REAL POLYNOMIALS WITH CONSTRAINED REAL DIVISORS. I.
FUNDAMENTAL GROUPS

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Abstract. In the late 80s, V. Arnold and V. Vassiliev initiated the topological study
of the space of real univariate polynomials of a given degree $d$ and with no real roots
of multiplicity exceeding a given positive integer. Expanding their studies, we consider
the spaces $\mathcal{P}_{d}^{\Theta}$ of real monic univariate polynomials of degree $d$ whose real divisors avoid
sequences of root multiplicities, taken from a given poset $\Theta$ of compositions which is closed
under certain natural combinatorial operations.

In this paper, we calculate the fundamental group of $\mathcal{P}_{d}^{\Theta}$ and of some related topolog-
ical spaces. We show that the groups $\pi_{1}(\mathcal{P}_{d}^{\Theta})$ are free with rank bounded from above by
a quadratic function in $d$. The mechanism that generates the groups $\pi_{1}(\mathcal{P}_{d}^{\Theta})$ is similar to
the one that produces the braid group as the fundamental group of the space of complex
monic degree $d$ polynomials with no multiple roots.

We further show that the groups $\pi_{1}(\mathcal{P}_{d}^{\Theta})$ admit an interpretation as special bordisms
of immersions of 1-manifolds into the cylinder $S^{1} \times \mathbb{R}$, whose images avoid the tangency
patterns from $\Theta$ with respect to the generators of the cylinder.

1. Introduction

1.1. Motivation and Outline of Results. In [Ar], V. Arnold proved the following The-
orems A–D, which were later generalized by V. Vassiliev, see [Va]. These results are the
main source of motivation and inspiration for our study. In the formulations of these theo-
rems, we keep the original notation of [Ar], which we will abandon later on. In what follows,
theorems, conjectures, etc., labeled by letters, are borrowed from the existing literature,
while those labeled by numbers are hopefully new.

**Theorem A.** The fundamental group of the space of smooth functions $f : S^{1} \to \mathbb{R}$ without
critical points of multiplicity higher than 2 on a circle $S^{1}$ is isomorphic to the group of
integers $\mathbb{Z}$.

The space of smooth functions $f : \mathbb{R} \to \mathbb{R}$ without critical points of multiplicity higher
than 2 and which, for arguments $|x| > 1$, coincide either with $x$ or with $x^{2}$ also have the
fundamental group $\mathbb{Z}$.

**Theorem B.** The latter fundamental group is naturally isomorphic to the group $\mathcal{B}$ of $A_{3}$-
obordism classes of embedded closed plane curves without vertical $^{1}$ tangential inflections.

The generator of $\mathcal{B}$ is shown as the kidney-shaped loop in Figure 4(a).

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1In our convention, the curves in the $tx$-plane do not have inflections with respect to the coordinate line
$t = const$. 


\[ \text{Date: January 29, 2020.} \]
Remark 1.1. The multiplication of the cobordism classes in $\mathcal{B}$ is defined as the disjoint union of curves, embedded in the half-planes $\{(t, x) | t < 0\}$ and $\{(t, x) | t > 0\}$, and the inversion is the change of sign of $t$.

For $1 \leq k \leq d$, let $G^d_k$ be the space of real monic polynomials $x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{R}[x]$ with no real roots of multiplicity greater than $k$.

**Theorem C.** If $k < d < 2k + 1$, then $G^d_k$ is diffeomorphic to the product of a sphere $S^{k-1}$ by an Euclidean space. In particular, for all $i$ and $k < d < 2k + 1$,

$$\pi_i(G^d_k) \simeq \pi_i(S^{k-1})$$

An analogous result holds for the space of polynomials whose sum of roots vanishes, i.e., polynomials with the vanishing coefficient $a_{d-1}$.

**Theorem D.** The homology groups with integer coefficients of the space $G^d_k$ are nonzero only for dimensions which are the multiples of $k-1$ and less or equal to $d$. For $(k-1)r \leq d$, we have

$$H_{r(k-1)}(G^d_k) \simeq \mathbb{Z}.$$  

The main goal of this paper and its sequel [KSW] is to generalize Theorems A–D to the situation where the multiplicities of the real roots avoid a given set of patterns $\Theta$. In our more general situation, the fundamental group of such polynomial spaces can be non-trivial and deserves a separate study, which is carried out below. We will see that the mechanism by which these fundamental groups are generated is similar to the one that produces the braid groups as the fundamental groups of spaces of complex degree $d$ monic polynomials with no multiple roots.

Besides the studies of V. Arnold [Ar] and V. Vassiliev [Va], our second major motivation for this paper comes from results of the first author connecting the cohomology of spaces of real polynomials that avoid certain patterns of root multiplicities with certain characteristic classes, arising in the theory of traversing flows, see [Ka], [Ka1], and [Ka3]. For traversing vector flows on compact manifolds $X$ with boundary $\partial X$ and with a priori forbidden tangency patterns of their trajectories to $\partial X$, the spaces of polynomial avoiding the same patterns play a fundamental role. The role is similar to the one played by Graßmannians in the category of vector bundles.

Let $\mathcal{P}_d$ denote the space of real monic univariate polynomials of degree $d$. Given a polynomial $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathcal{P}_d$, we define its real divisor $D_\mathbb{R}(P)$ as the multiset

$$x_1 = \cdots = x_{i_1} < x_{i_1+1} = \cdots = x_{i_1+i_2} < \cdots < x_{i_{k-1}+1} = \cdots = x_{i_k}$$

of the real roots of $P(x)$. The tuple $\omega = (\omega_1, \ldots, \omega_k)$ is called the (ordered) real root multiplicity pattern of $P(x)$. Let $\mathcal{R}^\omega_d$ be the set of all polynomials with root multiplicity pattern $\omega$, and let $\overline{\mathcal{R}}^\omega_d$ be the closure of $\mathcal{R}^\omega_d$ in $\mathcal{P}_d$.

For a given collection $\Theta$ of root multiplicity patterns, we consider the union $\mathcal{P}^\Theta_d$ of the subspaces $\mathcal{R}^\omega_d$ over all $\omega \in \Theta$. We denote by $\mathcal{P}^{\Theta\overline{\Theta}}_d$ its complement $\mathcal{P}_d \setminus \mathcal{P}^\Theta_d$. We restrict our studies to the case when $\mathcal{P}^\Theta_d$ is closed in $\mathcal{P}_d$. We then call such $\Theta$ closed.
We observe in Lemma 2.1 that, for a most closed Θ, the space \( P_d^{\Theta} \) is contractible. Thus it makes more sense to consider its one-point compactification \( \bar{P}_d^{\Theta} \). For a closed Θ, the latter is the union of the one-point compactifications \( \bar{R}_d^{\omega} \) of the \( R_d^{\omega} \) for \( \omega \in \Theta \) with the points at infinity identified. By Alexander duality on \( \bar{P}_d^{\Theta} \approx S^d \), we get in reduced (co)homology

\[
H^j(\bar{P}_d^{\Theta}; \mathbb{Z}) \approx H_{d-j-1}(P_d^{\Theta}; \mathbb{Z}).
\]

This implies that the spaces \( P_d^{\Theta\omega} \) and \( \bar{P}_d^{\Theta\omega} \) carry the equivalent (co)homological information.

**Example 1.2.** For Θ comprising all \( \omega \)'s with at least one component greater than or equal to \( k \), we have that Θ is closed and \( P_d^{\Theta\omega} \cong C_k^d \) (see Theorem C and Theorem D).

In this paper and its sequel [KSW], for a closed Θ, we aim at describing the topology of \( \bar{P}_d^{\Theta} \) and \( \bar{P}_d^{\Theta\omega} \) in terms of combinatorial properties of Θ.

**Outline of the results:** In § 2 we prove our main result Theorem 2.14, generalizing Theorem C in the case \( k = 2 \). It states that if Θ is closed and if for all \( \omega = (\omega_1, \ldots, \omega_l) \in \Theta \) we have \( |\omega'| := \omega_1 + \cdots + \omega_l - \ell > 1 \), then the fundamental group \( \pi_1(P_d^{\Theta\omega}) \) is a free group. In § 2.3.1 we preview a result from [KSW], where we will show that the number of its generators can be computed in terms of a certain combinatorial differential complex \( (\mathbb{Z}[\Theta_{(d)}], \partial) \) whose homology coincides with \( H_*(P_d^{\Theta\omega}; \mathbb{Z}) \). Thus the rank of \( \pi_1(P_d^{\Theta\omega}) \) equals the rank of the abelian group \( \bar{H}_{d-2}(P_d^{\Theta\omega}; \mathbb{Z}) \cong \bar{H}_{d-2}(\mathbb{Z}[\Theta_{(d)}], \partial) \). In Corollary 2.18, we provide an analog of Theorem A and show that the fundamental group of the space of real monic polynomials of a fixed odd degree \( d > 1 \) with no real critical points of multiplicity higher than 2 is isomorphic to \( \mathbb{Z} \).

When Θ consists of all \( \omega \)'s with the property \( |\omega'| \geq 2 \), then we show in Theorem 2.4, without using the results from [KSW], that the number of generators of the free group \( \pi_1(P_d^{\Theta\omega}) \) is equal to \( \frac{d(d-2)}{4} \) for an even \( d \), and to \( \frac{(d-1)^2}{4} \) for an odd \( d \). Moreover, we show that in this case \( P_d^{\Theta\omega} \) is homotopy equivalent to a wedge of circles and hence a \( K(\pi, 1) \)-space.

In § 3 a cobordism theory is developed which allows to prove an analog of Theorem B. In Theorem 3.3 it is shown that the free group \( \pi_1(P_d^{\Theta\omega}) \), where Θ consists of all \( \omega \)'s with the property \( |\omega'| \geq 2 \), admits an interpretation as special bordisms of immersions of 1-manifolds into the cylinder \( S^1 \times \mathbb{R} \), immersions whose images avoid the tangency patterns from Θ with respect to the generators of the cylinder.

### 1.2. Cell structure on the space of real univariate polynomials

Let us first introduce a well-known stratification of the space of real univariate polynomials of a given degree.

For any real polynomial \( P(x) \), we have already defined its real divisor \( D_\mathbb{R}(P) \), i.e. the ordered set of its real zeros, counted with their multiplicities. Denote by \( D_\mathbb{C}(P) \) its complex conjugation-invariant non-real divisor in \( \mathbb{C} \setminus \mathbb{R} \), i.e. the set of its non-real roots with their multiplicities. (The standard divisor \( D(P) \) of \( P(x) \) is the multiset of all its complex roots, i.e. \( D(P) = D_\mathbb{R}(P) + D_\mathbb{C}(P) \)).
We have already associated to a polynomial \( P(x) \in \mathbb{R}[x] \) its real root multiplicity pattern \( \omega_P := (\omega_1, \ldots, \omega_\ell) \). The combinatorics of multiplicity patterns will play the key role in our investigations. Let us fix some terminology and notation for multiplicity patterns.

**Definition 1.3.** A sequence \( \omega = (\omega_1, \ldots, \omega_\ell) \) of positive integers is called a composition of the number \(|\omega| := \omega_1 + \cdots + \omega_\ell\). We also allow the empty composition \( \omega = (\) of the number \(|()| = 0\). We call \(|\omega|\) the norm, and \(|\omega|' := |\omega| - \ell\) the reduced norm of \(\omega\).

Evidently, for a given composition \(\omega\), the stratum \(\mathcal{P}_d^\omega\) is empty if and only if either \(|\omega| > d\), or \(|\omega| \leq d\) and \(|\omega| \neq d\) mod 2.

**Notation 1.4.** We denote by \(\Omega\) the set of all compositions of natural numbers. For a given positive integer \(d\), we denote by \(\Omega_{(d]}\) the set of all compositions \(\omega\), such that \(|\omega| \leq d\) and \(|\omega| \equiv d\) mod 2. Finally, denote by \(\Omega_{(d],[\sim]_\ell}\) the subset of \(\Omega_{(d]}\), consisting of all compositions in \(\Omega_{(d]}\) whose reduced norm is greater than or equal to \(\ell\). Analogously, we define \(\Omega_{(d],[\sim]_\ell}\) as the subset of \(\Omega_{(d]}\), consisting of all compositions in \(\Omega_{(d]}\) whose reduced norm is equal to \(\ell\).

Let us define two (sequences of) operations on \(\Omega\) that will govern our subsequent considerations, see also [Ka].

