

GIAMBELLI-TYPE FORMULA FOR SUBBUNDLES OF THE TANGENT BUNDLE

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ABSTRACT. Consider a generic n -dimensional subbundle \mathcal{V} of the tangent bundle TM on some given manifold M . Given \mathcal{V} one can define different degeneracy loci $\Sigma_r(\mathcal{V})$, $r = (r_1 \leq r_2 \leq r_3 \leq \dots \leq r_k)$ on M consisting of all points $x \in M$ for which the dimension of the subspace $\mathcal{V}^j(x) \subset TM(x)$ spanned by all length $\leq j$ commutators of vector fields tangent to \mathcal{V} at x is less than or equal to r_j . Under a certain transversality assumption we 'explicitly' calculate the \mathbb{Z}_2 -cohomology classes of M dual to $\Sigma_r(\mathcal{V})$ using determinantal formulas due to W. Fulton and the expression for the Chern classes of the associated bundle of the free Lie algebras in terms of the Chern classes of \mathcal{V} .

1. PRELIMINARIES AND RESULTS

1.1. History and motivation. The question of the existence of a nontrivial sub-bundle of the tangent bundle on a given manifold is a geometric problem of a long-standing interest. (Such sub-bundles are often called *distributions* and we will freely use both terms below.) In the basic nontrivial case of rank 2 sub-bundles first important results in the area go back to the classical treatise [10]. Apparently the best achievements in this problem were obtained in late 60's by E. Thomas in [19],[21], see also a well-written survey [20]. Later some of his results were re-discovered by Y. Matsushita, [15]. Not much has been done in this area since then. One of the few recent exceptions is [11]. A rather detailed information is available about the existence of (oriented) sub-bundles of rank 2. For rank 3 and higher only the first obstruction to the latter problem is known, see [21]. The algebraic invariants for this result come from the Stiefel-Whitney classes of elements of $\mathbb{K}\tilde{\mathcal{O}}(M)$ which is the reduced real K -theory group of the manifold in question. Starting with late 70's the interest in the geometric properties of sub-bundles of the tangent bundle very stimulated by the development of the singularity theory and the revival of the interest in nonholonomic mechanics. A nice source of information about this topic is [16]. In particular, if a given manifold admits a sub-bundle of rank at least 2 one can construct at each point of the manifold an incomplete flag whose i th subspace is the linear span of the commutators of length at most i of the vector fields tangent to the sub-bundle. The ranks of these subspaces will (in general) depend on the point, see below. For a small generic perturbation of the original sub-bundle the ranks of the subspaces of the incomplete flags will stay constant almost everywhere on the manifold and depend only on the rank of the original sub-bundle and the dimension of the manifold. Generalizing the question we started with one can formulate the following problem

Problem. *When does a manifold admit a distribution whose associated flags have constant (and maximal possible) ranks throughout the manifold?*

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Being in general even more difficult than (still unsolved) initial question the latter problem has a nice answer in the case of oriented rank 2-distributions on oriented 4-manifolds. A rank 2-distribution on a 4-manifold whose associated flag has the set of ranks $(2, 3, 4)$ at each point is called the *Engel* distribution. Such a distribution has remarkable properties in many different aspects, see e.g. [3]-[5]. Namely, combining the results of [3] (see also [13]) together with the recent [23], [24] we get

Theorem. *An orientable 4-manifold admits an orientable Engel distribution if and only if the manifold is parallellizable.*

The aim of the present paper is to develop the basics of the obstruction theory adjusted to the above problem.

1.2. Standard definitions and notation. The following notions are standard, see e.g. [6]. Let M be an m -dimensional manifold and $\mathcal{V} \subset TM$ be an n -dimensional sub-bundle (a rank n -distribution) on TM . Given \mathcal{V} one associates at each point $x \in M$ its derived flag $f\mathcal{V}_x = \{\mathcal{V}_x = \mathcal{V}_x^1 \subseteq \mathcal{V}_x^2 \subseteq \dots \subseteq \mathcal{V}_x^j \subseteq \dots\}$, where $\mathcal{V}_x^j = \{\mathcal{V}_x^{j-1} + [\mathcal{V}_x^{j-1}, \mathcal{V}_x]\}_x$. If at each point $x \in M$ there exists a positive integer $k(x)$ such that the subspace $\mathcal{V}_x^{k(x)}$ coincides with TM_x then \mathcal{V} is called (maximally) *nonholonomic*. Let $n_j(x)$ denote the dimension of \mathcal{V}_x^j . The set of numbers $(n_1(x), \dots, n_k(x))$ is called *the growth vector* of \mathcal{V} at x . For a given nonholonomic \mathcal{V} the minimal number k such that $\mathcal{V}_x^k = TM_x$ at all points x is called *the degree of nonholonomicity*. (A sub-bundle \mathcal{V} is called *regular* if its $n_j(x)$ do not depend on x and the corresponding set of numbers n_1, n_2, \dots, n_k is called *the growth vector* of a regular sub-bundle \mathcal{V} .) Let \mathfrak{Lie}_n denote the free Lie algebra with n generators and let \mathfrak{Lie}_n^k be its linear subspace spanned by all elements of length k . Let $d(n, k)$ be the dimension of \mathfrak{Lie}_n^k . An n -dimensional \mathcal{V} with the degree of nonholonomicity k is called *a maximal growth sub-bundle* or *a mg-distribution* if $n_j = \sum_{i \leq j} d(n, i)$ for all $j < k$ and $n_k = m$. The growth vector with the entries $(n = d(n, 1), d(n, 1) + d(n, 2), \dots, d(n, 1) + d(n, 2) + \dots + d(n, k), m)$ is called *the maximal growth vector*.

Remark. According to [4] a germ of a generic distribution has maximal growth, where 'generic' means belonging to some open everywhere dense subset in C^∞ -Whitney topology. Thus locally a typical sub-bundle is a mg-distribution while globally there are (many) topological obstructions to the existence of mg-distributions on a given M . The problem we are addressing in the present paper can be reformulated as constructing obstructions to the existence of mg-distributions on a given manifold.

Examples. A contact structure is a regular mg-distribution. A 2-dimensional mg-subbundle on M_4 is called *an Engel distribution*, see above and [3]. It is the only example besides contact structures and their even-dimensional analogs with the stable local normal form.

1.3. Degeneracy loci. Given a generic n -distribution $\mathcal{V} \subset TM$ one expects that globally M contains a (typically reducible) degeneracy locus Σ consisting of all such points x where the growth vector $(n_1(x), \dots, n_k(x))$ is lexicographically smaller than the maximal one. Given a growth vector $r = (r_1 \leq r_2 \leq r_3 \leq \dots \leq r_k)$ denote by Σ_r the subset of all $x \in M$ satisfying the conditions $n_1(x) \leq r_1, \dots, n_k(x) \leq r_k$. Such Σ_r can be considered as degeneracy loci in the standard meaning of intersection theory, see §1, comp. [7], ch. 14. Namely, each n -distribution $\mathcal{V} \subset TM$ induces the associated fiber bundle $\mathfrak{Lie}(\mathcal{V}) \rightarrow M$ where the fiber \mathfrak{Lie}_x is the free Lie algebra generated by the subspace \mathcal{V}_x . $\mathfrak{Lie}(\mathcal{V})$ has an obvious grading $\mathfrak{Lie}(\mathcal{V}) = \bigoplus_{k=1}^{\infty} \mathfrak{Lie}^k(\mathcal{V})$ coming from the grading $\mathfrak{Lie}_n = \bigoplus_{k=1}^{\infty} \mathfrak{Lie}_n^k$. Moreover one can define a natural map

$\Phi : \mathfrak{L}\mathfrak{ie}(\mathcal{V}) \rightarrow TM$ of vector bundles sending each $\oplus_{i=1}^k \mathfrak{L}\mathfrak{ie}^i(\mathcal{V})$ onto \mathcal{V}^k . (The map Φ is not unique in very much the same way as the identifying map between TM and its nilpotentization, comp. [6].) This allows us to apply to the map Φ the determinantal formulas of [8] under the assumption that the considered Σ_r has the expected (co)dimension, i.e. the same codimension as the corresponding degeneracy locus for a generic map of flag bundles of the dimensions prescribed by $\mathfrak{L}\mathfrak{ie}_n$. Algebraically, in order to be able to apply these formulas one also needs to express the Chern classes of $\mathfrak{L}\mathfrak{ie}(\mathcal{V})$ in terms of that of \mathcal{V} .

Remark. W. Fulton in [8] has generalized a large number of previously known special cases of determinantal formulas giving the cohomology classes of different degeneracy loci for the maps of vector bundles to a very general situation of maps of flagged vector bundles. Such formulas could be traced back (through the works of Porteus and Thom as well as Laksov-Kempf and many other authors) to the pioneering results of G. Giambelli on the degrees of different strata in the spaces of matrices. For this reason in a number of publications the authors name similar determinantal formulas after Giambelli, see e.g. [8], §7. For detailed account on determinantal formulas and degeneracy loci we recommend [9] and for the information on Giovanni Giambelli see [14].

