, 90 | 1, 10, 49, 142, 229, 81 | 1, 10, 50, 145, 225, 81 |
| 1, 11, 60, 199, 393, 360 | 1, 11, 60, 197, 392, 363 | 1, 11, 60, 197, 392, 363 | 1, 11, 61, 203, 397, 351 |

# 7 Concluding remarks and open questions

Here we present some remarks about our experimental results as well as a list of open problems about deformed zonotopal algebras $\mathcal{C}_G^f$.

**1.** The Hilbert series of the graded algebra $\mathcal{C}_G^u$ is a specialization of the Tutte polynomial of the graph $G$. In particular, it is the same for all graphs with the same graphical matroid, e.g. for all trees with $n$ vertices. This is not the case for the filtered algebra $\mathcal{C}_G^f$ already for the simplest non-homogeneous function $f = u + u^2$. For example, for the trees

$T_1 =$  and $T_2 =$  ,

we have

$$\mathcal{H}_{T_1}^{u+u^2} = 1 + 8t + 31t^2 + 61t^3 + 27t^4$$

and

$$\mathcal{H}_{T_2}^{u+u^2} = 1 + 8t + 30t^2 + 62t^3 + 27t^4.$$

However, finding a graph-theoretical interpretation of the Hilbert function of $\mathcal{C}_G^f$ seems a difficult problem even for such simple graphs as the chain $A_n$ or the cycle $C_n$.

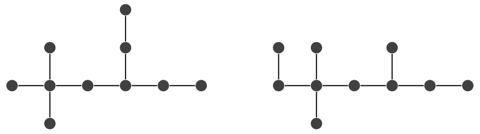
For these two families the Hilbert series for $f = u + u^2$ is generic. Thus a natural question is to find a graph-theoretical interpretation of $\mathcal{H}_G^f$ for a generic polynomial $f$ for arbitrary graphs.

**2.** According to Theorem 3.8 the filtered algebra $\mathcal{C}_G^f$ for non-degenerate $f$ is a complete graph invariant. So it is natural to ask whether graphs can be distinguished by the collection of all Hilbert series $\mathcal{H}_G^f$ for $f \in \mathbf{k}[u]$. Our experimental results show that it is the case for all graphs with at most 6 and for trees with up to 7 vertices.

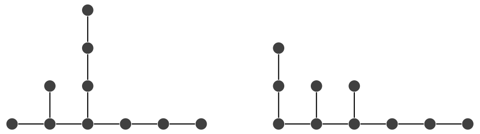
However, we found four pairs of non-isomorphic simple graphs for which the Hilbert series $\mathcal{H}_G^f$ coincide for a large number of polynomials $f$. This suggests that the answer to the question is negative in general. To confirm this we will need to analyze the parametric Gröbner bases for the corresponding algebras in complete generality, so it is still possible that their Hilbert functions will be different for some values of parameters of $f$.

These four pairs of graphs shown below are the only potential counterexamples among all trees with up to 10 vertices and among all unicyclic graphs with up to 8 vertices.

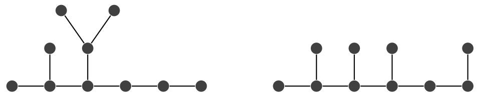
(i)



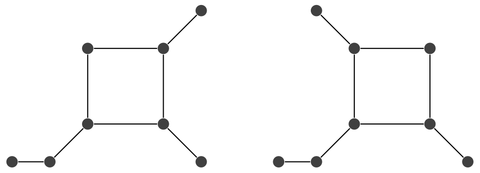
(ii)



(iii)



(iv)



**3.** If the maximum vertex degree of $G$ is 3 then one should only test a one-parameter family of functions $f$. From Corollary 5.9 it follows, that all Hilbert functions $\mathcal{H}_G^f$ will be the same except, possibly, for finitely many points which are roots of some univariate polynomial. Is it possible to find this polynomial from $G$ and in this way to describe the Hilbert stratification of the respective weighted projective line of parameters?

**4.** In several examples presented above the Hilbert stratification of the parameter space is coordinate-wise, i.e. the Hilbert function depends only on which coefficients of the polynomial $f(u)$ are vanishing and which are non-vanishing. But this is not always the case. As we see in e.g. Example 6.14 and some other cases $f(u) = u + u^2 + 2u^3$ gives a different Hilbert function than $f(u) = u + u^2 + au^3$ for $a \neq 2$. It turns out that $f(u) = u + u^2 + 2u^3$ is the truncation of $(u + u^2)^{-1} = u + u^2 + 2u^3 + 5u^4 + \dots$ whose coefficients are the Catalan numbers. The latter function is special for several graphs with maximal valency of vertices exceeding 3. Similar phenomenon happens for certain graphs with $f = e^u$ and $f = \log u$ (truncated at the maximal valency of the vertices in a graph). A natural question is to find explanations of the speciality of these functions which should be related to the defining relations of $\mathcal{C}_G^f$ presented in Corollary 4.16.

**5.** Our experiments with Maculay2 show that for many graphs $G$ and functions $f$, the Hilbert function of the algebra $\mathcal{C}_G^f$ is log-concave. In an earlier preprint version of this paper we conjectured that the log-concavity of the Hilbert functions holds for all graphs $G$ and all non-degenerate functions $f$.

Recently a truly remarkable proof of this fact in the graded case (i.e., $f = u$) was found in [1].

However, the latter conjecture does not hold in its complete generality. In fact, for the complete graph $G = K_5$ and every function considered in § **??** log-concavity fails. For example, for $f = u + u^4$, $\mathcal{H}_G^f = (1, 5, 14, 30, 53, 73, 60, 41, 9, 4, 1)$ which is not log-concave, since $9^2 = 81 < 41 \cdot 4 = 164$.

It seems interesting to find families of graphs $G$ and nondegenerate functions $f \neq u$ for which the Hilbert function $\mathcal{H}_G^f$ is log-concave and, in particular, to prove log-concavity for chain and cycle graphs in case $f = u + u^2$.

**6.** If $G_1$ is a subgraph of $G_2$ then the parameter space of $\mathcal{A}_{G_1}$ is an obvious subspace of the parameter space $\mathcal{A}_{G_2}$. Therefore in addition to the Hilbert stratification of $\mathcal{A}_{G_1}$ one has the second stratification on $\mathcal{A}_{G_1}$ induced from $\mathcal{A}_{G_2}$. While for some graphs the second stratification subdivides the first one, in general, this is not the case as shown by the graphs $K_5 - e$ and $K_5 - \{e, f\}$. A similar inclusion of the parameter spaces happens when when pass from a graph to its minor. A natural question to ask is what happens with the Hilbert stratification when we pass to subgraphs or minors of a given graph.

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