The **merge operations** \(M_j : \Omega \to \Omega\) sends \(\omega = (\omega_1, \ldots, \omega_\ell)\) to the composition
\[
M_j(\omega) = (M_j(\omega)_1, \ldots, M_j(\omega)_{\ell-1}),
\]
where, for any \(j \geq \ell\), one has \(M_j(\omega) = \omega\), and for \(1 \leq j < \ell\), one has
\[
M_j(\omega)_i = \omega_i \quad \text{if} \quad i < j,
M_j(\omega)_j = \omega_j + \omega_{j+1},
M_j(\omega)_i = \omega_{i+1} \quad \text{if} \quad i + 1 < j \leq \ell - 1.
\]

Similarly, we define the **insertion operations** \(I_j : \Omega \to \Omega\) that sends \(\omega = (\omega_1, \ldots, \omega_\ell)\) to the composition
\[
I_j(\omega) = (I_j(\omega)_1, \ldots, I_j(\omega)_{\ell+1}),
\]
where for any \(j > \ell + 1\), one has \(I_j(\omega) = \omega\), and for \(1 \leq j \leq \ell + 1\), one has
\[
I_j(\omega)_i = \omega_i \quad \text{if} \quad i < j,
I_j(\omega)_j = 2,
I_j(\omega)_i = \omega_{i-1} \quad \text{if} \quad j \leq i \leq \ell + 1.
\]

The next proposition collects some basic properties of \(\mathcal{R}_d^\omega\), see [Ka, Theorem 4.1] for details.

**Proposition E.** Take \(d \geq 1\) and \(\omega = (\omega_1, \ldots, \omega_\ell) \in \Omega_{(d]}\). Then \(\mathcal{R}_d^{\omega'} \subset \mathcal{P}_d\) is an (open) cell of codimension \(|\omega|'\). Moreover, \(\mathcal{R}_d^\omega\) is the union of the cells \(\{\mathcal{R}_d^{\omega'}\}_{\omega'}\), taken over all \(\omega'\) that are obtained from \(\omega\) by a sequence of merging and insertion operations. In particular,

(a) the cell \(\mathcal{R}_d^\omega\) has (maximal) dimension \(d\) if and only if \(\omega = (1, 1, \ldots, 1)\) for \(0 \leq \ell \leq d\)
and \(\ell \equiv d\mod 2\).
(b) the cell $\mathcal{R}_{\omega}^d$ has dimension 1 if and only if $\omega = (d)$. In this case, $\mathcal{R}^{(d)} = \mathcal{R}^d = \{(x - a)^d \mid a \in \mathbb{R}\}$.

Geometrically speaking, if a point in $\mathcal{R}_{\omega}^d$ approaches the boundary $\mathcal{R}_{\omega}^d \setminus \mathcal{R}_{\omega}^d$, then either there is at least one value of $j$ such that the distance between the $j^{th}$ and $(j + 1)^{st}$ distinct real roots goes to 0, or there are two complex-conjugate real roots that converge to a double real root, which then either is $j^{th}$ largest or adds 2 to the multiplicity of the $j^{th}$ largest real root.

The first situation corresponds to the application of the merge operation $\mathcal{M}_j$ to $\omega$, and the second one to the application of the insertion $\mathcal{I}_j$ or the application of $\mathcal{I}_j$, followed by $\mathcal{M}_j$.

Note that the norm $|\omega|$ is preserved under the merge operations, while the insert operations increase $|\omega|$ by 2 and thus preserve its parity.

By Proposition E, the merge and insert operations can be used to define a partial order "$\succ$" on the set $\Omega$ of all compositions, reflecting adjacency of the non-empty open cells $\mathcal{R}_{\omega}^d$.

**Definition 1.5.** For $\omega, \omega' \in \Omega$, we say that $\omega'$ is smaller than $\omega$ (notation "$\omega \succ \omega'$"), if $\omega'$ can be obtained from $\omega$ by a sequence of merge and insert operations $\{\mathcal{M}_j\}, j \geq 1$, and $\{\mathcal{I}_j\}, j \geq 0$. For a given $\omega \succ \omega'$, if there is no $\omega''$ such that $\omega \succ \omega'' \succ \omega'$, then we say that $\omega \succ \omega'$ is a cover relation, or that $\omega'$ is covered by $\omega$.

From now on, we will consider a subset $\Theta \subseteq \Omega$ as a poset, ordered by $\succ$. As an immediate consequence of Proposition E, we get the following statement.

**Corollary F.** For $\Theta \subseteq \Omega_{\langle d \rangle}$,

(i) $\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$ if and only if, for any $\omega \in \Theta$ and $\omega' \in \Omega_{\langle d \rangle}$, the relation $\omega' \prec \omega$ implies $\omega' \in \Theta$;

(ii) if $\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$, then the one-point compactification $\overline{\mathcal{P}}_d^\Theta$ carries the structure of a compact CW-complex with open cells $\{\mathcal{R}_{\omega}^d\}_{\omega \in \Theta}$, labeled by $\omega \in \Theta$, and the unique 0-cell, represented by the point $\bullet$ at infinity.

Recall that we have called $\Theta \subseteq \Omega_{\langle d \rangle}$ closed if $\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$. Hence F has the following immediate reformulation.

$\Theta \subseteq \Omega_{\langle d \rangle}$ is closed

for any $\omega \in \Theta$ and $\omega' \in \Omega_{\langle d \rangle}$, the relation $\omega' \prec \omega$ implies $\omega' \in \Theta$.

Now, we are in position provide a precise formulation of the main questions motivating this paper and its sequel [KSW]:

**Problem 1.6.** For a given closed poset $\Theta \subseteq \Omega_{\langle d \rangle}$,

$\triangleright$ calculate the homotopy groups $\pi_i(\overline{\mathcal{P}}_d^\Theta)$ and $\pi_i(\mathcal{P}_d^\Theta)$ in terms of the combinatorics of $\Theta$;\footnote{In full generality, this goal is as illusive as computing the homotopy groups of spheres; in fact, for special $\Theta$'s, the two problems are intimately linked, as Theorem C testifies.}
Lemma 2.1. For any poset $\Theta$ pair $(\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$ of roots of $P$, the restriction $q : P \times [0, \infty) \to P$ of the map $p$ follows.\[\square\]

In this paper we concentrate on the fundamental groups of $\overline{P}_d^\Theta$ and $P_c^\Theta$. Questions about the (co)homology of $\overline{P}_d^\Theta$ and $P_c^\Theta$ will be addressed in [KSW]. However, in this paper, we will need one special case of computation from [KSW], namely, of $H_*(\overline{P}_d^\Theta; \mathbb{Z})$ for $* = d - 2$.

Acknowledgements. The second author wants to acknowledge the financial support of his research by the Swedish Research council through the grant 2016-04416. The third author wants to thank department of Mathematics of Stockholm University for its hospitality in May 2018. He also was partially supported by an NSF grant DMS 0932078, administered by the Mathematical Sciences Research Institute while the author was in residence at MSRI during the complimentary program 2018/19. During both visits substantial progress on the project was made.

We also would like to thank the referee for the suggestions which led to an improvement of our presentation and sharpening of our results.

2. Computing $\pi_1(\overline{P}_d^\Theta)$ and $\pi_1(P_c^\Theta)$

2.1. Homotopy type of $P_c^\Theta$ and the fundamental group $\pi_1(\overline{P}_d^\Theta)$. The following simple statement gives us a start on the homotopy type of the polynomial spaces under consideration. In the proof, we use the map $q : P \times [0, \infty) \to P$ which sends each pair $(P(x), \lambda)$, where $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$ and $\lambda \in [0, 1]$, to the polynomial $x^d + a_{d-1} \lambda x^{d-1} + \cdots + a_0 \lambda^d$. Hence, this transformation amounts to the multiplication of all roots of $P(x)$ by $\lambda$.

Lemma 2.1. For any poset $\Theta \subseteq \Omega_{(d]}$ that contains $(d)$, the space $P_c^\Theta \subseteq P$ is contractible. In particular, for any closed poset $\Theta \subseteq \Omega_{(d]}$, the space $P_c^\Theta$ is contractible.

Proof. For $P(x) \in P_c^\Theta$ and $\lambda \geq 0$, the roots of $q(P(x), \lambda)$ are the roots of $P(x)$, being multiplied by $\lambda$. So it follows that $q(P(x), \lambda) \in P_c^\omega$ for $P(x) \in P_c^\omega$. Obviously, $q(P(x), 0) = x^d \in P_c^\Theta$. Thus, since $(d) \in \Theta$, the restriction of $q$ to $[0, 1] \times P_c^\Theta$ is well-defined homotopy between the identity map and the constant map that sends $P_c^\Theta$ to $x^d$. The assertion now follows.\[\square\]

In contrast to $P_c^\Theta$, its one-point compactification $\overline{P}_d^\Theta$ often has non-trivial topology for closed posets $\Theta$. A simple example of such a situation is $P_c^{\Omega_{(d]}} = P_c^\omega \cong S^d$. Other examples, including the case treated in Theorem C and Theorem D, show that $P_c^{\Theta}$ can have non-trivial topology as well.

Besides the map $q$, the following map $p$ has been frequently used in the literature in the study of the topology of spaces of univariate polynomials. The map $p : P \to P$ sends $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{R}[x]$ to $P(x - \frac{a_{d-1}}{d})$. The map $p$ preserves the stratification $(\overline{R}_d^{\omega})_\omega$ and is a fibration with the fiber $\mathbb{R}$. Thus, for a closed poset $\Theta \subseteq \Omega_{(d]},$ the restriction $p|_{P_c^\Theta} : P_c^\Theta \to P_c^\Theta$ is also a fibration with fiber $\mathbb{R}$. Its image $P_c^{\Theta_0}$ consists
of all polynomials in $\mathcal{P}_d^\Theta$ with the vanishing coefficient at $x^{d-1}$, i.e. with vanishing root sum. Therefore, we get a homeomorphism $\mathcal{P}_d^\Theta \cong \tilde{\Sigma} \mathcal{P}_d^\Theta$. Here $\tilde{\Sigma} X$ denotes the reduced suspension of a space $X$.

Since $q(P(x), \lambda)$ amounts to multiplying the roots of $P(x)$ by $\lambda$, their sum is preserved. In particular, $q$ preserves $\mathcal{P}_d^\Theta$. By the map $a_0 + \cdots + a_{d-1}x^{d-1} + x^d \mapsto (a_0, \ldots, a_{d-1})$ we identify $\mathcal{P}_d$ with Euclidean $d$-space. Clearly, for $\lambda \geq 0$ and fixed $0 \neq P(x) \in \mathcal{P}_d$, the norm $\|q(P(x), \lambda)\|$ is strictly monotone in $\lambda$. Since $\|q(P(x), 0)\| = 0$, there is a unique $\lambda_P > 0$ for which $\|q(P(x), \lambda_P)\| = 1$. Let $S^{d-1} \subset \mathcal{P}_d$ be the unit sphere. Therefore, on $\mathcal{P}_d^\Theta \setminus \{0\}$ the map $P(x) \mapsto q(P(x), \lambda_P)$ is a deformation retraction to the closed subspace $S^{d-1} \cap \mathcal{P}_d^\Theta$ of $S^{d-1}$. Thus, we get a homeomorphism $\tilde{\mathcal{P}}_{d,0}^\Theta \cong \tilde{\Sigma} (S^{d-1} \cap \mathcal{P}_d^\Theta)$. Note, that here we consider $\tilde{\Sigma} \emptyset$ as the discrete two-point space. This analysis implies the following claim.

**Theorem 2.2.** For any closed poset $\Theta \subseteq \Omega(d)$, we get $\pi_1(\mathcal{P}_d^\Theta) = 0$, unless $\Theta = \{(d)\}$. If $\Theta = \{(d)\}$ then $\mathcal{P}_d^\Theta \cong S^1$.

**Proof.** By the arguments preceding the theorem, we have $\tilde{\mathcal{P}}_{d,0}^\Theta \cong \tilde{\Sigma} \Sigma (S^{d-1} \cap \mathcal{P}_d^\Theta)$. Therefore, $\mathcal{P}_d^\Theta$ is simply connected, unless $S^{d-1} \cap \mathcal{P}_d^\Theta$ is empty. But this can only happen if $\Theta = \{(d)\}$. It is easily seen that $\hat{\mathcal{R}}_d^{(d)} = \{(x - \alpha)^d \mid \alpha \in \mathbb{R}\} \cong \mathbb{R}$. Hence its one-point compactification is $S^1$.

Note, that the argument employed in the proof of Theorem 2.2 also implies that

$$H_i(S^{d-1} \cap \mathcal{P}_d^\Theta, \mathbb{Z}) \approx H_{i+2}(\hat{\mathcal{P}}_d^\Theta, \mathbb{Z})$$

for all $i \geq 0$.

### 2.2. The fundamental group of the complement of the codimension two skeleton.

In this section, we study and determine the fundamental group $\pi_1(\mathcal{P}_d^{\Omega(d), \lvert \cdot \rvert \geq 2})$. Recall, that we collect in $\Omega(d), \lvert \cdot \rvert \geq 2$ all $\omega \in \Omega(d)$ such that $\lvert \omega \rvert' \geq 2$. Thus $\mathcal{P}_d^{\Omega(d), \lvert \cdot \rvert \geq 2}$ is the complement of the codimension 2 skeleton of $\mathcal{P}_d$ in our cellulation.