The contents of the paper is as follows. In §2 we construct the map $\Phi : \mathfrak{L}\mathfrak{ie}(\mathcal{V}) \rightarrow TM$. In §3 we find the explicit formula for the Chern character of the bundle $\mathfrak{L}\mathfrak{ie}^k(\mathcal{V})$ in terms of the Chern character of \mathcal{V} which turns out to be similar to the formula for the dimensions of $\mathfrak{L}\mathfrak{ie}_n^k$. In principle, this allows us in principle to calculate the Chern classes of $\mathfrak{L}\mathfrak{ie}^k(\mathcal{V})$ for any reasonable specific example by inverting Newton polynomials, see Appendix. In §4 we recall the appropriate determinantal formula for maps of flag bundles, adjust it to our needs and calculate some examples. Section §5 is devoted to some generalities on derived flags $f\mathcal{V}$ and the standard stratification of the spaces of matrices as well as counterexamples to transversality in big codimensions. In §6 we enumerate all potentially admissible defect vectors occurring for a generic distribution and prove the necessary transversality result showing that Σ_r have the expected (co)dimension in the cases $n = 2$, $m \leq 8$ and $n \geq 3$, $m \leq \frac{n(n+1)(2n+1)}{6}$. In §7 we briefly discuss some further directions of study and possible generalizations of the transversality theorem. The main result of the paper is formula (11) justified by the transversality theorem for the above mentioned values of (n, m) . Finally, appendix contains the Mathematica code which explicitly calculates the necessary Chern classes of the homogeneous components of the free Lie algebra $\mathfrak{L}\mathfrak{ie}_n^k$ up to order 4.

In short, the main proposition of the paper can be summarized as follows. In order to calculate the universal formula for the cohomology class dual to Σ_r for a given growth vector r one has to substitute the expressions for the Chern (or rather for the Stiefel-Whitney) classes of $\oplus_{i=1}^k \mathfrak{L}\mathfrak{ie}^i(\mathcal{V})$ and TM into the appropriate Giambelli-type formula.

The starting point of this note was the special case of 2-sub-bundles on M_4 considered in [13]. Later the authors realized that analogous computations over \mathbb{Z}_2 can be carried out in the general setup described in the present paper. Sincere thanks are due to R. Montgomery for important discussions of the subject and numerous remarks which substantially improved the quality of exposition. The second author is grateful to IHES(Paris), MPIM(Bonn) and MSRI(Berkeley) where parts of this project were carried out.

2. Σ_r AS CLASSICAL DEGENERACY LOCI

2.1. Associated bundle of free Lie algebras. Given $\mathcal{V} \subset TM$ let us now define the map $\Phi : \mathfrak{Lie}(\mathcal{V}) \rightarrow TM$ such that each sub-bundle $\oplus_{i=1}^j \mathfrak{Lie}^i(\mathcal{V})$ is mapped onto \mathcal{V}^j . In fact we define another filtered vector bundle $N(\mathcal{V})$ whose associated graded bundle is isomorphic to $\mathfrak{Lie}(\mathcal{V})$ with the canonical map $\Psi : N(\mathcal{V}) \rightarrow TM$. As a result one gets a (non-unique) map $\Phi : \mathfrak{Lie}(\mathcal{V}) \rightarrow TM$ defined up to a filtered isomorphism between $N(\mathcal{V})$ and $\mathfrak{Lie}(\mathcal{V})$.

2.2. Universal map. Let $\mathcal{V} \subset TM$ be an n -sub-bundle and k be its degree of nonholonomicity.

Theorem 1. *There exists a globally defined map of vector bundles*

$$\Phi_k : \oplus_{i \leq k} \mathfrak{Lie}^i(\mathcal{V}) \rightarrow TM$$

such that for all $x \in M$ and $j \leq k$, the subspace $\mathcal{V}^j(x)$ coincides with the image of $\Phi_j(x) : \oplus_{i \leq j} \mathfrak{Lie}^i(\mathcal{V})(x) \rightarrow T_x M$, where Φ_j is the restriction of Φ_k to $\oplus_{i \leq j} \mathfrak{Lie}^i(\mathcal{V}) \subset \oplus_{i \leq k} \mathfrak{Lie}^i(\mathcal{V})$.

For the local version of this theorem, see [6].

Proof. We construct an auxiliary flag of vector bundles

$$N^1(\mathcal{V}) \subset N^2(\mathcal{V}) \subset \dots \subset N^k(\mathcal{V}),$$

the map $\Psi_k : N^k(\mathcal{V}) \rightarrow TM$ satisfying the statement of Theorem, and a canonical isomorphism

$$N^j(\mathcal{V})/N^{j-1}(\mathcal{V}) \cong \mathfrak{Lie}^j(\mathcal{V})$$

which shows that the flag of bundles $N(\mathcal{V})$ is isomorphic to the flag of bundles $\mathfrak{Lie}(\mathcal{V})$.

Recall the notion of a Hall basis in the free Lie algebra \mathfrak{Lie}_n with n generators, see [2]. Namely, \mathfrak{Lie}_n has the following standard graded basis H called *Hall family* or *Hall basis*. Given a linearly ordered set V (of cardinality n) let us define the following linearly ordered subset H in the free monoid Mon_V .

1) if $u, v \in H$ and $l_{ng}(u) < l_{ng}(v)$ then $u < v$ where l_{ng} denotes the usual length of a word in Mon_V ; 2) $V = H^1 \subset H$ and H^2 consists of the set of all ordered pairs $(v_1, v_2) \in H$ where $v_1 < v_2$; 3) each element of H of length at least 3 has the form $a(bc)$ where $a, b, c \in H$, $bc \in H$, $b \leq a < bc$ and $b < c$. (Obviously, $H = \bigcup_{k=1}^{\infty} H^k$ where H^k is the set of all length k elements in H .)

Example. If $V = \{u < v\}$ then $H^1 = \{u; v\}$; $H^2 = \{(u, v)\}$; $H^3 = \{(u(u, v)); (v(u, v))\}$; $H^4 = \{(u(u(u, v))); (v(u(u, v))); (v(v(u, v)))\}$; $H^5 = \{(u(u(u(u, v)))); (v(u(u(u, v)))); (v(v(u(u, v))))\}$.

Now the construction of the flag of bundles $N(\mathcal{V})$ is as follows. Let W be the sheaf of free Lie algebras associated to the sheaf of local sections of the bundle $\mathcal{V} \subset TM$. The elements of W are \mathbb{R} -linear combinations of Lie monomials of sections of \mathcal{V} . Denote by \mathcal{A} the sheaf of rings of smooth functions on M . Define the homomorphism $D : W \rightarrow der \mathcal{A}$ as follows. If $v \in \mathcal{V} \subset W$ is of degree 1 we put

$$D_v f = v f, \quad (1)$$

i.e. the usual Lie derivative of the function f along the vector field v . Then, we assign by induction

$$D_{[a,b]} f = D_b(D_a f) - D_a(D_b f). \quad (2)$$

The operation D is well defined and D_u is the derivation of \mathcal{A} for every $u \in W$. Consider the sheaf $\mathcal{A} \otimes_{\mathbb{R}} W$ of \mathcal{A} -modules. We introduce the Lie algebra structure on $\mathcal{A} \otimes W$ as

$$[f \otimes u, g \otimes w] = fg \otimes [u, w] + f D_u g \otimes w - g D_w f \otimes u \quad (3)$$

(the Jacobi identity is verified by direct computation). Finally, define $N(\mathcal{V})$ to be the quotient Lie algebra of $\mathcal{A} \otimes W$ by the ideal generated by all relations of the form

$$f \otimes v = 1 \otimes fv, \quad f \in \mathcal{A}, \quad v \in \mathcal{V} \subset W. \quad (4)$$

Having (4) in mind we drop the sign of the tensor product in the notation of the elements in $N(\mathcal{V})$. The filtration on W by the length of Lie monomials gives the natural filtration $N^j(\mathcal{V})$ on the sheaf $N(\mathcal{V})$ of \mathcal{A} -modules. We claim that *all $N^j(\mathcal{V})$ are locally free sheafs of \mathcal{A} -modules of finite ranks*. Indeed, let e_1, \dots, e_n be the set of local sections of \mathcal{V} over some open domain $U \subset M$ such that these sections form a basis in each fiber $\mathcal{V}(x)$, $x \in U$. Then it follows from (1)–(4) that every section u of W over U can be represented as

$$u = \sum f_l h_l(e_1, \dots, e_n),$$

where f_l are some functions and $h_l \in H$ are the elements of Hall basis of the free Lie algebra $\mathfrak{L}\mathfrak{ie}_n$. Moreover, this representation is unique, i.e. the set of sections $h_l(e_1, \dots, e_n)$, $l \leq \dim \bigoplus_{i \leq j} \mathfrak{L}\mathfrak{ie}_n^i$ forms the set of free generators of the \mathcal{A} -module $N^j(\mathcal{V})$. Thus the \mathcal{A} -module $N^j(\mathcal{V})$ is the module of sections of some vector bundle which we also denote as $N^j(\mathcal{V})$.