We associate a graph $\mathfrak{G}_d$ with the cellular space $\mathcal{P}_d^{\Omega(d), \lvert \cdot \rvert \geq 2}$. The set of vertices of $\mathfrak{G}_d$ is the union of the sets $\Omega(d), \lvert \cdot \rvert = 1$ and $\Omega(d), \lvert \cdot \rvert = 0$. We connect vertices $\omega \in \Omega(d), \lvert \cdot \rvert = 1$ and $\omega' \in \Omega(d), \lvert \cdot \rvert = 0$ by an edge $\{\omega, \omega'\}$, if the $(d - 1)$-cell $\hat{\mathcal{R}}_d^{\omega'}$ lies in the boundary of the closure $\hat{\mathcal{R}}_d^{\omega'}$ of $\hat{\mathcal{R}}_d^{\omega}$. In particular, the edges of $\mathfrak{G}_d$ correspond to single insertion and merging operations, applied to compositions from $\Omega(d), \lvert \cdot \rvert = 0$. As usual, we identify the graph $\mathfrak{G}_d$ with the 1-dimensional simplicial complex, defined by its vertices and edges (see Figure 3 for the example of $\mathfrak{G}_6$).

We embed the graph $\mathfrak{G}_d$ in $\mathcal{P}_d^{\Omega(d), \lvert \cdot \rvert \geq 2}$ by mapping the vertex of $\mathfrak{G}_d$, labeled by $\omega \in \Omega(d), \lvert \cdot \rvert = 1$, to a preferred point $w_\omega$ in the $(d - 1)$-cell $\hat{\mathcal{R}}_d^{\omega}$ and the vertex of $\mathfrak{G}_d$, labeled by $\omega \in \Omega(d), \lvert \cdot \rvert = 0$, to a preferred point $e_\omega$ in the $d$-cell $\hat{\mathcal{R}}_d^{\omega}$. Then we identify each edge $\{\omega, \omega'\}$ of $\mathfrak{G}_d$, where $\omega \in \Omega(d), \lvert \cdot \rvert = 1$, $\omega' \in \Omega(d), \lvert \cdot \rvert = 0$, with a smooth path $[w_\omega, e_\omega]$ such that the semi-open segment $(w_\omega, e_\omega) \subset \hat{\mathcal{R}}_d^{\omega}$. In addition, we can choose the paths so that $[w_{\omega_1}, e_{\omega_1}] \cap [w_{\omega_2}, e_{\omega_2}] = e_\omega$ for any pair $\omega_1, \omega_2 < \omega'$ and $\omega_1 \neq \omega_2$. Moreover, for each
ω ∈ Ω(d,|·|'=1), we may arrange for the two paths, [wω, eω] and [wω, eω'], to share the tangent vector at their common end wω, so that the path [eω', eω'] is transversal to the hypersurface \( \mathbb{R}^d_x \) at wω. This construction produces an embedding \( E : \mathcal{G}_d \to \mathcal{P}_d^{\Omega(d,|·|'\geq 2)} \).

In what follows, we do not distinguish between \( \mathcal{G}_d \) and its image \( E(\mathcal{G}_d) \subseteq \mathcal{P}_d^{\Omega(d,|·|'\geq 2)} \).

**Lemma 2.3.** The graph \( \mathcal{G}_d \) is homotopy equivalent to a wedge of \( \frac{d(d-2)}{4} \) circles if \( d \) is even, and of \( \frac{(d-1)^2}{4} \) circles if \( d \) is odd.

**Proof.** A simple calculation shows \( |\Omega(d,|·|'=0)| = \left\lfloor \frac{d}{2} \right\rfloor + 1 \), and

\[
|\Omega(d,|·|'=1)| = \sum_{k=1}^{\lfloor \frac{d}{2} \rfloor} (2k - 1) = \left\lfloor \frac{d}{2} \right\rfloor^2.
\]

For each \( k \in [2, d] \), the vertex \((1, \ldots, 1) \in \Omega(d,|·|'=0)\) is contained in the \( k - 1 \) edges that lead to the vertices \((1, \ldots, 1, 2, 1, \ldots, 1) \in \Omega(d,|·|'=1), \) where \( 0 \leq s \leq k - 2 \leq d - 2 \), and in the \( k + 1 \) edges that lead to \((1, \ldots, 1, 2, 1, \ldots, 1) \in \Omega(d,|·|'=1), \) where \( 0 \leq s \leq k \leq d - 1 \).

For \( d \) even, each case yields \( \frac{d^2}{4} \) edges. Since the graph \( \mathcal{G}_d \) is easily seen to be connected, it is homotopy equivalent to a wedge of circles. Now a simple calculation of the Euler-characteristic \( \chi(\mathcal{G}_d) \) yields \( 1 - \left( \frac{d}{2} + 1 + \frac{d^2}{4} \right) + 2 \frac{d^2}{4} = \frac{d(d-2)}{4} \) circles. The calculation for odd \( d \) is analogous. \( \square \)

Our next result is inspired by Arnold’s Theorem A. In Figure 1, for \( d = 6 \), we illustrate the result by exhibiting the cell structure in \( \mathcal{P}_d^{\Omega(6,|·|'\geq 2)} \) and its graph \( \mathcal{G}_6 \).

**Theorem 2.4.** The space \( \mathcal{P}_d^{\Omega(d,|·|'\geq 2)} \) is homotopy equivalent to a wedge of \( \frac{d(d-2)}{4} \) circles for \( d \equiv 0 \mod 2 \), and to a wedge of \( \frac{(d-1)^2}{4} \) circles for \( d \equiv 1 \mod 2 \).

In particular, the fundamental group \( \pi_1(\mathcal{P}_d^{\Omega(d,|·|'\geq 2)}) \) is the free group on \( \frac{d(d-2)}{4} \) for even \( d \) and \( \frac{(d-1)^2}{4} \) generators for odd \( d \), and \( \mathcal{P}_d^{\Omega(d,|·|'\geq 2)} \) is the corresponding \( K(\pi, 1) \)-space.

**Proof.** As an open subset of \( \mathbb{R}^d \), the space \( \mathcal{P}_d^{\Omega(d,|·|'\geq 2)} \) is paracompact.

Let us consider a finite open cover \( \mathcal{X} := \{X_\omega\}_{\omega \in \Omega(d,|·|'=1)} \) of the space \( \mathcal{P}_d^{\Omega(d,|·|'\geq 2)} \). By definition, each set \( X_\omega \) consists of the \((d-1)\)-cell \( \mathbb{R}^d_\omega \) union with the two adjacent \( d \)-cells that contain \( \mathbb{R}^d_\omega \) in their boundary.

Each \( X_\omega \) is open in \( \mathcal{P}_d^{\Omega(d,|·|'\geq 2)} \). Indeed, any point \( x \in X_\omega \) either lies in: (1) one of the two \( d \)-cells and thus has an open neighborhood in \( \mathcal{P}_d^{\Omega(d,|·|'\geq 2)} \) (contained in that \( d \)-cell), or (2) \( x \in \mathbb{R}^d_\omega \), in which case it has on open neighborhood \( X_\omega \) in \( \mathcal{P}_d^{\Omega(d,|·|'\geq 2)} \).

By [Ka, Lemma 2.4] the attaching maps \( \phi : D^d \to \mathcal{P}_d \) of the \( d \)-cells are injective on the \( \phi \)-preimage of each open \((d-1)\) cell in \( \mathcal{P}_d \). This implies that \( X_\omega \) retracts to the \( \mathbb{R}^d_\omega \), which,
in turn, is contractible. For \( i \geq 2 \) and for pairwise distinct compositions \( \omega_1, \ldots, \omega_i \in \Omega_{d, k} \), the intersection \( \bigcap_{j=1}^{i} X_{\omega_j} \) is either empty, or is one of the open \( d \)-cells \( \mathcal{R}_{d, k}^{\omega_j} \) for some \( \omega \in \Omega_{d, k} \). It follows that, for \( i \geq 1 \) and for compositions \( \omega_1, \ldots, \omega_i \in \Omega_{d, k} \), the intersection \( \bigcap_{j=1}^{i} X_{\omega_j} \) is either empty or contractible.

The preceding arguments show that the assumption of [Ha, Corollary 4G.3] are satisfied for the open covering \( \mathcal{X} \) of \( P^\mathcal{C}_{d,k} \). Hence \( P^\mathcal{C}_{d,k} \) is homotopy equivalent to the nerve \( N_X \) of the covering \( \mathcal{X} \). We can identify \( N_X \) with the simplicial complex, whose simplices are the non-empty subsets \( A \) of \( \Omega_{d, k} \) such that \( \bigcap_{\omega \in A} X_{\omega} \) is non-empty. So the maximal simplices of the nerve \( N_X \) are in bijection with the elements of \( \Omega_{d, k} \).

The maximal simplex, corresponding to \( \omega \in \Omega_{d, k} \), contains all \( \omega \in \Omega_{d, k} \) for which \( \mathcal{R}_{d, k}^{\omega} \subseteq X_{\omega} \). The intersection of the two maximal simplices, corresponding to \( \omega, \omega'' \in \Omega_{d, k} \), is labeled by all \( \omega \in \Omega_{d, k} \) for which there are edges from \( \omega \) to \( \omega' \) and from \( \omega'' \) to \( \omega \) in \( \mathcal{G}_d \).

The graph \( \mathcal{G}_d \) can be covered by 1-dimensional subcomplexes \( Y_{\omega} \), \( \omega \in \Omega_{d, k} \), where \( Y_{\omega} \) is the union of the two edges in \( \mathcal{G}_d \), containing \( \omega \). It is easy to check that the nerve of this covering \( Y := \{ Y_{\omega} \}_{\omega \in \Omega_{d, k}} \) is again \( N_X \). In fact, under the embedding \( \mathcal{E} : \mathcal{G}_d \to P^\mathcal{C}_{d,k} \), one has \( Y_{\omega} = X_{\omega} \cap \mathcal{E}(\mathcal{G}_d) \).

By [Bj, Theorem 10.6], the nerve \( N_Y \) and the graph \( \mathcal{G}_d \) are homotopy equivalent.

Moreover, by the proof of [Ha, Corollary 4G.3], the following claim is valid. Consider an embedding \( Y \hookrightarrow X \) of a paracompact space \( Y \) into a paracompact space \( X \) and a locally finite open covering \( \mathcal{X} = \{ X_{\alpha} \}_{\alpha} \) of \( X \). Put \( \mathcal{Y} = \{ Y_{\alpha} := X_{\alpha} \cap Y \}_{\alpha} \). If, for any nonempty intersection \( \bigcap_{\alpha} X_{\alpha_i} \), the intersection \( \bigcap_{\alpha} Y_{\alpha_i} \neq \emptyset \), and both intersections are contractible, then the nerves \( N_X \) and \( N_Y \) are naturally isomorphic (as simplicial complexes), and \( Y \hookrightarrow X \) is a homotopy equivalence.

We conclude that the embedding \( \mathcal{E} : \mathcal{G}_d \to P^\mathcal{C}_{d,k} \) is a homotopy equivalence. The result now follows from Lemma 2.3.

\[ \square \]

**Remark 2.5.** Observe that \( \mathcal{G}_d \) splits naturally into \( \lfloor \frac{d}{2} \rfloor \) subgraphs \( \mathcal{G}_{d,i} \), each containing only two vertices of the form \( (1, \ldots, 1) \) and \( (1, \ldots, 1) \) and \( i+1 \) edges, labeled by \( (1, \ldots, 1, 2) \), \( (1, \ldots, 1, 2, 1) \), \ldots, \( (2, 1, \ldots, 1) \). Here \( i = 0, 2, \ldots, d \) for \( d \) even, and \( i = 1, 3, \ldots, d \) for \( d \) odd. Since each vertex \( (1, \ldots, 1) \) is a cut vertex (i.e. its removal disconnects \( \mathcal{G}_d \) ), \( \pi_1(G) \) is the free product \( \prod_i \pi_1(\mathcal{G}_{d,i}) \). Notice that \( \pi_1(\mathcal{G}_{d,i}) \) is a free group on \( i \) generators.

Let us now give a slightly different and perhaps more natural interpretation of the computation of \( \pi_1(P^\mathcal{C}_{d,k}) \). We use the notations from the proof of Theorem 2.4.
Figure 1. The cell structure in the open domain $P^c_{6, \geq 2} \subset P_6$, defined in §1.1, and the graph $\mathfrak{S}_6$, dual to it. The diagram shows some intermediate stages of the retraction of $P^c_{6, \geq 2}$ onto $\mathfrak{S}_6$.

Figure 2. Diagram A shows the graph $\mathfrak{S}_6$, drawn as a poset, where the edges indicate the elementary merge and insert operations. Diagram B shows the same graph in the way that makes the count of its fundamental cycles easy.

For $2k \leq d$ and $i \in [0, k - 1]$, consider the codimension one “wall” $W_{k,i} := \tilde{R}^i_d$, where $\omega = (1, \ldots, 1, 2, 1, \ldots, 1) \in \Omega_{(d), \sim'}=1$. The walls divide $P^c_{d, \sim'}=2$ into open $d$-cells $E_{2k} := \tilde{R}^i_d$, where $\omega = (1, \ldots, 1) \in \Omega_{(d), \sim'=0}$ and $2k \in [0, d]$. 
We orient each wall $W_{k,i}$ in such a way that crossing it in the preferred normal direction increases the number of simple real roots by 2.

Consider an oriented loop $\gamma : S^1 \to \mathcal{P}_d^{c\Omega[|d|]|\sim|\geq 2}$. By the general position arguments, we may assume that $\gamma$ is smooth and transversal to each wall $W_{k,i}$. In particular, since $S^1$ is compact, the intersection of $\gamma(S^1)$ with each wall $W_{k,i}$ is a finite set. As we move along $\gamma$, we record each transversal crossing $a_\omega \in \gamma \cap \mathcal{R}^d_{k,i} := \gamma \cap W_{k,i}$ with the $+$ sign if $\gamma$ crosses the wall in the positive direction, and with the $-$ sign otherwise.

In other words, let us introduce the alphabet $A^\pm$ whose letters $a_\omega$’s without signs are indexed by the $\omega$’s that have a single entry 2 (see Figure 1 and Figure 2). Equivalently, these letters $a_\omega$’s are the labels of the edges of the graph $\mathcal{G}_d$. The signs capture possible orientations of these edges (or the coorientations of the walls $W_{k,i}$, as described above). The sign “$+$” corresponds to the orientation from the vertex with less ones towards the vertex with more ones, and the sign “$-$” corresponds to the opposite orientation.

For $d$ even, the total number of letters $a_\omega^\pm$’s in $A^\pm$ is equal to $2(1+3+5+\cdots+(d-1)) = \frac{d^2}{2}$; for $d$ odd, this number equals $2(2+4+6+\cdots+(d-1)) = \frac{d^2-1}{2}$. As we mentioned already, each (generic) loop $\gamma : S^1 \to \mathcal{P}_d^{c\Omega[|d|]|\sim|\geq 2}$ generates a cyclic word $\alpha(\gamma)$ in the alphabet $A^\pm$.

Although the alphabet $A^\pm$ contains more letters than there are generators in $\pi_1(\mathcal{P}_d^{c\Omega[|d|]|\sim|\geq 2})$ and not all words in $A^\pm$ correspond to closed loops, it is easy to fix this problem, as we will do below. The alphabet $A^\pm$ is very natural in our context and is especially useful in describing the relations that define $\pi_1(\mathcal{P}_d^{c\Theta})$ for an arbitrary closed poset $\Theta \subset \Omega[|d|]$ (see Lemma 2.11).

**Definition 2.6.** A word $w$ in the alphabet $A^\pm$ is called reducible if it contains at least one occurrence of either a pair of consecutive letters $(a_\omega^+, a_\omega^-)$ or $(a_\omega^-, a_\omega^+)$. By removing such a pair of letters from $w$, we obtain its reduction. If $\bar{w}$ is obtained from $w$ by a sequence of reductions and cannot be further reduced then we call $\bar{w}$ a complete reduction of $w$. We say that two words $w_1$ and $w_2$ are equivalent if they share a complete reduction.

Using standard arguments from the theory of words one can easily check that the complete reduction of a word is uniquely defined.

**Definition 2.7.** For $d$ even, we say that a word $w$ in the alphabet $A^\pm$ is admissible if it satisfies the following two conditions:

(a) it starts with the letter $(2)^+$ and ends with the letter $(2)^-$;
(b) any two consecutive letters $(a_{\omega_1}^\pm, a_{\omega_2}^\pm)$ have the property that the number of 1’s in $\omega_1$ and $\omega_2$ either coincide or differs by two. In the former case, the signs of the letters should be different and in the latter case, the signs should be as follows. If $\omega_1$ has less ones than $\omega_2$, then we only allow the pair $(a_{\omega_1}^+, a_{\omega_2}^-)$ and if $\omega_1$ has more ones than $\omega_2$, then we only allow the pair $(a_{\omega_1}^-, a_{\omega_2}^+)$.

For $d$ odd, condition (b) is exactly the same, while condition (a) is substituted by:

(c) $w$ starts with the letters $(12)^+$ or with $(21)^+$ and ends with the letters $(12)^-$ or $(21)^-$.
Examples of admissible words:

\[ w_1 = \{(2)^+, (112)^+, (121)^-, (211)^+, (1112)^+, (1211)^-, (121)^-, (2)^-\}; \]

\[ w_2 = \{(21)^+, (1112)^+, (112111)^+, (111211)^-, (1211)^-, (121)^-\}. \]

We observe that, for any admissible word \( w \), its complete reduction \( \bar{w} \) is admissible as well.

**Lemma 2.8.** The set of equivalence classes of admissible words over the alphabet \( A^\pm \) form a group \( G_d \) with respect to concatenation if one takes the equivalence class of the empty word as the unit element in \( G_d \).

**Proof.** The proof is straightforward: indeed, it is easy to check that concatenation of two admissible words is again an admissible word. The associativity law for concatenations is trivially satisfied. Finally, for each admissible word \( w \), one can determine its inverse by reading \( w \) from right to left and reversing the signs of all its letters. \( \square \)

Let us now reinterpret \( \pi_1(\mathcal{P}^{c\Theta}_{d}[\sim']_{\geq 2}) \) in terms of words over the alphabet \( A^\pm \). To do this, let us fix a base point for this fundamental group to lie in the connected space of polynomials with the least possible number of real roots. In other words, for \( d \) even, we put our base point among the polynomials with no real roots and, for \( d \) odd, among the polynomials with one real root. Let us denote the base point by \( p_{\text{base}} \). In terms of the graph \( G_d \) it means that for \( d \) even, we start at the vertex (0) and for \( d \) odd, at the vertex (1).

**Lemma 2.9.** The homotopy classes of loops \( \gamma \subset \mathcal{P}^{c\Theta}_{d}[\sim']_{\geq 2} \) based at \( p_{\text{base}} \) are in one-to-one correspondence with the equivalence classes of admissible words in the alphabet \( A^\pm \). In other words, \( \pi_1(\mathcal{P}^{c\Theta}_{d}[\sim']_{\geq 2}) \simeq G_d \), implying that \( G_d \) is the free group on \( \frac{d(d-2)}{4} \) generators for \( d \equiv 0 \) mod 2 and on \( \frac{(d-1)^2}{4} \) generators for \( d \equiv 1 \) mod 2.

**Sketch of Proof:** By Theorem 2.4, \( \pi_1(\mathcal{P}^{c\Theta}_{d}[\sim']_{\geq 2}) \simeq \pi_1(\mathcal{G}_d) \). Our choice of \( p_{\text{base}} \in \mathcal{P}^{c\Theta}_{d}[\sim']_{\geq 2} \) exactly corresponds to the choice of the vertex (0) as the base point \( \beta_{\text{base}} \in \mathcal{G}_d \) for \( d \) even and the vertex (1) as the base vertex \( \beta_{\text{base}} \in \mathcal{G}_d \) for \( d \) odd. The above conditions for admissibility of words over the alphabet \( A^\pm \) exactly describe possible sequences of oriented edges occurring when one takes closed paths in \( \mathcal{G}_d \) starting and ending at \( \beta_{\text{base}} \). Finally, two such paths in \( \mathcal{G}_d \) are homotopy equivalent if and only if the complete reductions of their sequences of oriented edges coincide. \( \square \)

2.3. The fundamental group \( \pi_1(\mathcal{P}^{c\Theta}_{d}) \) for an arbitrary closed poset \( \Theta \subset \Omega_{(d),[\sim']_{\geq 2}} \).

We start with the following simple statement.

**Lemma 2.10.** Let \( \Theta \subset \Omega_{(d)} \) be a closed poset such that \( \Theta \subset \Omega_{(d),[\sim']_{\geq k}} \). Then the homotopy groups \( \pi_i(\mathcal{P}^{c\Theta}_{d}) \) vanish for all \( i < k - 1 \). In particular, \( \pi_1(\mathcal{P}^{c\Theta}_{d}) \) vanishes, provided \( \Theta \subset \Omega_{(d),[\sim']_{\geq 3}} \). As a special case, we have \( \pi_1(\mathcal{P}^{c\Omega}_{d}[\sim']_{\geq 3}) = 0 \).
Figure 3. The normal disks to the strata $P_6^{1311}$ (diagram A), $P_6^{2121}$ (diagram B), and $P_6^{1221}$ (diagram C). Traveling along the boundaries of these disks, gives rise to the relations as in the three bullets of Lemma 2.11. Note the two homotopic loops in diagram C: the small loop has 4 intersections with the walls, the big one only 2.

**Proof.** We observe that if $\Theta \subset \Omega_{\{d_i, |\omega|' \geq k\}}$, then $\text{codim}(P_d^\Theta, P_d) \geq k$. Therefore, by the general position argument, $\pi_i(P_d^\Theta) = 0$ for all $i < k - 1$. In particular, $\pi_1(P_d^\Theta) = 0$, provided that $\Theta \subset \Omega_{\{d_i, |\omega|' \geq k\}}$.

Further, we observe that, by Alexander duality and the Hurewicz Theorem, for any closed $\Theta \subset \Omega_{\{d_i\}}$, a minimal generating set of $\pi_1(P_d^\Theta)$ contains at least $\text{rank}(\bar{H}_{d-2}(P_d^\Theta; \mathbb{Z}))$ elements.

Given a closed poset $\Theta \subseteq \Omega_{\{d_i\}}$, whose elements have reduced norms that exceed 1, we consider two disjoint sets:

$$\Lambda(\Theta) := \Omega_{\{d_i, |\omega|' = 2\} \cap \Theta}, \quad \Lambda(\mathbf{c}\Theta) := \Omega_{\{d_i, |\omega|' = 2\} \setminus \Theta}.$$  

By definition, $\Lambda(\Theta)$ consist of $\omega$'s which either have a single entry 3 and some number of 1's or two entries 2 and some number of 1's.

For each $\tau \in \Lambda(\mathbf{c}\Theta)$, we denote by $\tau^{\triangleright}$ the set of elements in $\Omega_{\{d_i, |\omega|' = 1\}}$ that are bigger than $\tau$.

Consider a loop $\gamma$ in $P_d^{\mathbf{c}\Theta}$. It bounds a 2-disk $D$ in $P_d$. By a general position argument, we may assume that $D$ avoids all the strata $R_d^\omega$ with $|\omega|' \geq 3$ and, if it intersects a stratum $\hat{R}_d^\omega$ with $|\omega|' = 2$, then it hits it transversally. For such a stratum $\hat{R}_d^\omega$ and each intersection point $x \in D \cap \hat{R}_d^\omega$, we consider a small 2-disk $D_x \subset D$, normal to $\hat{R}_d^\omega$, and the loop $\kappa_x := \partial D_x \subset P_d^{\Omega(d_i, |\omega|' \geq 3)}$ (see Figure 3). Since $\hat{R}_d^\omega$ is contractible, the homotopy class of $\kappa_x$ or its inverse depends only on $\omega$ for all $x \in D \cap \hat{R}_d^\omega$. Therefore, $\gamma$ is homotopy equivalent in $P_d^{\mathbf{c}\Theta}$ to a product of loops $\{\hat{\kappa}_\omega, \tilde{\kappa}_\omega^{-1}\}_\omega$, where $\omega \in \Omega_{\{d_i, |\omega|' = 2\}}$ and $\tilde{\kappa}_\omega := \beta^{-1} \circ \kappa_\omega \circ \beta$ is a loop that starts at a base point $\ast$, follows a path $\beta \subset P_d^{\mathbf{c}\Theta}$ from $\ast$ to a point on $\kappa_\omega$, traverses $\kappa_\omega$ once, and returns to the base point following $\beta^{-1}$. Evidently, if $\omega \in \Lambda(\mathbf{c}\Theta)$, the loop $\kappa_\omega$ is contractible in $P_d^{\mathbf{c}\Theta}$. Thus any loop $\gamma$ is homotopy equivalent to a product of loops $\{\hat{\kappa}_\omega, \tilde{\kappa}_\omega^{-1}\}_{\omega \in \Lambda(\Theta)}$, considered as words in a new alphabet.
These considerations lead to the following generalization of Theorem 2.4, which describes \( \pi_1(\mathcal{P}_d^\Theta) \) in terms of generators and relations for an arbitrary closed poset \( \Theta \subset \Omega_{(d)} \).

**Lemma 2.11.** For any closed poset \( \Theta \subset \Omega_{(d)} \setminus \sim' \geq 2 \), the fundamental group \( \pi_1(\mathcal{P}_d^\Theta) \) is a quotient of the group \( \mathcal{G}_d \) (defined in Lemma 2.8) modulo the set \( \{ R_\tau \mid \tau \in \Lambda(\Theta) \} \) of relations, given below.

For \( \tau \in \Lambda(\Theta) \), the relation \( R_\tau \) is given by a word \( \alpha(\tau) \) of length \( \leq \#(\tau \uparrow) \) in the letters \( \{ a_\omega^+, a_\omega^- \}_{\omega \in \tau \uparrow} \). The recipe for writing \( R_\tau \) is as follows.