Observe, that if $[u, w] \in N(\mathcal{V})$ has degree j then $[fu, gw] - fg[u, w]$ has degree strictly less than j . Therefore, the homomorphism $N^j(\mathcal{V})/N^{j-1}(\mathcal{V}) \rightarrow \mathfrak{L}\mathfrak{ie}^j(\mathcal{V})$ is well defined. Moreover, the arguments above show that this homomorphism is, in fact, an isomorphism of vector bundles.

The homomorphism of Lie algebras $\Psi : N(\mathcal{V}) \rightarrow Vect(M)$ is now obvious. It sends a formal Lie bracket of vector fields in \mathcal{V} to the corresponding commutator of these vector fields. Formulas (1)–(4) show that this homomorphism is well defined and $\Psi(N^j(\mathcal{V})(x))$ coincides with $\mathcal{V}^j(x)$ by definition. Theorem 1 is proved. \square

Remark. The vector bundle $N^j(\mathcal{V})$ can be also described, as a usual vector bundle by trivializations and transition functions. Trivializations of $N^j(\mathcal{V})$ correspond to the trivializations of \mathcal{V} and are given by the sections $h_l(e_1, \dots, e_n)$, $l \leq \dim \bigoplus_{i \leq j} \mathfrak{L}\mathfrak{ie}_n^i$, where e_1, \dots, e_n are sections giving some local basis of \mathcal{V} . If $\{e'_1, \dots, e'_n\}$ is another basis such that $e_i = \sum a_{ir} e'_r$ then to find transition functions for $N^j(\mathcal{V})$ one should express $h_l(\sum a_{1p} e'_p, \dots, \sum a_{np} e'_p)$ using (1)–(4) as a linear combination of $h_i(e'_1, \dots, e'_n)$ with some functional coefficients.

3. ON THE CHERN CLASSES OF THE BUNDLE OF FREE LIE ALGEBRAS

Let $E \rightarrow M$ be a complex vector bundle of dimension n over a smooth compact manifold M (not necessarily a sub-bundle of TM). For any linear representation of the group $GL(n, \mathbb{C})$ in \mathbb{C}^m one can associate in a natural way to the bundle $E \rightarrow M$ the corresponding m -dimensional bundle over M . For example, the bundles $E \otimes E$, E^* , $\Lambda^2 E$ etc. are associated with the obvious representations of $GL(n, \mathbb{C})$ in $\mathbb{C}^n \otimes \mathbb{C}^n$, \mathbb{C}^{n*} , $\Lambda^2 \mathbb{C}^n$ respectively.

Given a basis $\{e_1, \dots, e_n\}$ in \mathbb{C}^n let $\mathfrak{L}\mathfrak{ie}_n$ denote the free Lie algebra with the generators $\{e_1, \dots, e_n\}$, and let $\mathfrak{L}\mathfrak{ie}_n^k$ be its k th homogeneous component. A linear change of the above basis acts naturally on the spaces $\mathfrak{L}\mathfrak{ie}_n^k$. Denote by $\mathfrak{L}\mathfrak{ie}_n^k(E)$ or simply by $\mathfrak{L}\mathfrak{ie}^k(E)$ the bundle over M associated with this action. The relation between the characteristic classes of the bundles E and $\mathfrak{L}\mathfrak{ie}^k(E)$ is described in the following theorem. (This question was already proposed in [22].)

Let $ch(E) \in H^*(M)$ be the Chern character of a bundle E . For any element $\eta = \eta_0 + \eta_1 + \eta_2 + \dots \in H^*(M)$ and a number d set $(\eta)_d = \eta_0 + \eta_1 d + \eta_2 d^2 + \dots$, where η_i is homogeneous of degree $2i$.

Theorem 2, see e.g. [17]. *The Chern character of the bundle $\mathfrak{L}\mathfrak{ie}_n^k$ is given by the formula*

$$ch(\mathfrak{L}\mathfrak{ie}_n^k) = \frac{1}{k} \sum_{d|k} \mu(d) \left(ch(E)^{\frac{k}{d}} \right)_d. \quad (5)$$

Here the summation is taken over the set of all divisors of k and μ is the Möbius function. By taking the component of degree 0 in this formula we get the well-known expression for the dimension of $\mathfrak{L}\mathfrak{ie}_n^k$ (see [Bou])

$$dim(\mathfrak{L}\mathfrak{ie}_n^k) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}.$$

Proof. The main observation in this argument is that the total tensor algebra of \mathbb{C}^n is isomorphic, as the $GL(n, \mathbb{C})$ -module, to the universal enveloping algebra of $\mathfrak{L}\mathfrak{ie}_n$. Therefore, by Poincaré-Birkhoff-Witt theorem one has

$$T^*(E) \cong S^*(\mathfrak{L}\mathfrak{ie}(E)) = S^*(\mathfrak{L}\mathfrak{ie}^1) \otimes S^*(\mathfrak{L}\mathfrak{ie}^2) \otimes \dots$$

Applying the Chern character to both sides of the latter equality we get

$$\frac{1}{1 - ch(E)t} = \prod_{n=1}^{\infty} s(\mathfrak{L}\mathfrak{ie}^k)(t^k),$$

where t is a formal parameter, $s_i(V) = ch(S^i V)$ is the Chern character of the i th symmetric power of V , and $s(V)$ is a formal series $s(V)(t) = \sum s_i(V)t^i$. Now, applying $-t d \log$ to both sides we get

$$\frac{ch(E)t}{1 - ch(E)t} = - \sum_{n=1}^{\infty} k t^k (d \log s(\mathfrak{L}\mathfrak{ie}^k))(t^k). \quad (6)$$

Observe now that $-d \log s(V)(t) = (ch(V))_1 + (ch(V))_2 t + (ch(V))_3 t^2 + \dots$. To prove that we can assume (using the splitting principle) that the bundle V is represented as $V = V_1 \oplus \dots \oplus V_m$, where the 1-dimensional bundle V_i has the Chern character h_i . Then

$$\begin{aligned} -d \log s(V) &= -d \log(s(V_1)s(V_2)\dots s(V_m)) = -d \log \prod \frac{1}{1 - h_i t} = \sum \frac{h_i}{1 - h_i t} = \\ &= \sum h_i + \sum h_i^2 t + \sum h_i^3 t^2 + \dots = (ch(V))_1 + (ch(V))_2 t + (ch(V))_3 t^2 \dots, \end{aligned}$$

since $h_i^k = e^{kt_i} = \sum \frac{t_i^l k^l}{l!}$, where t_i denotes the first Chern class of V_i . Therefore, (6) is equivalent to

$$\frac{ch(E)t}{1 - ch(E)t} = \sum (ch(\mathfrak{L}\mathfrak{ie}^1))_k t^k + 2 \sum (ch(\mathfrak{L}\mathfrak{ie}^2))_k t^{2k} + 3 \sum (ch(\mathfrak{L}\mathfrak{ie}^3))_k t^{3k} + \dots,$$

Comparing the terms of the same degree in t we get

$$ch(E)^k = \sum_{d|k} d (ch(\mathfrak{L}\mathfrak{ie}^d))_{\frac{k}{d}}.$$

If we now multiply the l th homogeneous component of this equality by k^{-l} then after this re-scaling we get

$$(ch(E)^k)_{\frac{1}{k}} = \sum_{d|k} d (ch(\mathfrak{L}\mathfrak{ie}^d))_{\frac{1}{d}}.$$

Applying to the latter equality the Möbius inversion formula we obtain

$$k (ch(\mathfrak{L}\mathfrak{ie}^k))_{\frac{1}{k}} = \sum_{d|k} \mu(d) (ch(E)^{\frac{k}{d}})_{\frac{d}{k}}.$$

which (after another re-scaling) gives the required formula of Theorem 2. \square

Examples. The relation between the Chern classes and the Chern character gives the possibility to compute the Chern classes of $\mathfrak{L}ie_n^k$. For $k \leq 4$ (taking in account only terms of degree at most 4 in the characteristic classes) we obtain the following explicit formulas for the total Chern class of $\mathfrak{L}ie_n^k$

$$\begin{aligned}
c(\mathfrak{L}ie_n^1) &= c(E) = 1 + c_1 + c_2 + c_3 + c_4 + \dots, \\
c(\mathfrak{L}ie_n^2) &= 1 + (-1 + n)c_1 + \left(\left(1 - \frac{3n}{2} + \frac{n^2}{2}\right)c_1^2 + (-2 + n)c_2 \right) + \\
&\quad + \left(\left(-1 + \frac{11n}{6} - n^2 + \frac{n^3}{6}\right)c_1^3 + (4 - 4n + n^2)c_1c_2 + (-4 + n)c_3 \right) + \\
&\quad + \left(\left(1 - \frac{25n}{12} + \frac{35n^2}{24} - \frac{5n^3}{12} + \frac{n^4}{24}\right)c_1^4 + (-6 + 8n - \frac{7n^2}{2} + \frac{n^3}{2})c_1^2c_2 + \right. \\
&\quad \left. + (3 - \frac{5n}{2} + \frac{n^2}{2})c_2^2 + (9 - 6n + n^2)c_1c_3 + (-8 + n)c_4 \right) + \dots, \\
c(\mathfrak{L}ie_n^3) &= 1 + (-1 + n^2)c_1 + \left((2 - n - \frac{3n^2}{2} + \frac{n^4}{2})c_1^2 + (-3 + n^2)c_2 \right) + \\
&\quad + \left((-4 + 3n + \frac{17n^2}{6} - n^3 - n^4 + \frac{n^6}{6})c_1^3 + (12 - 4n - 5n^2 + n^4)c_1c_2 + (-9 + n^2)c_3 \right) + \\
&\quad + \left((8 - \frac{15n}{2} - \frac{61n^2}{12} + \frac{7n^3}{2} + \frac{47n^4}{24} - \frac{n^5}{2} - \frac{5n^6}{12} + \frac{n^8}{24})c_1^4 + \right. \\
&\quad \left. + (-36 + 19n + \frac{35n^2}{2} - 5n^3 - 4n^4 + \frac{n^6}{2})c_1^2c_2 + \right. \\
&\quad \left. + (18 - 6n - \frac{7n^2}{2} + \frac{n^4}{2})c_2^2 + (36 - 6n - 11n^2 + n^4)c_1c_3 + (-27 + n^2)c_4 \right) + \dots, \\
c(\mathfrak{L}ie_n^4) &= 1 + (-n + n^3)c_1 + \left((1 + n - n^2 - \frac{n^3}{2} - n^4 + \frac{n^6}{2})c_1^2 + (-2(n + n^3))c_2 \right) + \\
&\quad + \left((-4 - \frac{n}{3} + 2n^2 + \frac{8n^3}{3} + \frac{3n^4}{2} - n^5 - \frac{n^6}{2} - \frac{n^7}{2} + \frac{n^9}{6})c_1^3 + \right. \\
&\quad \left. + (8 + 4n - 4n^2 - n^3 - 3n^4 + n^6)c_1c_2 + (-4n + n^3)c_3 \right) + \\
&\quad + \left((13 - 2n - \frac{77n^2}{12} - \frac{35n^3}{4} - \frac{3n^4}{4} + \frac{5n^5}{2} + \frac{55n^6}{24} + n^7 - \frac{n^8}{2} - \frac{n^9}{4} - \frac{n^{10}}{6} + \frac{n^{12}}{24})c_1^4 + \right. \\
&\quad \left. + (-48 + 12n^2 + 18n^3 + 7n^4 - 5n^5 - \frac{3n^6}{2} - 2n^7 + \frac{n^9}{2})c_1^2c_2 + \right. \\
&\quad \left. + (24 + 4n - 7n^2 - \frac{n^3}{2} - 2n^4 + \frac{n^6}{2})c_2^2 + \right. \\
&\quad \left. + (24 + 8n - 5n^2 - n^3 - 5n^4 + n^6)c_1c_3 + (-8n + n^3)c_4 \right) + \dots,
\end{aligned}$$

Remark. Substituting w_i instead of c_i in the above formulas and reducing coefficients mod 2 one gets the expression for the total Stiefel-Whitney class of $\mathfrak{L}ie_n^k$ in the case of a real n -dimensional bundle E . Note, that the coefficients of the above polynomials have integer values for any n and therefore their values mod 2 are well defined.

4. DETERMINANTAL FORMULA AND ITS APPLICATION

4.1. Determinantal formula. First we recall a certain formula borrowed from [8]. Assume that we have a flag $A_1 \subset A_2 \subset \dots \subset A_l$ of the complex vector bundles over a manifold M with the ranks $a_1 \leq a_2 \leq \dots \leq a_l$ resp. and a map

$$h : A_1 \subset A_2 \subset \dots \subset A_l \rightarrow B$$

to a manifold B of dimension b . Assume furthermore that the set of nonnegative integers $\kappa_1, \dots, \kappa_l$ satisfies the inequalities

$$0 < a_1 - \kappa_1 < a_2 - \kappa_2 < \dots < a_l - \kappa_l, \quad \kappa_1 < \kappa_2 < \dots < \kappa_l < b. \quad (7)$$

Let $\Omega_\kappa \subset M$ be the degeneracy locus defined by the conditions $\text{rk}(h : A_i \rightarrow B) \leq \kappa_i$, $i = 1, \dots, l$. that is the set of all points $x \in M$ where all the previous conditions are valid. Now consider the Young diagram $(p_1^{m_1}, \dots, p_l^{m_l})$ where

$$\begin{aligned}
p_1 &= a_l - \kappa_l, p_2 = a_{l-1} - \kappa_{l-1}, \dots, p_l = a_1 - \kappa_1, \\
m_1 &= b - \kappa_l, m_2 = \kappa_l - \kappa_{l-1}, \dots, m_l = \kappa_2 - \kappa_1.
\end{aligned}$$

Its dual diagram is $\mu = (q_1^{n_1}, \dots, q_l^{n_l})$ where

$$q_1 = b - \kappa_1, \dots, q_l = b - \kappa_l,$$

$$n_1 = a_1 - \kappa_1, \dots, n_l = (a_l - \kappa_l) - (a_{l-1} - \kappa_{l-1}).$$

Let $cd(\kappa) = |\lambda|$, $s\lambda = b - \kappa_1$, $s\mu = a_l - \kappa_l$. Finally, set

$$\rho(i) = \max\{s \in [1, l] : i \leq b - \kappa_s = m_1 + \dots + m_{l+1-s}\}, i = 1, \dots, s\lambda,$$

$$\rho'(i) = \min\{s \in [1, l] : i \leq a_s - \kappa_1 = n_1 + \dots + n_s\}, i = 1, \dots, s\mu.$$

Proposition 1, see 10.2. of [8]. *If the codimension of Ω_r equals $cd(r)$ then the \mathbb{Z} -cohomology class $[\Omega_r]_{\mathbb{Z}}$ of M dual to Ω_r is given by*

$$[\Omega_r]_{\mathbb{Z}} = \det(c_{\lambda_i - i + j}(A_{\rho(i)}^* - B^*))_{1 \leq i, j \leq s\lambda} = \det(c_{\mu_i - i + j}(B - A_{\rho'(i)}))_{1 \leq i, j \leq s\mu}, \quad (8)$$

where $'*$ ' denotes the dual bundle.

4.2. Real case. Consider a fixed flag of vector spaces

$$\mathbb{R}^{a_1} \subset \mathbb{R}^{a_2} \subset \dots \subset \mathbb{R}^{a_l}.$$

Denote by $Mat(a_l, b)$ the space of all linear maps $\mathbb{R}^{a_l} \rightarrow \mathbb{R}^b$ identified with the space of $(a_l \times b)$ -matrices. Two elements $u, v \in Mat(a_l, b)$ are called *equivalent* if for all $i = 1, \dots, l$ the restriction of u and v to \mathbb{R}^{a_i} have the same rank. The set of all pairwise equivalent elements in $Mat(a_l, b)$ will be called a *stratum*. Obviously, one obtains in this way a finite stratification of $Mat(a_l, b)$.

Using the same notation as above consider now a map h of *real* vector bundles

$$h : A_1 \subset A_2 \subset \dots \subset A_l \rightarrow B.$$

Let $x \in M$ be a point of the base and $U \ni x$ be its small neighborhood such that the bundles are trivial and trivialized over U . Then the map h over U is given by a family of matrices

$$h_U : U \rightarrow Mat(a_l, b).$$

Definition. The map h is called *transversal* at the point x if the map h_U is transversal to the stratum containing the point $h_U(x)$. The map h is called *transversal* if it is transversal at every point $x \in M$.

Note that the transversality condition does not depend on the trivialization of the bundles chosen over U . Thom's transversality theorem implies that a generic map h is transversal at every point x .

Corollary 1. *If a map h is transversal then for any growth vector r its degeneracy locus Ω_r is a closed (possibly empty) sub-variety of M . The dual \mathbb{Z}_2 -cohomology class given by the intersection index with the smooth part of Ω_r is well-defined and given by the analog of formula (8) where the Chern classes substituted by the corresponding Stiefel-Whitney classes.*

Proof. One should follow step-by-step the proof of Proposition 10.2 of [8]. In fact, one can show that the proof of formula (8) can be reduced to the following basic results:

1) the axioms of the Chern classes the most important of which being the Whitney formula $c(C \oplus F) = c(E)c(F)$.

2) the fact that the cohomology ring of $\mathbb{C}P^n$ is given by $H^*(\mathbb{C}P^n) = \mathbb{Z}[c_1]/c_1^{n+1}$, where c_1 is the first Chern class of the tautological bundle over $\mathbb{C}P^n$.