- For each \( \tau = (\ldots 3 \ldots) \in \Lambda(\Theta) \) that contains a single entry 3, the set \( \tau \uparrow \) consists of two elements, and the corresponding relation is \( R_\tau := \{ a_\omega^+ a_\omega^- = 1 \} \), where \( a_\omega^+ \) and \( a_\omega^- \) correspond to
  \[
  \omega_1 = (\ldots 12 \ldots) \text{ and } \omega_2 = (\ldots 21 \ldots).
  \]

- For each \( \tau = (\ldots 2 \ldots 2 \ldots) \in \Lambda(\Theta) \) that contains two non-adjacent 2's, the set \( \tau \uparrow \) consists of four elements
  \[
  \omega_1 = (\ldots 11 \ldots 2 \ldots), \quad \omega_2 = (\ldots 2 \ldots 11 \ldots), \quad \omega_3 = (\ldots 2 \ldots 2' \ldots), \quad \omega_4 = (\ldots 2' \ldots 2 \ldots),
  \]
  and the corresponding relation is given by \( R_\tau := \{ a_\omega^+ a_\omega^- a_\omega^+ a_\omega^- = 1 \} \).

- For each \( \tau = (\ldots 22 \ldots) \in \Lambda(\Theta) \) that contains two adjacent 2's, the set \( \tau \uparrow \) consists of three elements
  \[
  \omega_1 = (\ldots 2 \ldots), \quad \omega_2 = (\ldots 211 \ldots), \quad \omega_3 = (\ldots 112 \ldots),
  \]
  and the corresponding relation is given by \( R_\tau := \{ a_\omega^+ a_\omega^- = 1 \} \).

**Remark 2.12.** Note, that in the first and the third case, the above relations can be rewritten as \( a_\omega^+ = a_\omega^- \) where \( \omega_1 \) and \( \omega_2 \) are some edges that belong to the same subgraph \( \mathfrak{S}_{d,i} \) of \( \mathfrak{S}_d \), defined in Remark 2.5. In the second case, it can be written as \( a_\omega^+ a_\omega^- = a_\omega^- a_\omega^+ \) where \( \omega_3 \) and \( \omega_4 \) are some edges in \( \mathfrak{S}_{d,i} \) and \( \omega_1 \) and \( \omega_2 \) are some edges in \( \mathfrak{S}_{d,i+2} \).

**Remark 2.13.** Geometrically (see Figure 3), these relations arise by taking the boundary \( S^1 \) of a small disk \( D^2 \), normal to the cell \( \hat{R}_d^\omega \) in \( \mathcal{P}_d \). Following the loop \( S^1 \), we register its transversal intersections with the codimension 1 cells \( \{ \hat{R}_d^\omega \}_{\omega \in \tau \uparrow} \), thus creating the word \( \alpha(\tau) \).

**Proof of Lemma 2.11.** First, we notice that by a general position argument, any loop in \( \mathcal{P}_d \) (after a small deformation) may be assumed to have empty intersection with the set \( \mathcal{P}_d^{\Omega_{(d)} \setminus \sim' \geq 3} \), formed by the cells of codimension at least 3.

For each \( \omega \) such that \( |\omega'| = 2 \), we consider a closed regular neighborhood \( U_\omega \) of the cell \( \hat{R}_d^\omega \) in the space \( \mathcal{P}_d^{\Omega_{(d)} \setminus \sim' \geq 2} \). We may assume that for distinct \( \omega \)'s from \( \Lambda(\Theta) \) these neighborhoods are disjoint.

\(^3\)In each of the following bullets, “...” stands for portions of the compositions \( \omega \) that do not change.

\(^4\)Thus, for any \( \tau = (\ldots 3 \ldots) \in \Lambda(\Theta) \), we may exclude the generator \( (\ldots 12 \ldots) \), retain the generator \( (\ldots 21 \ldots) \), and drop the relation \( R_\tau \).
Let \( X := \mathcal{P}_d^{\Omega(d) \mid \sigma \geq 2} \). For \( d \equiv 0 \mod 2 \), by Theorem 2.4, \( \pi_1(X) \) is a free group on \( \frac{d(d-2)}{4} \) generators. Adding a component \( \check{R}_d^\omega \), where \( \omega \in \Lambda(\mathfrak{c}^\Theta) \), to the locus \( X \), produces a new space \( Y \). The spaces \( X, U_\omega \), and \( X \cap U_\omega \) are path-connected. Thus, by the Seifert–van Kampen Theorem, \( \pi_1(Y) \approx \pi_1(X)*_{\pi_1(X \cap U_\omega)} \pi_1(U_\omega) \), the free product \(*\) of the groups \( \pi_1(X) \) and \( \pi_1(U_\omega) \) amalgamated over \( \pi_1(X \cap U_\omega) \). Since \( U_\omega \) is homotopy equivalent to the cell \( \check{R}_d^\omega \), we get \( \pi_1(U_\omega) = 0 \). At the same time, \( X \cap U_\omega = U_\omega \setminus \check{R}_d^\omega \) is homotopy equivalent to a circle \( \check{S}_1^1 \), the boundary of a small disk \( D_2^\omega \), normal to the stratum \( \check{R}_d^\omega \). Therefore, \( \pi_1(Y) \approx \pi_1(X)/[\check{S}_1^1] \). We can recycle this construction by now taking \( Y \) for the role of \( X \) and adding another cell \( \check{R}_d^\omega \), where \( |\omega_1|' = 2 \) and \( \omega_1 \neq \omega \), to produce a new space \( Y_1 \). By a similar argument, we conclude that \( \pi_1(Y_1) \equiv \pi_1(Y)*_{\pi_1(Y \cap U_{\omega_1})} \pi_1(U_{\omega_1}) \). By the reasoning as above, we get

\[
\pi_1(Y_1) \approx \pi_1(Y)/[\check{S}_1^1] \approx \pi_1(X)/[[\check{S}_1^1], [\check{S}_1^1]].
\]

Eventually, we will conclude that

\[
\pi_1(Y_k) \approx \pi_1(X)/[[\check{S}_1^1], [\check{S}_1^1], \ldots, [\check{S}_1^1]],
\]

where \( \{\omega, \omega_1, \ldots, \omega_k\} = \Lambda(\mathfrak{c}^\Theta) \).

It remains to observe that the relations \( \{R_\omega = \alpha(\check{S}_1^1)\}_{\omega \in \Lambda(\mathfrak{c}^\Theta)} \) that correspond to the loops \( \{\check{S}_1^1\}_{\omega \in \Lambda(\mathfrak{c}^\Theta)} \) are exactly the ones that are listed in the three bullets of the lemma and are labeled by the poset \( \omega \) (see Figure 3). We observe that all three types of relations satisfy condition (b) of admissibility (see Definition 2.7), and therefore they are well-defined as relations in \( G_d \).

The considerations of the case when \( d \) is odd follow parallel arguments. \( \square \)

The next theorem, whose proof is based on Lemma 2.11, is somewhat surprising.

**Theorem 2.14.** (i) For any closed poset \( \Theta \subset \Omega(d) \mid \sigma \geq 2 \), the fundamental group \( \pi_1(\mathcal{P}_d^{\Theta}) \) is a free group on \( \mu := \text{rank}(\check{H}_{d-2}(\mathcal{P}_d^{\Theta}; \mathbb{Z})) \) generators.

(ii) For \( d \equiv 0 \mod 2 \), \( \mu \in [0, d(d-2)/4] \) and, for \( d \equiv 1 \mod 2 \), \( \mu \in [0, (d-1)^2/4] \).

**Proof.** Using Lemma 2.11, we will show that for a closed poset \( \Theta \subset \Omega(d) \), the fundamental group \( \pi_1(\mathcal{P}_d^{\Theta}) \) can be interpreted as the fundamental group of a connected graph \( \check{G}_{\Theta,d} \subset \check{G}_d \) which contains the vertex \( \beta_{\text{base}} \). This circumstance immediately implies that \( \pi_1(\mathcal{P}_d^{\Theta}) \) is free.

Indeed, by Lemma 2.11, \( \pi_1(\mathcal{P}_d^{\Theta}) \) is the quotient of \( \check{G}_d \) modulo a set of relations of three types. The group \( \check{G}_d \) is the group of homotopy classes of based paths in \( \check{G}_d \), starting and ending at \( \beta_{\text{base}} \). Obviously, for any connected subgraph \( \check{G}' \subset \check{G}_d \) containing the vertex \( \beta_{\text{base}} \), \( \check{G}_d \) contains as a subgroup the set of homotopy classes of all based paths in \( \check{G}' \) starting and ending in \( \beta_{\text{base}} \). This subgroup consists of all admissible words \( w \) in the alphabet \( \mathbb{A}^\pm \), whose complete reductions \( \check{w} \) use only oriented edges that belong to \( \check{G}' \).

Now we observe that relations of the first and of the third type are of the form \( a_{\omega_1}^\pm = a_{\omega_2}^\pm \), which means that in any admissible word over the alphabet \( \mathbb{A}^\pm \), representing an element of \( \pi_1(\mathcal{P}_d^{\Theta}) \), every occurrence of \( a_{\omega_1}^\pm \) can be replaced by \( a_{\omega_2}^\pm \) and every occurrence of \( a_{\omega_2}^- \)

---

5 The homotopy type of \( Y \) is the result of attaching a 2-handle to \( X \) along its boundary.
can be replaced by $a_{\omega_1}^-$. Since the edges $\omega_1$ and $\omega_2$ belong to the same subgraph $\mathcal{G}_{d,i}$ (see Remark 2.5), the word obtained from an arbitrary admissible word by applying the latter substitutions will still be admissible. By the previous observation, applying these substitutions to all possible admissible words we obtain all possible admissible words that avoid the letters $a^{\pm}_{\omega_2}$, which means that they will be describing all closed paths based at $\beta_{\text{base}}$ in the subgraph of the initial graph, obtained by removal of the edge $\omega_2$. Performing this deletion operation for each relation of the first and the third type, we obtain the set of all admissible words that describe all closed paths with the base point $\beta_{\text{base}}$ in a certain connected subgraph of the initial graph $\mathcal{G}_d$, containing $\beta_{\text{base}}$. Its connectivity follows from the fact that relations of the first and the third type always contain two edges from the same $\mathcal{G}_{d,i}$, which means that when only one such edge will be left it will be impossible to remove it, and the subgraph will stay connected.

Let us now consider relations of the second type. Such a relation has the form

\[(2.1) \quad a^+_{\omega_1} a^-_{\omega_2} = a^-_{\omega_4} a^+_{\omega_3},\]

where $\omega_1$ and $\omega_2$ belong to some subgraph $\mathcal{G}_{d,i+2}$ while $\omega_1$ and $\omega_2$ belong to $\mathcal{G}_{d,i}$. If, while reducing relations of the first and the third type, we have already identified $\omega_1$ with $\omega_2$ or $\omega_3$ with $\omega_4$, then we are in the previous situation and can remove an extra edge from the subgraph, obtained in the previous step. Finally, if we have a full relation (2.1), then we get

\[a^+_{\omega_1} = a^+_{\omega_4} a^+_{\omega_3} a^+_{\omega_2} \quad \text{and} \quad a^-_{\omega_1} = a^-_{\omega_2} a^-_{\omega_4} a^-_{\omega_3}.\]

Again, these two equations mean that, in any admissible word, every occurrence of $a^+_{\omega_1}$ and $a^-_{\omega_1}$ can be replaced by the right-hand side of the respective formula given above. Once more, an easy check shows that the application of these substitutions preserves the admissibility of a word. Moreover, the set of all admissible words that are obtained after the substitutions will be describing all closed paths (based at $\beta_{\text{base}}$) in the subgraph of the graph under consideration, obtained by removal of the edge $\omega_1$.

Summa summarum, for any closed $\Theta \subset \Omega_{(d), |\sim'| \geq 2}$, the set of words obtained by applying the relations from Lemma 2.11 to the set of all admissible words in the alphabet $A^{\pm}$ coincides with the set of all admissible words whose letters correspond to all edges of a certain subgraph $\mathcal{G}_{\Theta,d}$ of $\mathcal{G}_d$. Thus $\pi_1(\mathcal{P}_{d}^{e\Theta})$ is a free group and by the Hurewicz theorem its rank is the rank of the cohomology group $\tilde{H}^1(\mathcal{P}_{d}^{e\Theta}; \mathbb{Z})$. This completes the proof of (i).

Claim (ii) follows by combining Theorem 2.4 with the arguments from the proof of Lemma 2.11. □

Remark 2.15. • By the Alexander duality, the cohomology group $\tilde{H}^1(\mathcal{P}_{d}^{e\Theta}; \mathbb{Z})$ is isomorphic to the homology group $\tilde{H}_{d-2}(\mathcal{P}_{d}^{e\Theta}; \mathbb{Z})$. A combinatorially defined complex $(\mathbb{Z}[\Theta[d]], \partial)$ whose homology coincide with $\tilde{H}_{*}(\mathcal{P}_{d}^{e\Theta}; \mathbb{Z})$ is described in the sequel paper [KSW]. We include in § 2.3.1 below a brief description of $(\mathbb{Z}[\Theta[d]], \partial)$ as a preview of [KSW].