3) the construction of the Gysin map $\phi_* : H^*(X) \rightarrow H^*(Y)$ for the proper map $\phi : X \rightarrow Y$ of smooth manifolds, and the Gysin formula $\phi_*(\phi^*a \cup b) = a \cup \phi_*(b)$, $a \in H^*(X)$, $b \in H^*(Y)$.

4) the relation

$$p_*(1/(1 - c_1(S))) = c^{-1}(E^*),$$

where $\pi : E \rightarrow M$ is any complex vector bundle, $p : P \rightarrow M$ is its projectivization, and finally S is the natural tautological linear sub-bundle in the bundle $p^*\pi$ over P .

All these assertions have their real analogs with \mathbb{Z} -cohomology replaced by \mathbb{Z}_2 -cohomology, the Chern classes by the Stiefel-Whitney classes, $\mathbb{C}P^n$ by $\mathbb{R}P^n$ etc. The proof of the corollary follows. \square

4.3. Application to sub-bundles. If we drop the restrictions (7) then for a given n -sub-bundle $\mathcal{V} \subset TM_m$ and a given growth vector $r = (r_1 = n \leq r_2 \leq \dots \leq r_k = m)$ the degeneracy locus Σ_r is the subset $\Omega_r \subset M$ for the map

$$\Phi_k : \mathfrak{S}\mathfrak{ie}^1(\mathcal{V}) \subset \dots \subset \bigoplus_{i=1}^k \mathfrak{S}\mathfrak{ie}^i(\mathcal{V}) \rightarrow TM.$$

Let us denote $L_j(\mathcal{V}) = \bigoplus_{i=1}^j \mathfrak{S}\mathfrak{ie}^i(\mathcal{V})$ and $\partial(n, j) = \dim(\bigoplus_{i=1}^j \mathfrak{S}\mathfrak{ie}^i(\mathcal{V})) = \sum_{i=1}^j d(n, i)$. In order to apply Fulton's formula (8) we must get rid of the redundant subspaces, i.e. those subspaces whose rank conditions are automatically satisfied due to the rank conditions imposed on the previous subspaces. (In other words, the i th subspace is redundant if $\partial(n, i) - r_i = \partial(n, i-1) - r_{i-1}$.) We call by a *reduced index set* $I = (i_1, \dots, i_l)$ the maximal subset of indices for which the corresponding r_{i_j} satisfy the conditions

$$r_{i_1} < \dots < r_{i_l} < m, \quad 0 < \partial(n, i_1) - r_{i_1} < \partial(n, i_2) - r_{i_2} < \dots < \partial(n, i_l) - r_{i_l},$$

i.e. both ranks and coranks are strictly increasing.

One gets the following Young diagram $\lambda(r) = (p_1(r)^{m_1(r)}, \dots, p_l(r)^{m_l(r)})$ where

$$p_1(r) = \partial(n, i_l) - r_{i_l}, \quad p_2(r) = \partial(n, i_{l-1}) - r_{i_{l-1}}, \quad \dots, \quad p_l(r) = \partial(n, i_1) - r_{i_1}, \quad (9)$$

$$m_1(r) = m - r_{i_l}, \quad m_2(r) = r_{i_l} - r_{i_{l-1}}, \quad \dots, \quad m_l(r) = r_{i_2} - r_{i_1}.$$

Its dual diagram is $\mu(r) = (q_1(r)^{n_1(r)}, \dots, q_l(r)^{n_l(r)})$ where

$$q_1(r) = m - r_{i_1}, \quad \dots, \quad q_l(r) = m - r_{i_l},$$

$$n_1(r) = \partial(n, i_1) - r_{i_1}, \quad \dots, \quad n_l(r) = (\partial(n, i_l) - r_{i_l}) - (\partial(n, i_{l-1}) - r_{i_{l-1}}). \quad (10)$$

Finally, we set $cd(r) = |\lambda(r)| = |\mu(r)|$ is the area of either of these Young diagrams.

Analogously, $s\lambda(r) = m - r_{i_1}$, $s\mu(r) = \partial(n, i_l) - r_{i_l}$ and

$$\rho_r(i) = \max\{s \in [1, l] : i \leq m - r_{i_s} = m_1(r) + \dots + m_{l+1-s}(r)\}, \quad i = 1, \dots, s\lambda(r)$$

$$\rho'_r(i) = \min\{s \in [1, l] : i \leq \partial(n, i_s) - r_{i_1} = n_1(r) + \dots + n_s(r)\}, \quad i = 1, \dots, s\mu(r).$$

Definition. The number $cd(r)$ is called the *expected codimension* of Σ_r .

Main result. *If $\text{codim}(\Sigma_r(\mathcal{V}))$ coincides with its expected codimension $cd(r)$ then the \mathbb{Z}_2 -cohomology class $[\Sigma_r]_{\mathbb{Z}_2}$ of the base manifold M dual to Σ_r is given by*

$$\begin{aligned} [\Sigma_r]_{\mathbb{Z}_2} &= \det(w_{\lambda_i(r)-i+j}(L(\mathcal{V})_{\rho_r(i)}^* - TM^*))_{1 \leq i, j \leq s\lambda(r)} = \\ &= \det(w_{\mu_i(r)-i+j}(TM - L(\mathcal{V})_{\rho'_r(i)}))_{1 \leq i, j \leq s\mu(r)}, \end{aligned} \quad (11)$$

where w_l are the Stiefel-Whitney classes.

Examples, compare [13]. Consider a generic 2-sub-bundle in TM_4 . There are 3 possible non-maximal growth vectors $(2, 2, 4)$, $(2, 3, 3)$, $(2, 2, 3, 4)$. The coincidence of the actual and the expected codimensions in this case follows from the normal forms in [26]. (In the case $(2, 2, 2, \dots)$ the codimension is ≥ 5 .)

1) Case $r = (2, 2, 4)$. The reduced index set is $I = \{2\}$, i.e. we have to consider only the map $\Phi_2 : L_2(\mathcal{V}) \rightarrow TM$ of the usual bundles and determine the locus of points where $rk(\Phi_2) \leq 2$. One has $rk(L_2(\mathcal{V})) = 3$, $rk(TM) = 4$,

$\lambda(r) = (1^2)$, $s\lambda(r) = 2$, $cd(r) = 2$. Finally, $\mu(r) = 2$, $s\mu(r) = 1$ and $\rho'(1) = 1$. Therefore,

$$[\Sigma_{(2,2,4)}]_{\mathbb{Z}_2} = w_2(TM - L_2(\mathcal{V})) = w_2(M) + w_2(\mathcal{V}) + w_1^2(\mathcal{V}).$$

2) Case $r = (2, 3, 3)$. The reduced index set is $I = \{3\}$, i.e. we have to consider only the map $\Phi_3 : L_3(\mathcal{V}) \rightarrow TM$ of the usual bundles and determine the locus of points where $rk(\Phi_3) \leq 3$. One has $rk(L_3(\mathcal{V})) = 5$, $rk(TM) = 4$, $\lambda(r) = (2^1)$, $s\lambda(r) = 1$, $cd(r) = 2$. Finally, $\mu(r) = (1^2)$, $s\mu(r) = 2$ and $\rho'(1) = 1$, $\rho'(2) = 1$. Therefore,

$$\begin{aligned} [\Sigma_{(2,3,3)}]_{\mathbb{Z}_2} &= \begin{vmatrix} w_1(TM - L_3(\mathcal{V})) & w_2(TM - L_3(\mathcal{V})) \\ w_0(TM - L_3(\mathcal{V})) & w_1(TM - L_3(\mathcal{V})) \end{vmatrix} = \\ &= w_1^2(M) + w_1^2(\mathcal{V}) + w_2(M) + w_1(M)w_1(\mathcal{V}). \end{aligned}$$

3) Case $r = (2, 2, 3, 4)$. The reduced index set is $I = \{2, 3\}$, i.e. we have to consider the map $\Phi : L_2(\mathcal{V}) \subset L_3(\mathcal{V}) \rightarrow TM$. One has $\lambda(r) = (2, 1)$, $s\lambda(r) = 2$, $cd(r) = 3$. Now, $\mu(r) = (2, 1)$, $s\mu(r) = 2$ and $\rho'(1) = 1$, $\rho'(2) = 2$. Therefore,

$$\begin{aligned} [\Sigma_{(2,2,3,4)}]_{\mathbb{Z}_2} &= \begin{vmatrix} w_2(TM - L_2(\mathcal{V})) & w_3(TM - L_2(\mathcal{V})) \\ w_0(TM - L_3(\mathcal{V})) & w_1(TM - L_3(\mathcal{V})) \end{vmatrix} = \\ &= w_1(M)w_2(M) + w_2(M)w_1(\mathcal{V}) + w_1^3(\mathcal{V}) + w_3(M). \end{aligned}$$

The above answers are obtained by the standard manipulations with the total Chern class of TM and $\mathfrak{L}\mathfrak{e}_n^k$.