• By Theorem 2.14, for any $\Theta \subset \Omega_{(d), |\sim'| \geq 2}$, $\pi_1(\mathcal{P}_{d}^{e\Theta}) \approx F_\mu$ is a free group on $\mu$ generators. But in general, $\mathcal{P}_{d}^{e\Theta}$ is not a $K(F_\mu, 1)$-space. Note that the (reduced)
cohomology of a free group $F_\mu$ is non-trivial only in dimension 1, where it is isomorphic to $\mathbb{Z}^\mu$. Therefore, for a general closed poset $\Theta \subset \Omega_{(d)}$, $|\sim'| \geq 2$, the space $P^c_{d} \subset \Theta$ cannot be of the $K(\pi, 1)$-type, unless the homology of complex $(\mathbb{Z}[\Theta_{(d)}], \partial)$ is concentrated in dimension $d - 2$.

For example, for $\Theta = \{\omega \in \Omega_{(8)} | \omega \leq (1, 3) \text{ or } \omega \leq (3, 1)\}$, our computations of the differential complex $(\mathbb{Z}[\Theta_{(8)}], \partial)$ show that $\check{H}_5(P^c_8) \approx \mathbb{Z} \neq 0$, see [KSW]. This implies that $P^c_8(31, (13))$ is simply connected but is not contractible.

The same reasoning shows that other examples of spaces $P^c_{d+1}$ that are not $K(\pi, 1)$ are provided by Arnold’s theorem Theorem D.

For any closed poset $\Theta \subset \Omega$ and any $d > 0$, we denote by $P^c_{d+1}$ the space of polynomials of degree $d + 1$, whose derivatives belong to $P^c_{d}$.

**Corollary 2.16.** For any closed poset $\Theta \subset \Omega$ and any $d > 0$, the space $P^c_{d+1}$ is homotopy equivalent to the space $P^c_{d}$. In particular,

(i) $P^c_{d+1}$ is homotopy equivalent to a wedge of $\frac{d(d-2)}{4}$ circles for $d \equiv 0 \mod 2$, and to a wedge of $\frac{(d-1)^2}{4}$ circles for $d \equiv 1 \mod 2$.

(ii) for any closed $\Theta \subset \Omega_{(d)}$, the fundamental group $\pi_1(P^c_{\Theta})$ is free.

**Proof.** The homotopy equivalence follows from the simple fact that $\frac{d}{dx} : P^c_{d+1} \to P^c_{d}$ is a trivial fibration with fiber $\mathbb{R}$. Then (i) follows from Theorem 2.4 and (ii) from Theorem 2.14.

**Proposition 2.17.** Let $\Theta_3 \subset \Omega_{(d)}$ be the smallest closed poset that contains all $\omega \in \Omega_{(d)}$ with all entries 1 or 3 and a single entry 2. Then $P^c_{\Theta_3}$ is the space of monic polynomials of degree $d$ whose real roots have multiplicity 2 at most and for any $d \geq 3$, $\pi_1(P^c_{\Theta_3}) = \mathbb{Z}$.

**Proof.** By definition $P^c_{\Theta_3}$ is the space of all monic polynomials of degree $d$ whose real roots have at most multiplicity 2.

Using the relations in the last two bullets of Lemma 2.11, we aim to deduce that $\pi_1(P^c_{\Theta_3}) \approx \mathbb{Z}$. We first consider the case when $d$ is even and reduce the free group $\pi_1(P^c_{\Theta_3})$ on $\frac{d(d-2)}{4}$ generators by the relations coming from all compositions $\tau = (1, \ldots, 1, 2, 1, \ldots, 1) \in \Lambda(\Theta_3)$ for $i, j \geq 0$. Again consider the graph $\mathcal{G}_d$, shown for $d = 6$ in Figure 2. Recall that $\mathcal{G}_d$ naturally splits into $\left\lfloor \frac{d}{2} \right\rfloor$ subgraphs $\mathcal{G}_{d,i}$ each containing only two vertices of the form $(1, \ldots, 1)$ and $(1, \ldots, 1)$ and $i+1$ edges, labeled by $(1, \ldots, 1, 2)$, $(1, \ldots, 1, 2, 1)$, ..., $(2, \ldots, 1, 1)$. Here $i = 0, 2, \ldots, d$ for $d$ even, and $i = 1, 3, \ldots, d$ for $d$ odd.

Next, for a given $i \geq 2$, consider $(i - 1)$ compositions of the form

$(2.2) \quad (1, \ldots, 1, 2, 2), (1, \ldots, 1, 2, 2, 1), (1, \ldots, 1, 2, 1, 1)$, ..., $(2, 2, 1, \ldots, 1).$
Using the relations from the third bullet of Lemma 2.11 with the latter \((i - 1)\) compositions applied to admissible words in the alphabet \(A^\pm\) we get two sequences of equalities

\[
(1, \ldots, 1, 2) = (1, \ldots, 1, 2, 1, 1) = (1, \ldots, 1, 2, 1, 1, 1, 1) = \cdots
\]

and

\[
(1, \ldots, 1, 2, 1) = (1, \ldots, 1, 2, 1, 1, 1) = (1, \ldots, 1, 2, 1, 1, 1, 1, 1) = \cdots
\]

Again, we employ the graph \(G_d\), used in the proof of Theorem 2.14. We get that the subgraph \(G_{d,i} \subset G_d\) may be reduced, modulo the relations coming from the compositions \((2.2)\), to the graph \(\tilde{G}_{d,i}\) with vertices \((1, \ldots, 1)\) and \((1, \ldots, 1)\), connected by only two edges. Each edge is labeled by the collection of all compositions appearing in the respective equality above.

Applying this procedure for all \(i \geq 2\), we reduce the graph \(G_d\) to \(\tilde{G}_d := \bigcup_i \tilde{G}_{d,i}\) in which each pair of neighbouring vertices is connected by two edges only. Next, we want to reduce the graph \(\tilde{G}_d\) to a single loop by using compositions of the form \(\tau = (\ldots 2 \ldots 2 \ldots) \in \Lambda(c\Theta_3)\) with two non-adjacent 2’s. We observe that the relation in the second bullet of Lemma 2.11 provides an equality of two loops, each consisting of two edges; one of these loops belongs to the graph \(\tilde{G}_{d,i}\) and the second one belongs to the graph \(\tilde{G}_{d,i+2}\) for some value of \(i\). Recall that, for an even (odd) \(d\), \(i\) runs over the set of non-negative even (odd) numbers not exceeding \(d\).

To perform this reduction, we distinguish the case of odd and even \(d\). Namely, if \(d \geq 4\) is even and we want to collapse the (loop of the) graph \(\tilde{G}_{d,d}\) to the graph \(\tilde{G}_{d,d-2}\), we have to use the relation that comes from the composition \((1, 2, 1, \ldots, 1, 2)\). Similarly, to collapse \(\tilde{G}_{d,d-2}\) to \(\tilde{G}_{d,d-4}\), we use the relation coming from the composition \((1, 2, 1, \ldots, 1, 2)\), etc. At the end, we are left with the graph \(\tilde{G}_{d,2}\) which consists of two vertices connected by a pair of edges. Its fundamental group is obviously isomorphic to \(\mathbb{Z}\).

Analogously, for odd \(d \geq 3\), we collapse the (loop of the) graph \(\tilde{G}_{d,d}\) to the graph \(\tilde{G}_{d,d-2}\), by using the relation that comes from the composition \((2, 1, \ldots, 1, 2)\). Further, to collapse \(\tilde{G}_{d,d-2}\) to \(\tilde{G}_{d,d-4}\), we use the relation coming from the composition \((2, 1, \ldots, 1, 2)\), etc. At the end, we are left with the graph \(\tilde{G}_{d,1}\), which is again a single loop with two vertices. \(\square\)

**Corollary 2.18.** The fundamental group of the space of real monic polynomials of a fixed odd degree \(d > 1\) with no real critical points of multiplicity higher than 2 is isomorphic to \(\mathbb{Z}\).
Proof. Follows immediately from Corollary 2.16 and Proposition 2.17. □

2.3.1. Preview: Computation of \( \bar{H}_*(\bar{P}_d^\Theta) \) and the number of generators of \( \pi_1(P_c^d \Theta) \). In this section we provide a preview of a result from [KSW] and show that it makes the number of generators of the free group \( \pi_1(P_c^d \Theta) \). Let \( \Theta \subseteq \Omega_{[d]} \) be a closed poset. Consider the \( \mathbb{Z} \)-module \( \mathbb{Z}[\Theta] \), freely generated by the elements of \( \Theta \). The function \( d - |\sim'| : \Theta_{[d]} \to \mathbb{Z}_+ \) defines a grading of the module \( \mathbb{Z}[\Theta_{[d]}] \).

Using the merge operator \( M_* \) and the insert operator \( I_* \) from § 1.2, we define two homomorphisms, \( \partial_M : \mathbb{Z}[\Theta_{[d]}] \to \mathbb{Z}[\Theta_{[d]}] \) and \( \partial_I : \mathbb{Z}[\Theta_{[d]}] \to \mathbb{Z}[\Theta_{[d]}] \). For \( \omega = (\omega_1, \ldots, \omega_s) \) we set

\[
\partial_M(\omega) := - \sum_{k=1}^{s-1} (-1)^k M_k(\omega)
\]

and

\[
\partial_I(\omega) := \sum_{k=0}^{s} (-1)^k I_k(\omega),
\]

The homomorphisms \( \partial_M \) and \( \partial_I \) are differentials and so is the homomorphism

\[
\partial := \partial_M + \partial_I : \mathbb{Z}[\Theta_{[d]}] \to \mathbb{Z}[\Theta_{[d]}].
\]

The result we borrow from ?? is the following.

Theorem 2.19 ([KSW]). The homology of \((\mathbb{Z}[\Theta_{[d]}], \partial)\) coincides with the reduced homology \( \bar{H}_*(\bar{P}_d^\Theta ; \mathbb{Z}) \).

As an immediate corollary we get the announced computability result.

Corollary 2.20. For any closed poset \( \Theta \subset \Omega_{[d]} \cap \Omega_{|\sim'| \geq 2} \), the number of generators in \( \pi_1(P_c^d \Theta) \) is equal to

\[
\text{rank}(\ker\{ \partial : \mathbb{Z}[\Theta_{[d]} \cap \Omega_{|\sim'|=2}] \to \mathbb{Z}[\Theta_{[d]} \cap \Omega_{|\sim'|=3}] \}).
\]

Proof. By Theorem 2.19 the homology of \((\mathbb{Z}[\Theta_{[d]}], \partial)\) coincides with the reduced homology \( \bar{H}_*(\bar{P}_d^\Theta ; \mathbb{Z}) \). In particular, the \((d-2)^{nd} \) homology of \((\mathbb{Z}[\Theta_{[d]}], \partial)\) is isomorphic to \( \bar{H}_1(P_{c,d}^\Theta ; \mathbb{Z}) \). Using that, \( |\omega'| \geq 2 \) for any \( \omega \in \Theta \), we get

\[
\bar{H}_1(P_{c,d}^\Theta ; \mathbb{Z}) \approx \bar{H}_{d-2}(\mathbb{Z}[\Theta_{[d]}], \partial) = \ker\{ \partial : \mathbb{Z}[\Theta_{[d]} \cap \Omega_{|\sim'|=2}] \to \mathbb{Z}[\Theta_{[d]} \cap \Omega_{|\sim'|=3}] \}.
\]

So \( \pi_1(P_{c,d}^\Theta) \) is a free group on

\[
\mu := \text{rank}(\ker\{ \partial : \mathbb{Z}[\Theta_{[d]} \cap \Omega_{|\sim'|=2}] \to \mathbb{Z}[\Theta_{[d]} \cap \Omega_{|\sim'|=3}] \})
\]

generators. □
Figure 4. A set \{a, b, c, d, e, f\} of six generators, freely generating the bordism group \(B(S^1 \times \mathbb{R}; e\Omega(6) \geq 2) \approx \pi_1(\mathcal{P}^e\Omega^{6, \sim \gamma \geq 2}_6)\), is shown as collections of curves in the cylinder with the coordinates \((\psi, x) \in S^1 \times \mathbb{R}\). Each collection of curves is generated by a specific map \(\gamma : S^1 \to \mathcal{P}^e\Omega^{6, \sim \gamma \geq 2}_6\) as the set of pairs \((\psi, x)\) with the property \(\gamma(\psi)(x) = 0\). Each line \(\{\psi = const\}\) is either transversal to the collection, or is quadratically tangent to it. No double tangent lines are permitted. Each collection of curves is equipped with the circular word (written under each of the six diagrams) that reflects the transversal intersections of the loop \(\gamma(S^1)\) with the discriminant variety \(D_6 \subset \mathcal{P}_6\).

3. \(\pi_1(\mathcal{P}^e\Omega^{6, \sim \gamma \geq 2}_d)\) AND COBORDISMS OF PLANE CURVES WITH RESTRICTED VERTICAL TANGENCIES

Results and constructions in this section are similar to the ones from Arnold’s Theorem B.