5. TRANSVERSALITY PROPERTY FOR SUB-BUNDLES AND SOME GENERAL PROPERTIES ON $f\mathcal{V}$

In order to be able to apply formula (11) to sub-bundles one needs to show that a certain transversality property is valid for the map $\Phi_k : \oplus_{i \leq k} \mathfrak{L}\mathfrak{e}^i(\mathcal{V}) \rightarrow TM$, see §2. This condition can be formulated as follows. The total space $Hom(\oplus_{i \leq k} \mathfrak{L}\mathfrak{e}^i(\mathcal{V}), TM)$ has a natural stratification according to different degenerations of the growth vector. The transversality property says that the section of the above bundle determined by the map Φ_k is transversal to each stratum of this natural stratification.

Naturally one wants to know if the transversality property is valid for generic n -dimensional sub-bundles $\mathcal{V} \subset TM$. The conjecture stated below claims that this is indeed the case. (Up to codimension $m - \sqrt{m}$ a similar statement is shown to be valid in [6].)

Since the transversality property is essentially local let us formulate it in local terms.

5.1. Local problem. Take $M = \mathbb{R}^m$ with a fixed system coordinates x_1, \dots, x_m and consider the set Ω^0 of germs of n -sub-bundles in \mathbb{R}^m such that for any $\mathcal{V} \in \Omega^0$ the subspace $\mathcal{V}(0)$ at the origin is spanned by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. The set Ω^0 can be identified with the set of all n -tuples of vector-fields v_1, \dots, v_n of the form $v_i = \frac{\partial}{\partial x_i} + \sum_{j=n+1}^m a_{i,j}(x_1, \dots, x_m) \frac{\partial}{\partial x_j}$. Indeed, fixing the standard Euclidean structure on \mathbb{R}^m we can uniquely lift the vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ to any sub-bundle $\mathcal{V} \in \Omega^0$ and get the n -tuple of vector fields $v_1(\mathcal{V}), \dots, v_n(\mathcal{V})$ with the above properties.

Remark. For each k we have the derived map $\Psi_k : \mathcal{V} \rightarrow Fl_k(\mathcal{V})$ where $Fl_k(\mathcal{V}) = (\mathcal{V} = \mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots \subset \mathcal{V}_k)$. Each $Fl_k(\mathcal{V})$ is the image under the canonical map of the \mathcal{A} -module $N_k(\mathcal{V})$, see §2. Fixing the standard Euclidean structure, the n -tuple of vector fields ∂_i and the Hall basis we obtain the standard set of sections for all $\mathcal{V} \in \Omega^0$ and in all $N_k(\mathcal{V})$. This gives us a non-canonical isomorphism of \mathcal{V}_1 and \mathcal{V}_2 and $N_k(\mathcal{V}_1)$ and $N_k(\mathcal{V}_2)$ as the \mathcal{A} -modules for any sub-bundles $\mathcal{V}_1, \mathcal{V}_2$ and any positive integer k . Localizing we can identify the jets of the sub-bundle \mathcal{V}

with the jets of the n -tuples of vector fields $v_1(\mathcal{V}), \dots, v_n(\mathcal{V})$ and the jets of $Fl_k(\mathcal{V})$ with the jets of the $\partial(n, k)$ -tuples of vector fields obtained from $v_1(\mathcal{V}), \dots, v_n(\mathcal{V})$ by applying the commutations prescribed by the elements in the chosen Hall basis. (Recall that $\partial(n, k) = \sum_{i=1}^k d(n, i)$.) The map Ψ_k induces the well-defined map $\Psi_k^i : j^{k+i}(\mathcal{V}) \rightarrow j^i(Fl_k(\mathcal{V}))$ of the corresponding jets.

Remark. The 0-jet of $Fl_k(\mathcal{V})$ can be represented by a $m \times \partial(n, i)$ -matrix of the form

$$\begin{pmatrix} \vdots & n & \vdots & m-n & \vdots & d(n, 2) & \vdots & d(n, 3) & \vdots & \dots & \vdots & d(n, k) & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n & \vdots & \mathbf{1} & 0 & 0 & 0 & \dots & 0 & \vdots \\ m-n & \vdots & 0 & 0 & * & * & \dots & * & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (12)$$

Here $\mathbf{1}$ and 0 denote the identity and the zero matrices of the sizes given in the first row and the first column and '*' stands for arbitrary real entries.

Further notation. Let $Mat^0(n, m, k)$ denote the subset of all $m \times \partial(n, k)$ -matrices of the above form (12) and $J(n, m, k)$ denote the space of k -jets of $\mathcal{V} \in \Omega^0$. The space $J(n, m, k)$ is isomorphic to the space of all $n(m-n)$ -tuples of polynomials in m variables of degree $\leq k$, see above. Obviously, $\dim J(n, m, k) = (m-n)n \binom{m+k}{k}$ and $\dim Mat^0(n, m, k) = (m-n) \sum_{i=2}^k d(n, i)$. The main object of the remaining part of the paper is the polynomial map of affine spaces

$$\Psi_k^0 : J(n, m, k-1) \rightarrow Mat^0(n, m, k).$$

The space $Mat^0(n, m, k)$ has the following natural stratification. We start with the obvious inclusions $Mat^0(n, m, 1) \subset Mat^0(n, m, 2) \subset \dots \subset Mat^0(n, m, k)$. Now fixing the growth vector $r = (r_1 = n \leq r_2 \leq \dots \leq r_k \leq \partial(n, k))$ we define the subset $Mat_r^0(n, m, k) \subset Mat^0(n, m, k)$ of all matrices restriction of which to $Mat^0(n, m, i)$ have rank $\leq r_i$ for all $i = 1, \dots, k$. Rather obviously, the codimension of $Mat_r^0(n, m, k)$ in $Mat^0(n, m, k)$ equals $cd(r)$.

Finally we are in position to formulate the required transversality property.

Main Conjecture. *The subset of points x in $J(n, m, k)$ such that the map Ψ_k^0 is non-transversal to the stratum containing the point $\Psi_k^0(x) \in Mat^0(n, m, k)$ has the codimension in $J(n, m, k)$ strictly exceeding m .*

Thom's transversality theorem implies that the validity of the above conjecture immediately leads to the transversality assumption for generic n -dimensional sub-bundles in TM .

We were unable to prove the above conjecture in its complete generality but we were able to settle a number of cases given below.

Transversality Theorem. *The required transversality property holds either for $n \geq 3$ and $m \leq \frac{n(n+1)(2n+1)}{6} = \partial(n, 3)$ or for $n = 2$ and $m \leq 8 = \partial(2, 4)$. Namely, for $k = 2, 3$ and any $m \geq n$ the map Ψ_k^0 is a submersion. Also, for $n = 2$ and any $m \geq n$ and $k = 4$ the map Ψ_4^0 is a submersion.*

5.2. Violation of transversality for big codimensions. We will finish this section by pointing out that the behavior of the image $\Psi_k^0(J(n, m, k-1))$ w.r.t the natural rank stratification of the space $Mat^0(n, m, k)$ is highly nontrivial. The next 2 statements show that one can only hope that transversality holds only for the strata of relatively small codimension as stated in the main conjecture.

Lemma 1. *For any fixed $m \geq n$ and for $k > \text{const} \log_{\binom{2m}{n}} \binom{2m}{m}$ one has $\dim J(n, m, k-1) < \dim Mat^0(n, m, k)$ and therefore Ψ_k is not surjective.*

Proof. This is simply the dimension count since $\dim J(n, m, k-1) = (m-n)n \binom{m+k-1}{k-1}$ and $\dim Mat^0(n, m, k) = (m-n) \sum_{i=2}^k d(n, i)$ where $kd(n, k) = \sum_{j|k} \mu(j)n^{k/j}$ with $\mu(j)$ denoting the Möbius function. \square

Lemma 2. *For any $m \geq n \geq 3$ and $k \geq 4$ the map Ψ_k is never onto. The same holds for $n = 2$ and $k \geq 5$.*

Proof. Consider a matrix in $Mat^0(n, m, 4)$ of the form

$$\begin{array}{cccccccc} \vdots & n & \vdots & m-n & \vdots & d(n, 2) & \vdots & d(n, 3) & \vdots & d(n, 4) & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ n & \vdots & \mathbf{1} & 0 & 0 & 0 & * & \text{full} & \vdots & \cdot & \\ m-n & \vdots & 0 & 0 & 0 & 0 & * & \text{rank} & \vdots & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

Such a matrix is never in the image of Ψ_4 since the 4-th homogeneous component H^4 of the Hall basis contains elements of the form $((v_i, v_j), (v_p, v_q))$ which vanish since the first commutators of all basic vector fields vanish. (For $n = 2$ the same effect happens for $k \geq 5$.) \square

6. TRANSVERSALITY THEOREM AND DEFECT VECTORS

This section is devoted to the proof of transversality theorem of the previous section as well as to the description of generic defect vectors in our situation. Namely, let $w = (((v_{i_1}, v_{i_2})(v_{i_3}, v_{i_4})) \dots ((v_{i_{k-4}}, v_{i_{k-3}})(v_{i_{k-1}}, v_{i_k}))) \in H^k$ be any element of length k in the Hall basis.