Consider a smooth \(n\)-manifold \(Y\) and an immersion \(\beta : X \to \mathbb{R} \times Y\) of a smooth closed \(n\)-manifold \(X\) into the interior of \(\mathbb{R} \times Y\). We denote by \(\mathcal{L}\) the one-dimensional foliation, defined by the fibers of the projection map \(\pi : \mathbb{R} \times Y \to Y\). For each point \(x \in X\), we define the natural number \(\mu_\beta(x)\) as the multiplicity of tangency between the \(x\)-labeled...
branch of $\beta(X)$ — the $\beta$-image of the vicinity of $x$ in $X$ — and the leaf of $L$ through $\beta(x)$. In particular, if the branch is transversal to the leaf, then $\mu_\beta(x) = 1$.

We fix a natural number $d$ and assume that $\beta$ is such that each leaf $L_y$, $y \in Y$, hits $\beta(X)$ so that the following property holds:

\[
(3.1) \quad m_\beta(y) := \sum_{\{a \in L_y \cap \beta(X)\}} \left( \sum_{x \in \beta^{-1}(a)} \mu_\beta(x) \right) \leq d.
\]

We order the points $\{a_i\}$ of $L_y \cap \beta(X)$ by the values of their projections on $\mathbb{R}$ and introduce the combinatorial pattern $\omega^\beta(y)$ of $y \in Y$ as the sequence of multiplicities

$\{\omega_i(y) := \sum_{\{x \in \beta^{-1}(a_i)\}} \mu_\beta(x)\}_i$. We denote by $D^\beta(y)$ the real divisor of the intersection $L_y \cap \beta(X)$ with multiplicities $\{\omega_i(y)\}_i$.

**Proposition 3.1.** Given manifolds $X$ and $Y$ as above, for any immersion $\beta : X \to \mathbb{R} \times Y$ that satisfies ((3.1)), together with the parity condition $m_\beta(y) \equiv d \mod 2$, there exists a continuous map $\Phi : Y \to \mathcal{P}_d$ such that

$$\{(x, y) \in \mathbb{R} \times Y | \Phi(y)(x) = 0\} = \beta(X).$$

If, for a given closed poset $\Theta \subset \Omega_{(d)}$, the immersion $\beta$ is such that no $\omega^\beta(y)$ belongs to $\Theta$, then $\Phi$ maps $Y$ to $\mathcal{P}_d^{\Theta}$. 

**Proof.** We claim that the loci $\beta(X)$ may be viewed as the solutions of the equations $\{x^d + \sum_{j=0}^{d-1} a_j(y) x^j = 0\}_{y \in Y}$, where $\{a_j : Y \to \mathbb{R}\}_j$ are some smooth functions.

Let us justify this claim. By Lemma 4.1 from [Ka2] and Morin’s Theorem [Mor1, Mor2], if a particular branch $\beta(X)_\kappa$ of $\beta(X)$ is tangent to the leaf $L_{y_0}$ at a point $b = (\alpha, y_0)$ with the order of tangency $j$, then there is a system of local coordinates $(u, \tilde{y}, \tilde{z})$ in the vicinity of $b$ in $\mathbb{R} \times Y$ such that:

1. $\beta(X)_\kappa$ is given by the equation $\{u^j + \sum_{k=0}^{j-2} \tilde{y}_k u^k = 0\}$;
2. each nearby leaf $L_y$ is given by the equations $\{\tilde{y} = \text{const}, \tilde{z} = \text{const}'\}$.

Setting $u = x - \alpha$ and writing $\tilde{y}_k$’s as smooth functions of $y$, the same $\beta(X)_\kappa$ can be given by the equation

$$\{P_{\alpha, \kappa}(x, y) := (x - \alpha)^j + \sum_{j=0}^{j-2} a_{\kappa,k}(y) (x - \alpha)^k = 0\},$$

where $a_{\kappa,k} : Y \to \mathbb{R}$ are smooth functions vanishing at $y_0$. Therefore, there exists an open neighborhood $U_{y_0}$ of $y_0$ in $Y$ such that, in $\mathbb{R} \times U_{y_0}$, the locus $\beta(X)$ is given by the monic polynomial equation

$$\{P_{y_0}(x, y) := \prod_{(\alpha, y_0) \in L_{y_0} \cap \beta(X)} \left( \prod_{\kappa \in A_\alpha} P_{\alpha, \kappa}(x, y) \right) = 0\},$$

of degree $m_\beta(y_0) \leq d$ in $x$. Here the finite set $A_\alpha$ labels the local branches of $\beta(X)$ that contain the point $(\alpha, y_0) \in L_{y_0} \cap \beta(X)$. 


By multiplying \( P_{y_0}(x, y) \) with \((x^2 + 1)^{-\frac{d-m_\beta(y_0)}{2}}\), we get a polynomial \( \tilde{P}_{y_0}(x, y) \) of degree \( d \) which, for each \( y \in U_{y_0} \), shares with \( P_{y_0}(x, y) \) the zero set \( \beta(X) \cap (\mathbb{R} \times U_{y_0}) \), as well as the divisors \( D^\beta(y) \).

For each \( y \in Y \), consider the space \( \mathcal{X}_\beta(y) \) of monic polynomials \( \tilde{P}(x) \) of degree \( d \) such that their real divisors coincide with the \( \beta \)-induced divisor \( D^\beta(y) \). We view \( \mathcal{X}_\beta := \bigsqcup_{y \in Y} \mathcal{X}_\beta(y) \) as a subspace of \( Y \times \mathcal{P}_d \). It is equipped with the obvious projection \( p : \mathcal{X}_\beta \to Y \). The smooth sections of \( p \) are exactly the smooth functions \( \tilde{P}(x, y) \) that interest us. Each \( p \)-fiber \( \mathcal{X}_\beta(y) \) is a convex set. Now, given several smooth sections \( \{\sigma_i\}_i \) of \( p \), we conclude that \( \sum_i \phi_i \cdot \sigma_i \) is again a section of \( p \), provided that the smooth functions \( \phi_i : Y \to [0, 1] \) have the property \( \sum_i \phi_i \equiv 1 \). Note that \( \phi_i \cdot \sigma_i \notin \mathcal{P}_d \).

Since \( X \) is compact, \( \pi(\beta(X)) \subset Y \) is compact as well. So it admits a finite cover by the open sets \( \{U_{y_i}\}_i \) as above. Let \( \{\phi_i : Y \to [0, 1]\}_i \) be a smooth partition of unity, subordinated to this finite cover. Then the monic polynomial

\[
\tilde{P}(x, y) := \sum_i \phi_i(y) \cdot \tilde{P}_i(x, y)
\]

of degree \( d \) has the desired properties. In particular, its divisor is \( D^\beta(y) \) for each \( y \in Y \). Thus, using \( \tilde{P}(x, y) \), any immersion \( \beta : X \to \mathbb{R} \times Y \), such that no \( \omega^\beta(y) \) belongs to \( \Theta \), is realized by a smooth map \( \Phi : Y \to \mathcal{P}_d^\Theta \) for which \( \beta(X) = \{\Phi(y)(x) = 0\} \).

We denote by \( \mathcal{L} \) the foliation of the cylinder \( A = S^1 \times \mathbb{R} \), formed by the fibers \( \{\ell_\psi\}_{\psi \in S^1} \) of the obvious projection \( q : S^1 \times \mathbb{R} \to S^1 \), and by \( \mathcal{L}^* \) the 1-dimensional foliation of \( A \times [0, 1] \) by the fibers of the obvious projection \( Q : S^1 \times \mathbb{R} \times [0, 1] \to S^1 \times [0, 1] \). We pick a base point \( \psi_* \in S^1 \) and the leaf \( \ell_* \) of \( \mathcal{L} \) that corresponds to \( \psi_* \). Similarly, for each \( t \in [0, 1] \), we fix the base leaf \( \ell_*(t) \) of \( \mathcal{L}^* \).

We consider immersions \( \beta : M \to A \) of closed smooth 1-dimensional manifolds \( M \) such that:

1. (P1) for each \( \psi \in S^1 \), the multiplicity from (3.1) satisfies the inequality \( m_\beta(\psi) \leq d \);
2. (P2) no leaf \( \ell_\psi \) of \( \mathcal{L} \) has the combinatorial tangency pattern \( \omega^\beta(\psi) \in \Omega_{|d_i|, |\sim^i| \geq 2} \);
3. (P3) the map \( q \circ \beta : M \to S^1 \) has only Morse type singularities;
4. (P4) \( \beta(M) \cap \ell_* = \emptyset \).

The next definition lays the foundation for our notion of cobordism, which deviates from the usual cobordism theory.

**Definition 3.2.** We say that a pair of immersions \( \beta_0 : M_0 \to A \), \( \beta_1 : M_1 \to A \) is cobordant, if there exists a compact smooth orientable surface \( W \) with boundary \( \partial W = M_1 \bigsqcup (-M_0) \) and an immersion \( B : W \to A \times \mathbb{R} \) such that:

- \( B|_{M_0} = \beta_0 \) and \( B|_{M_1} = \beta_1 \);
- for each \( (\psi, t) \in S^1 \times [0, 1] \), the multiplicity \( m_B((\psi, t)) \leq d \) (see (3.1));
- for each \( (\psi, t) \in S^1 \times [0, 1] \), the tangency pattern \( \omega^B((\psi, t)) \) does not belong to \( \Omega_{|d_i|, |\sim^i| \geq 2} \);
- the composition of \( B : W \to S^1 \times [0, 1] \) with the obvious map \( \Pi : A \times [0, 1] \to [0, 1] \), is a Morse function with the regular values 0 and 1.
We denote by $B(A; c\Omega_{d,i,|i'\geq 2})$ the set of cobordism classes of such immersions $\beta : M \to A$.

Note that $B(A; c\Omega_{d,i,|i'\geq 2})$ is set of cobordism classes of immersed curves (and not the usual set/group of cobordisms of 1-manifolds with the target $A$).

In fact, the set $B(A; c\Omega_{d,i,|i'\geq 2})$ also carries a group structure, where the group operation $\beta \circ \beta'$ is defined as follows. Since $\beta(M) \cap \ell_* = \emptyset$ and $\beta'(M') \cap \ell_* = \emptyset$, we may view $\beta(M)$ as subset of the strip $[0,2\pi] \times \mathbb{R}$, and $\beta'(M')$ as subset of the strip $[2\pi,4\pi] \times \mathbb{R}$. Then $\beta(M) \mathbin{\coprod} \beta'(M') \subset [0,4\pi] \times \mathbb{R}$. We use the linear map $\lambda : [0,4\pi] \to [0,2\pi]$ to place the locus $\beta(M) \mathbin{\coprod} \beta'(M')$ in $[0,2\pi] \times \mathbb{R}$, and thus in $A$. Evidently, this operation produces a pattern $\beta(M) \circ \beta'(M')$ satisfying (P1)-(P4).

The next result is similar to Arnold’s Theorem B.

**Theorem 3.3.** The cobordism group $B(A; c\Omega_{d,i,|i'\geq 2})$, where $d \equiv 0 \mod 2$, is isomorphic to the fundamental group $\pi_1(P_d^{c\Omega_{d,i,|i'\geq 2}})$, and thus is a free group in $\frac{d(d-2)}{4}$ generators.

See Figure 4 for the case $d = 6$. In this case, $B(A; c\Omega_{(6),|i'\geq 2})$ is the free group on 6 generators.

**Proof.** Each continuous loop $\gamma : S^1 \to P_d$ produces a 1-dimensional locus $\Xi_\gamma$ (a collection of curves) in the cylinder $A = S^1 \times \mathbb{R}$ by the formula

$$\Xi_\gamma := \{(\psi, x) \in S^1 \times \mathbb{R} \mid \gamma(\psi)(x) = 0\}.$$

If $\gamma$ is a smooth loop in $P_d^{c\Omega_{d,i,|i'\geq 2}}$ that is either transversal to the non-singular part $D_d^0 \subset P_d^{c\Omega_{d,i,|i'\geq 2}}$ of the discriminant variety $D_d \subset P_d$, or is quadratically tangent to $D_d^0$, then the locus $\Xi_\gamma$ is the image of a compact 1-manifold $M$ under an immersion $\beta : M \to A$. Indeed, for each $\psi \in S^1$ such that $\gamma(\psi)$ has only simple real roots, $\Xi_\gamma$ is a disjoint union of several smooth arcs over the vicinity of $\psi$, and the projection $q : \Xi_\gamma \to S^1$ is a local diffeomorphism. So we only need to sort out what happens over the vicinity of $\psi$, such that $\gamma(\psi)$ has a single root $x_*$ of multiplicity 2. In the vicinity of $(\psi_*, x_*)$ in $A$, the intersections $\{\Xi_\gamma \cap \ell_\psi\}_\psi$ either:

1. have cardinality 2 for all $\psi \neq \psi_*$, or
2. have cardinality 2 for all $\psi > \psi_*$ and cardinality 0 all $\psi < \psi_*$, or
3. have cardinality 2 for all $\psi < \psi_*$ and cardinality 0 all $\psi > \psi_*$.  

The case (1) arises when $\gamma$ is quadratically tangent to $D_d^0$ at $\gamma(\psi_*)$. In such a case, locally $\gamma(\psi)(x) = ((x-x_*)^2 - (\psi - \psi_*)^2) \cdot T(x, \psi)$, where $T$ is an $x$-polynomial with smooth coefficients (in $\psi$) and simple roots, so that $T(x_*, \psi_*) \neq 0$. Then two local branches of $\Xi_\gamma$ intersect at $(\psi_*, x_*)$. Both branches are transversal to $\ell_{\psi_*}$, and the restriction of the map $q$ to each branch is a diffeomorphism. So the composition $q \circ \beta$ has no critical points in the vicinity of $\beta^{-1}((\psi_*, x_*))$.

When $\gamma$ is transversal to $D_d^0$ at $\gamma(\psi_*)$, the cases (2) and (3) are realized. Then the $q$-images of $\Xi_\gamma$, localized to the vicinity of $(\psi_*, x_*)$, are semi-open intervals, bounded by $\psi_*$. In such cases, locally $\gamma(\psi)(x) = ((x-x_*)^2 + (\psi - \psi_*)^2) \cdot T(x, \psi)$, where $T$ is an $x$-polynomial.
with smooth coefficients (in $\psi$) and simple roots, so that $T(x_*, \psi_*) \neq 0$. The locus $\Xi_\gamma$ is quadratically tangent to $\ell_\psi$ at $(\psi_*, x_*)$, and the Morse function $q \circ \beta$ attains its extremum at the unique point in $M$ whose $\beta$-image is $(\psi_*, x_*)$.

So the triple self-intersections of $\Xi_\gamma$, the double tangencies to the leaves $\{\ell_\psi\}_\psi$, and the cubic tangencies to $\{\ell_\psi\}_\psi$ are forbidden when $\gamma(S^1) \subset \mathcal{P}_d^{c\mathcal{E}_1(\Omega)}}$; they correspond to compositions $\omega_\psi \in \Omega(d_i, \cdot | \cdot)^{2}$. Thus, if $\gamma$ is a smooth loop in $\mathcal{P}_d^{c\mathcal{E}_1(\Omega)}}$ that is either transversal to $D^\circ$ or quadratically tangent to it, then $\Xi_\gamma \subset A$ satisfies (P1)-(P3).

If the image $\gamma(*)$ of the base point $\ast \in S^1$ belongs to the $d$-cell $R^0_d \subset \mathcal{P}_d$ that represents polynomials with no real roots, then property (P4) is also satisfied.

Of course, by a small homotopy, we may assume that $\gamma$ is transversal to $D^\circ_d$ and is a regular embedding when $d > 2$. For such a loop $\gamma$, $\Xi_\gamma$ does not have self-intersections (is a disjoint union of regularly embedded loops). However, even if the cobordism $B : W \to A \times [0, 1]$ between two regular embeddings, $\beta_0 : M_0 \to A \times \{0\}$ and $\beta_1 : M_1 \to A \times \{1\}$, itself is a regular embedding, some $t$-slices of $W$ will develop singularities (in particular, self-intersections). For that reason there is the need to consider immersions (and not regular embeddings only) in (P1)-(P4).

Conversely, assume $\beta : M \to A$ is an immersion satisfying (P1)-(P4). By Proposition 3.1, we may lift $\beta$ to a smooth loop $\gamma : S^1 \to \mathcal{P}_d^{c\mathcal{E}_1(\Omega)}}$ such that $\Xi_\gamma = \beta(M)$, $\gamma$ is transversal or quadratically tangent to $D^\circ_d$, and $\gamma(*)$ belongs to the cell with no real roots.

So the correspondence $\gamma \sim \Xi_\gamma$ is the candidate for realizing the group isomorphism $\Xi_\ast : \pi_1(\mathcal{P}_d^{c\mathcal{E}_1(\Omega)}}) \to B(A; c\mathcal{E}_1(\Omega)}}$.

We have shown already that any immersion $\beta : M \to A$ which satisfies (P1)-(P4) is realizable by a loop $\gamma : S^1 \to \mathcal{P}_d^{c\mathcal{E}_1(\Omega)}}$. It remains to prove that:

1. homotopic loops $\gamma_0, \gamma_1 : S^1 \to \mathcal{P}_d^{c\mathcal{E}_1(\Omega)}}$ produce cobordant patterns $\Xi_{\gamma_0}, \Xi_{\gamma_1}$ in $A$ (so that the correspondence $\Xi_\ast$ is well-defined);
2. if $\Xi(\gamma)$ is cobordant in $A$ to $\emptyset$, then $\gamma$ is contractible in $\mathcal{P}_d^{c\mathcal{E}_1(\Omega)}}$ (i.e., $\Xi_\ast$ is an injective map).

In order to validate these two claims, we consider the domain

$$\mathcal{E}_d := \{(\vec{a}, x) | P(x, \vec{a}) \leq 0\} \subset \mathcal{P}_d \times \mathbb{R}$$

where $P(x, \vec{a}) := x^d + \sum_{j=0}^{d-1} a_j x^j$. We denote by $\partial \mathcal{E}_d$ the boundary of $\mathcal{E}_d$, a smooth hypersurface. Let $\pi : \mathbb{R} \times \mathcal{P}_d \to \mathcal{P}_d$ denote the obvious projection. Then $\pi^{-1}(\vec{a}) \cap \partial \mathcal{E}$ is the support of the real divisor of the $x$-polynomial $P(x, \vec{a})$.

By a general position argument, we may assume that the homotopy $\Gamma : S^1 \times [0, 1] \to \mathcal{P}_d^{c\mathcal{E}_1(\Omega)}}$ that links $\gamma_0$ and $\gamma_1$ is smooth and transversal to all the strata $\{R^\omega_d\}_\omega$ of codimension $\leq 1$. Note, that by definition, it misses all the strata of codimension $\geq 2$. In other words, we may assume that $\gamma_0, \gamma_1$, and $\Gamma$ are transversal to the $(d - 1)$-cells $R^\omega_d$ for $\omega \in \Omega(d_i, \cdot | \cdot)^{1}$ that form the non-singular portion $\mathcal{D}_d := \mathcal{D}_d \cap \mathcal{P}_d^{c\mathcal{E}_1(\Omega)}}$ of the
discriminant variety $\mathcal{D}_d = \mathcal{P}_d^{\Omega(d),\cdot \prec 1} \subset \mathcal{P}_d$. For $d > 4$, by a general position argument, we may also assume that $\Gamma$ is a regular embedding.

Set $G := \Gamma^{-1}(\mathcal{D}_d^0)$, a collection of disjoint arcs and loops in the cylinder $S^1 \times [0, 1]$. We may perturb the obvious function $t : S^1 \times [0, 1] \to [0, 1]$ so that the new $\tilde{t} : S^1 \times [0, 1] \to [0, 1]$ has the following properties:

1. $\tilde{t}(S^1 \times \{0\}) = t(S^1 \times \{0\}) = 0$;
2. $\tilde{t}(S^1 \times \{1\}) = t(S^1 \times \{1\}) = 1$;
3. $\tilde{t}$ has no critical points;
4. the restriction $\tilde{t} : G \to [0, 1]$ is a Morse function with critical points in the interior of $G$.

From now and on, we retain the old notation “$t$” for this perturbation $\tilde{t}$. If $\Gamma$ is transversal to $\mathcal{D}_d^0$, then we claim that the map

\[ \Lambda := \Gamma \times \text{id}_{\mathbb{R}} : (S^1 \times [0, 1]) \times \mathbb{R} \to \mathcal{P}_d^{\text{cri}(d),\cdot \prec 2} \times \mathbb{R} \]

is transversal to the hypersurface $\partial \mathcal{E}_d \subset \mathcal{P}_d \times \mathbb{R}$. Indeed, consider a line $\ell_{\psi,t} := \pi^{-1}(\Gamma(\psi,t))$ and a point $\Lambda(\psi,t,x) \in \partial \mathcal{E}_d$. If $x$ is a simple real root of the polynomial $P(\cdot, \Gamma(\psi,t))$, then the line $\ell_{\psi,t}$ is transversal to the hypersurface $\partial \mathcal{E}_d$ at $\Lambda(\psi,t,x)$. If $x$ is a real root of multiplicity 2, then $\Gamma(\psi,t) \in \mathcal{D}_d^0$, and by the transversality of $\Gamma$ to $\mathcal{D}_d^0$, $\Lambda$ is transversal at the point $\Lambda(\psi,t,x)$ to $\partial \mathcal{E}_d$.

The $\Lambda$-transversality implies that $\Lambda^{-1}(\partial \mathcal{E})$ is a regularly embedded surface $W$ in the shell $(S^1 \times [0, 1]) \times \mathbb{R} \cong A \times [0, 1]$. With the help of $Q : A \times [0, 1] \to S^1 \times [0, 1]$ that takes each $(x,\psi,t)$ to $(\psi,t)$, the surface $W$ projects on the cylinder $S^1 \times [0, 1]$. The surface $W$ interacts with the leaves of $\mathcal{L}^*$ (the $Q$-fibers) in the ways that are described by the first three bullets in Definition 3.2.

To validate the last bullet from Definition 3.2, we need to check that the composition $t^\bullet : W \xrightarrow{Q} S^1 \times [0, 1] \xrightarrow{t} [0, 1]$ is a Morse function.

Consider the $(d-1)$-dimensional non-singular affine variety

\[ \mathcal{F}_d := \{(a,\bar{a}) \mid P(x,\bar{a}) = 0, \frac{\partial}{\partial x} P(x,\bar{a}) = 0\} \subset \partial \mathcal{E}_d \subset \mathcal{P}_d \times \mathbb{R}, \]

where $P(x,\bar{a}) := x^d + \sum_{j=0}^{d-1} a_j x^j$.

Note that $x(\mathcal{F}_d) = \mathcal{D}_d$, the discriminant variety. The locus $\mathcal{F}_d^0 := \pi^{-1}(\mathcal{D}_d^0)$ is an open and dense subset of $\mathcal{F}_d$, characterized by the inequality $\frac{\partial^2}{\partial x^2} P(x,\bar{a}) \neq 0$. The projection $\pi : \mathcal{F}_d^0 \to \mathcal{D}_d^0$ is a diffeomorphism.

Set $Z := W \cap \mathcal{F}_d$. By definition, $G := \pi(Z) \subset S^1 \times [0, 1]$. Moreover, since $\pi : \mathcal{F}_d^0 \to \mathcal{D}_d^0$ is a diffeomorphism, so is the map $\pi : Z \to G$. Using that $\Gamma$ is transversal to $\mathcal{D}_d^0 = \pi(\mathcal{F}_d^0)$, we get that $W$ is transversal to $\mathcal{F}_d$. Thus $Z$ is a smooth regular 1-dimensional submanifold of $W$. By its construction, $Z$ is exactly the folding locus of the map $Q : W \to S^1 \times [0, 1]$. Therefore, away from $Z$, the map $Q : W \to S^1 \times [0, 1]$ is a local diffeomorphism. Thus, away from $Z$, its composition $t^\bullet : W \xrightarrow{Q} S^1 \times [0, 1] \xrightarrow{t} [0, 1]$ is a non-singular function. As a result, the critical points of $t^\bullet : W \to [0, 1]$ are located along $Z$ and are among
the critical points of the function $t^*: Z \xrightarrow{\pi} G \xrightarrow{t} [0,1]$. Since $Z$ is the folding locus of $Q: W \to S^1 \times [0,1]$, the critical points of $t^*: W \to [0,1]$ are exactly the critical points of $t^*: Z \to [0,1]$, the later function being Morse by the construction of $t: S^1 \times [0,1] \to [0,1]$ and the fact that $\pi: Z \to G$ is a diffeomorphism. Therefore, all the critical points of $t^*: W \to [0,1]$ are of Morse type.

Remarkably, changes in the topology of the slices $\{(t^*)^{-1}(t) \cap W\}_{t \in [0,1]}$ and of the slices $\{(t^*)^{-1}(t) \cap Z\}_{t \in [0,1]}$ are synchronized in $t$.

So $W = \Lambda^{-1}(\partial E)$ delivers the desired cobordism between the loop patterns $W \cap (A \times \{0\})$ and $W \cap (A \times \{1\})$. As a result, the map $\Xi$ is well-defined and onto.

To validate (2), we use again Proposition 3.1 to produce a smooth ($\psi,t$)-parameter family of $x$-polynomials $\{P(x,\psi,t)\}$, whose roots form the surface $W$. The immersion (embedding when $d > 4$) $B: W \to A \times [0,1]$ bounds the given immersion $\beta: M \to A \times \{0\}$. For each $t \in [0,1]$, the $\psi$-family $\{P(x,\psi,t)\}$ gives rise to a loop $\gamma_t: S^1 \to P_\Omega^{\{d,|\sim|\geq 2\}}$ that depends continuously on $t$. Since for some $t^* \in [0,1]$, the $t^*$-slice of $W$ is empty, the loop $\gamma_{t^*}$ is a subset of $R_\Omega^{\{d\}}$ and thus is contractible in $P_\Omega^{\{d,|\sim|\geq 2\}}$. So the loop $\gamma_0$ is contractible in $P_\Omega^{\{d,|\sim|\geq 2\}}$. □

References


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