Definition. By the *depth* $dp_w(v_{i_j})$ of any variable v_{i_j} in w we denote the difference between the number of opening and closing parentheses preceding v_{i_j} . By the *depth* $dp(w)$ of w we mean $\max_j dp_w(v_{i_j})$.

Obviously, for any j one has $dp_w(v_{i_j}) \leq dp(w)$ and there exist at least 2 different values of j for which $dp_w(v_{i_j}) = dp(w)$.

Let H^p denote the p th homogeneous component of H and $H^{p,q}$ be its subset containing all elements of depth q . Obviously, q can vary between $\lceil \log_2(p) \rceil + 1$ and p . Let us restrict the map Ψ_k^0 to the union $\bigcup_{j=1}^k H^{j,j}$.

Lemma 3. *For any k and $m \geq n$ the restricted map*

$$\Psi_k^{res} : J(n, m, k-1) \rightarrow Mat_{res}^0(n, m, k)$$

is a submersion where $Mat_{res}^0(n, m, k) \subset Mat^0(n, m, k)$ contains only rows corresponding to the elements from $\bigcup_{j=1}^k H^{j,j}$.

Proof. We use induction on k .

Base of induction, $k = 2$. One has $(v_i, v_j)_l = a_{j,i,l} - a_{i,j,l}$ where $(v_i, v_j)_l$ is the value of the l th component of the commutator (v_i, v_j) at the origin, $a_{j,i,l}$ is the coefficient at x_i in the l th component of v_j and $a_{i,j,l}$ is the coefficient at x_j in the l th component of v_i . Since $a_{j,i,l}$ and $a_{i,j,l}$ are independent parameters in the pre-image the map from $J(n, m, 1)$ to $Mat_{res}^0(n, m, 2)$ is, obviously, a submersion.

Step of induction. Assume that the statement is proved for $j < k$. All elements in $H^{k,k}$ are in 1-1-correspondence with the commutators $(v_{i_1}(v_{i_2}(\dots(v_{i_{k-1}}, v_{i_k})))\dots)$ where $i_1 \geq i_2 \geq \dots \geq i_{k-1} < i_k$, see def. of the Hall basis. (In particular, the number $\#_k(n)$ of such elements equals $\sum_{j=0}^{n-1} \binom{j+k-2}{j} j$. For example, $\#_2(n) = \frac{n(n-1)}{2}$, $\#_3(n) = \frac{n(n-1)(n+1)}{3}$, $\#_4(n) = \frac{(n-1)n(n+1)(n+2)}{8}$, $\#_5(n) = \frac{n(n-1)^2(n+2)(n+3)}{30}$, $\#_6(n) = \frac{n^2(n^2-1)(n^2+n+2)}{16}$.)

One has

$$(v_{i_1}(v_{i_2}(\dots(v_{i_{k-1}}, v_{i_k})))\dots)_l = \text{const} + \frac{a_{i_k, i_1 i_2 \dots i_{k-1}, l}}{c(i_1, \dots, i_{k-1})} - \frac{a_{i_{k-1}, i_1 i_2 \dots i_{k-2} i_k, l}}{c(i_1, \dots, i_{k-2}, i_k)}.$$

Here $(v_{i_1}(v_{i_2}(\dots(v_{i_{k-1}}, v_{i_k})))\dots)_l$ is the value at 0 of the l th component of the commutator vector field corresponding to the considered element of $H^{k,k}$. The term 'const' in the r.h.s. depends only on the $(k-2)$ nd jet of the basic vector fields $v_1(\mathcal{V}), \dots, v_n(\mathcal{V})$; $c(i_1, \dots, i_{k-1})$ and $c(i_1, \dots, i_{k-2}, i_k)$ are factorial type constant factors. Finally, in the above formula we denote by $a_{i_k, i_1 i_2 \dots i_{k-1}, l}$ (resp. $a_{i_{k-1}, i_1 i_2 \dots i_{k-2} i_k, l}$) the variable coefficient at the product $x_{i_1} x_{i_2} \dots x_{i_{k-1}}$ in the l th component of the basic vector field v_{i_k} (resp. at the product $x_{i_1} x_{i_2} \dots x_{i_{k-2}} x_{i_k}$ in the l th component of the basic vector field $v_{i_{k-1}}$). One can easily check that the variable coefficient $a_{i_k, i_1 i_2 \dots i_{k-1}, l}$ appears only in the commutator $(v_{i_1}(v_{i_2}(\dots(v_{i_{k-1}}, v_{i_k})))\dots)_l$. Since all these variable coefficients are independent parameters in the pre-image we get the submersion of the space $\Omega_{\mathfrak{J}}$ on the space $\widetilde{Mat}_{res}^0(n, m, k)$. Here $\Omega_{\mathfrak{J}}$ is the subspace of all sub-bundles in Ω with some arbitrary fixed $(k-2)$ -jet \mathfrak{J} and $\widetilde{Mat}_{res}^0(n, m, k)$ is the last block in $Mat_{res}^0(n, m, k)$. \square

Corollary 2. *The above Lemma implies the transversality theorem of the previous section.*

Proof. Indeed, in both cases $n = 2$, $m \leq 8$ and $n \geq 3$, $m \leq \frac{n(n+1)(2n+1)}{6} = \partial(n, 3)$ one has $H^k = H^{k,k}$. \square

6.1. Description of generic defect vectors. We give a *conjectural* description of all possible degenerations of the associated flag bundle $f\mathcal{V}$ which might occur for a generic n -dimensional bundle \mathcal{V} in TM .

Definition. For any given growth vector $r = (r_1 = n, \dots, r_k = m)$ we call the vector $def(r) = (\partial(n, 1) - r_1 = 0; (\partial(n, 2) - r_2) - (\partial(n, 1) - r_1); \dots; (\partial(n, k-1) - r_{k-1}) - (\partial(n, k-2) - r_{k-2}); 0)$ the *defect vector* of r . (Up to the reduction of redundant indices this definition coincides with n_i 's in (9).) A stratum St_r is called *admissible* (resp. *potentially admissible*) if $\text{codim } St_r \leq m$ (resp. $cd(r) \leq m$) and *bounding* (resp. *potentially admissible*) if it is not admissible (resp. potentially admissible) but there is no non-admissible (resp. potentially nonadmissible) stratum containing St_r in its closure.

Lemma 4. *For any $n \geq 3$ and any $k \geq 1$ one has $d(n, k+1) > \partial(n, k) = \sum_{j=1}^k d(n, j)$. For $n = 2$ and any $k \geq 1$ one has $d(2, k+1) + d(2, k+2) > \partial(2, k) = \sum_{j=1}^k d(2, j)$.*

Proof. We will consider only the case $n \geq 3$. We construct for each element $a \in H^j$, $j \leq k$ the unique element in H^{k+1} , i.e. embed $\bigcup_{j=1, \dots, k} H^j$ into H^{k+1} , thus proving that $d(n, k+1) > \partial(n, k)$. Recall that the Hall family is linearly ordered and each element in the Hall family has the unique representation in the form $(a(bc))$ where a, b, c satisfy the conditions: a, b, c, bc belong to H , $a \geq b$ and $b < c$. For all $j < \lfloor \frac{k+1}{2} \rfloor$ we associate to any element $a \in H^j$ the element $(a(v_1(v_1(\dots(v_1, v_2)\dots)))) \in H^{k+1}$. Now let $j > \lfloor \frac{k+1}{2} \rfloor$ and a be some element in H^j . Then $a = (f, g)$ where $f < g$. Assume additionally that $2\text{lng}(f) + \text{lng}(g) \leq k+1$. Then we associate

to a the element $(h(f, g))$ where h is the maximal element in $H^{k+1-\text{lng}(h)-\text{lng}(g)}$. (Note that, by definition, $h \geq f$.) If $2\text{lng}(f) + \text{lng}(g) > k + 1$ then we choose for each $a = (f, g) \in H^j$ the element $(f(h, g))$ where h is the minimal element in $H^{k+1-\text{lng}(h)-\text{lng}(g)}$. The last case to consider is H^l when $k = 2l - 1$. One has that $\Lambda^2(H^l)$ is embedded into H^{2k} and under the assumption $n \geq 3$ one has $\dim \Lambda^2(H^l) \geq \dim H^l$. Combining all choices together we obtain a set-theoretic embedding of $\bigcup_{j=1}^k H^j$ into H^{k+1} . The result follows. More detailed consideration shows that $d(2, k+1) + d(2, k+2) > \partial(2, k)$. \square

Remark. The transversality property is equivalent to showing that the codimension of each potentially admissible (resp. potentially bounding) stratum St_r equals $cd(r)$ and therefore St_r is, in fact, admissible (resp. bounding).

Let us enumerate the defect vectors of all potentially admissible strata.

Lemma 5. *For $n \geq 3$ the defect vectors of all potentially admissible strata are as follows. We assume that $\partial(n, p) \leq m < \partial(n, p+1)$.*

case/posit	:	1	2	...	$l-1$	l	$l+1$...	$p-1$	p	$p+1$:
a)	:	0	0	...	0	1	0	...	0	χ	0	:
b)	:	0	0	...	0	1	0	...	0	0	χ	:
c)	:	0	0	...	0	0	0	...	0	χ	ν	:

where $l < p$ and the following restrictions are satisfied for each of the above cases

$$\begin{cases} a) & (m - \partial(n, p) + 1 + \chi)\chi \leq \partial_l - 1, \chi \geq 0; \\ b) & (m - \partial(n, p+1) + 1 + \chi)\chi \leq \partial_l - 1, \chi + 1 + m - \partial(n, p+1) \geq 0; \\ c) & (m - \partial(n, p) + \chi)\chi + (m - \partial(n, p+1) + \chi + \nu)\nu \leq m, \\ & (m - \partial(n, p+1) + \chi + \nu) \geq 0. \end{cases}$$

Proof. If the rank of the image of some $L_l(\mathcal{V}) = \bigoplus_{i=1}^l \mathfrak{L}\mathfrak{e}^i(\mathcal{V})$ where $l < p$ drops then it can drop exactly by 1. Indeed, assume that it drops by at least 2 then using (10) we get for the dual diagram μ_r that $q_1 \geq m - \partial(n, l) + 2$ and $n_1 \geq 2$. But by lemma 5.4. one has $2(m - \partial(n, l) + 2) > m$ which contradicts to the assumption $cd(r) \leq m$. Analogously, if $L_l(\mathcal{V})$, $l < p$ has the corank equal to 1 then the corank can possibly increase again (becomes > 1) only for either $L_p(\mathcal{V})$ or $L_{p+1}(\mathcal{V})$, see more detailed description below. Indeed, assume that the corank drops by 1 at $L_{l_1}(\mathcal{V})$ and by 2 at $L_{l_2}(\mathcal{V})$ where $l_1 < l_2 < k$. Then by (10) one has $q_1 = m - \partial(n, l_1) + 1$, $n_1 = 1$, $q_2 = m - \partial(n, l_2) + 2$, $n_2 = 1$. But again by 5.4. one gets $q_1 + q_2 > m$ which contradicts to $cd(r) \leq m$. Therefore the second rank drop can only occur either at position p or $p+1$. If we have that $L_l(\mathcal{V})$, $l < p$ has corank 1 then further corank drops at both positions p and $p+1$ are simultaneously impossible, i.e. only 1 extra drop is allowed. Indeed, assume that we have coranks 1, 2, 3 at positions $l, p, p+1$ resp. Then by (10) one gets $q_1 = m - \partial(n, l) + 1$, $n_1 = 1$; $q_2 = m - \partial(n, p) + 2$, $n_2 = 1$; $q_3 = 1$, $n_3 = \partial(n, p+1) - m - 1$. But $q_2 n_2 + q_3 n_3 = m - \partial(n, p) + 2 + \partial(n, p+1) - m - 1 = d(n, p) + 1$. Again, by 5.4. one gets $q_1 n_1 + q_2 n_2 + q_3 n_3 > m$. Thus we are left with the cases a) and b) if the rank drops in some position prior to p . The above list of inequalities follows from the expressions for the terms in the dual Young diagram $\mu = (q_1^{n_1}, q_2^{n_2})$ given below. In the case a) one gets $q_1 = m - \partial(n, l) + 1$, $n_1 = 1$, $q_2 = m - (\partial(n, p) - 1 - \chi)$, $n_2 = \chi$. In the case b)

one gets $q_1 = m - \partial(n, l) + 1, n_1 = 1, q_2 = m - (\partial(n, p + 1) - 1 - \chi), n_2 = \chi$. Finally, in the case c) one gets $q_1 = m - \partial(n, p) + \chi, n_1 = \chi, q_2 = m - \partial(n, p + 1) + \chi + \nu, n_2 = \nu$. The inequalities express the condition $cd(r) \leq m$ and the condition that the second rank drop actually occurs. One can easily show that a) are b) are pairwise excluding, i.e. for a given pair (n, m) either inequalities for a) or or b) can be satisfied under the assumption that $\chi > 0$. \square

Corollary 3. *The defect vectors for all potentially bounding strata have one of the following forms*

	⋮	1	2	...	l ₁ - 1	l ₁	l ₁ + 1	...	l ₂ - 1	l ₂	l ₂ + 1	...	p - 1	p	p + 1	⋮
.....																
a)	⋮	0	0	...	0	1	0	0	χ	0	⋮
b)	⋮	0	0	...	0	1	0	0	0	χ	⋮
c)	⋮	0	0	0	χ	ν	⋮
d)	⋮	0	0	...	0	1	0	1	0	0	0	⋮
e)	⋮	0	0	...	0	2	0	0	0	⋮
.....																

In the cases a)-c) (whose form coincides with that in Lemma 5) we additionally require that the inequalities from the formulation of Lemma 5 are violated for the considered values of ν and χ but satisfied for any smaller nonnegative values of these variables.

Proof. Obtained by a simple case-by-case consideration. \square

7. FINAL REMARKS

1. The basic problem related to the above topological obstructions is to what extent vanishing of these obstructions guarantees the existence of a maximal growth sub-bundle. The result of T. Vogel on the existence of Engel structures on parallelizable 4-manifolds brings a certain amount of optimism about this problem. For example,

Problem. Does every closed parallelizable m -dimensional manifold admits a maximal growth distribution of rank $1 < n < m$?

2. The developed theory is incomplete due to the lack of proof of the transversality conjecture. Essentially, to accomplish the proof one should only consider the defect vectors listed in the end of §5. The authors are convinced that progress in this direction will be intimately related to the detailed study of the combinatorics of different Hall bases.

3. The natural representation of $GL_n(\mathbb{C})$ in $\mathfrak{L}\mathfrak{ie}_n^i$ was extensively studied in the series of papers [22], [25], [12] and some others. There exists an interesting formula for the multiplicity of each irreducible representation (corresponding to some Young diagram μ) in $\mathfrak{L}\mathfrak{ie}_n^i$ analogous to the one for the dimension of $\mathfrak{L}\mathfrak{ie}_n^i$. This suggests the existence of a much more sophisticated theory of characteristic classes for sub-bundles in the tangent bundle since there exist many different natural filtrations in $\mathfrak{L}\mathfrak{ie}_n$ the most complicated of which coming from the direct sum of irreducible $GL_n(\mathbb{C})$ -representations. The 1st step in this direction will be to find analogs of Theorem 2.1 for these other filtrations and the second step to find the analogous transversality theorems. As an example of characteristic classes different from the ones considered above one can try to calculate the characteristic classes related to

the depth filtration of $\mathfrak{L}\mathfrak{i}\mathfrak{e}_n$ introduced in §5. For this case there exists a character formula in terms of plethysms analogous to (5) suggested to the authors by C. Reutenauer but it is unfortunately rather complicated.

4. Another natural question related to the transversality theorem is to understand for strata of what corank in the space $Mat^0(n, m, k)$ it holds and therefore to extend the determinantal formulas to the case of non-generic sub-bundles or families of sub-bundles.

Problem. Generalize the transversality theory to the case of non-generic sub-bundles.

8. APPENDIX

```
(* MATHEMATICA program for calculation of Chern classes of
  homogeneous components  $\mathfrak{L}\mathfrak{i}\mathfrak{e}_n^k$  of free Lie algebra bundles
  up to the order *)
order=4;

(* This function computes Chern character via total Chern class *)
classtochar[cc_]:=
  (resc[n-t D[Log[cc],t],-1]/.t^k_->t^k/k!)+0[t]^(order+1)//ExpandAll;

(* This function calculates total Chern class via Chern character *)
chartoclass[ch_]:=Exp[-Integrate[PolynomialQuotient[resc[ch,-1]/
  t^k_->t^k k!,t,t],t]+0[t]^(order+1)]//ExpandAll;

(* This function makes rescaling  $\eta \mapsto (\eta)_l$  *)
resc[eta_,l_]:=Normal[eta]/.t->1 t;

(* Total Chern class of original bundle *)
class=1+Sum[c[i]t^i,i,order]+0[t]^(order+1);

(* Chern character of original bundle *)
char=classtochar[class];

(* Calculates Chern character of  $\mathfrak{L}\mathfrak{i}\mathfrak{e}^k$  using Theorem 2.1. *)
charofL[k_]:=
  (Plus@@(MoebiusMu[#]resc[char^(k/#),#]/k)&/@(Divisors[k]))//ExpandAll;

(* Total Chern class of  $\mathfrak{L}\mathfrak{i}\mathfrak{e}^k$  as a series in t *)
classofL[k_]:=chartoclass[charofL[k]];
```

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