EULER CHARACTERISTICS FOR LINKS OF SCHUBERT CELLS IN THE SPACE OF COMPLETE FLAGS

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§1. Introduction

Let $\mathbf{F}_n$ be the space of complete flags in $\mathbf{k}^n$ (where $\mathbf{k}$ is $\mathbf{R}$ or $\mathbf{C}$). With an arbitrary complete flag $f \in \mathbf{F}_n$ we associate the standard Schubert cell decomposition Sch_f of the space $\mathbf{F}_n$ whose cells are enumerated by elements from $\mathbf{S}_n$ while the dimension over $\mathbf{k}$ of such a cell equals the number of inversions in the corresponding permutation (see for example [FF] §5.4).

Definition. The train $\mathbf{Tn}_f$ of the flag $f \in \mathbf{F}_n$ is the union of all cells of Sch_f of positive codimension.

Let c_σ be the cell of the decomposition Sch_f corresponding to the permutation σ and B a sufficiently small $n(n-1)/2$-dimensional (over $\mathbf{k}$) ball with the origin at some point of c_σ.

Definition. The manifold $A_\sigma = B \setminus \mathbf{Tn}_f$ is called the link of the cell c_σ. By χ_σ we denote the Euler characteristic of A_σ :

$$\chi_\sigma = \sum_k (-1)^k \dim H^k(A_\sigma).$$

In the complex case we introduce also the numbers

$$\chi_\sigma^{pq} = \sum_k (-1)^k \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^k(A_\sigma)$$

where Gr^W and Gr_F are the associated graded objects of the weight and the Hodge filtrations, respectively.

In this paper we describe a construction which enables us to reduce the calculation of χ_σ and χ_σ^{pq} for $\mathbf{F}_n$ to similar calculations for $\mathbf{F}_{n-1}$ and give the results of the calculations in low dimensions. We also formulate a relation between χ_σ for the real case and χ_σ^{pq} for the complex one and establish certain properties of the latter numbers.

§2. Sylvester manifolds, flags transversal to a given pair of flags and links of Schubert cells

2.1. Let M be an arbitrary matrix over the ring of polynomials in d variables with the coefficients from the field $\mathbf{k}$.

Definition. The Sylvester polynomial of the matrix M is the product of all its main minors; the Sylvester manifold of the matrix M is the complement in $\mathbf{k}^d$ to the set of zeros of the Sylvester polynomial of M.

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Let σ be an arbitrary permutation from $\mathbf{S}_n$. Assign to permutation σ a following matrix M_σ over the ring of polynomials in $n(n-1)/2$ variables: take the upper triangular matrix with unit diagonal elements and independent variables x_{ij} above the diagonal and permutate its columns with the help of σ.

EXAMPLE. Let $\sigma = (2, 3, 1, 4)$, then

$$M_\sigma = \begin{pmatrix} x_{13} & 1 & x_{12} & x_{14} \\ x_{23} & 0 & 1 & x_{24} \\ 1 & 0 & 0 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Sylvester polynomial of this matrix equals $x_{14}x_{34}(x_{12}x_{24} - x_{14})$.

2.2. Let $e_1, \dots, e_n$ be an arbitrary basis in $\mathbf{k}^n$.

DEFINITION. A complete flag is called **coordinate** if all its subspaces are spanned by the vectors of the basis and **standard coordinate** if each its i-dimensional subspace is spanned by $e_1, \dots, e_i$.

Coordinate flags are enumerated by permutations and each Schubert cell of the Schubert decomposition associated with a standard coordinate flag contains the unique coordinate flag while the permutations corresponding to these flag and cell coincide.

Given a permutation $\sigma = (i_1, i_2, \dots, i_n)$, we call the permutation $\bar\sigma = (i_n, \dots, i_2, i_1)$ transversal to σ. The coordinate flag $f_{\bar\sigma}$ corresponding to the permutation $\bar\sigma$ is the unique coordinate flag transversal to f_σ.

Let (f_1, f_2) be an arbitrary pair of flags in $\mathbf{F}_n$. Evidently, there exists a basis in $\mathbf{k}^n$ for which f_1 is the standard flag and f_2 is a coordinate flag. Each such basis can be obtained from another one via the multiplication by a nondegenerate lower triangular matrix. Thus a permutation $\sigma \in \mathbf{S}_n$ is assigned to each pair of flags from $\mathbf{F}_n$. The manifold of all flags transversal to a given pair of flags will be denoted by V_σ.

In what follows we shall often use the following fibration $\mathbf{GL}_n(\mathbf{k}) \to \mathbf{F}_n$. Fix some basis in $\mathbf{k}^n$ thus identifying $\mathbf{GL}_n(\mathbf{k})$ with the set of the nondegenerate $n \times n$-matrices and map each matrix onto the flag whose i-dimensional subspace is spanned by the first i rows of the matrix.

2.3. Lemma. *The manifolds A_σ and $V_{\bar\sigma}$ are diffeomorphic to the Sylvester manifold of the matrix M_σ.*

Proof. At first we prove the statement about $V_{\bar\sigma}$. The matrix M_σ defines the mapping of $\mathbf{k}^{n(n-1)/2}$ to $\mathbf{GL}_n(\mathbf{k})$. Let us ascertain now that the set of all flags transversal to $f_{\bar\sigma}$ can be identified with the image of this mapping. Suppose g is such a flag and $\bar\sigma = (i_1, \dots, i_n)$. Since the line of the flag g is transversal to the linear subspace spanned by $e_{i_1}, \dots, e_{i_{n-1}}$, it contains a unique vector whose projection along $\{e_{i_1}, \dots, e_{i_{n-1}}\}$ coincides with the basis vector e_{i_n}; we place the coordinates of this vector in the first row of our matrix. Since the 2-plane of the flag g is transversal to the subspace spanned by $e_{i_1}, \dots, e_{i_{n-2}}$, it contains a unique vector whose projection along $\{e_{i_1}, \dots, e_{i_{n-2}}\}$ coincides with the basis vector $e_{i_{n-1}}$; we place the coordinates of this vector in the second row of the matrix, etc. At the end of this process we obtain a matrix belonging, obviously, to the image of the mapping determined by M_σ. Consider now the condition that the flag g represented by this matrix is transversal to the standard coordinate flag (represented by the unit matrix). The transversality of the $(n-i)$-dimensional subspace of the standard flag and of the i-dimensional subspace of g is equivalent to the nondegeneracy of the matrix constituted of the first $n-i$ rows of the unitary matrix and the first i rows of the matrix representing g, i.e. to the nonvanishing of the i-th main minor of the latter.

Now we prove the statement concerning A_σ. Consider the set of all flags belonging to the $n(n-1)/2$-dimensional ball with the origin at f_σ. If the radius of this ball is sufficiently small then all these flags are transversal to $f_{\bar\sigma}$. Therefore by previous arguments the set of these flags is identified with the image of a small ball under the mapping to $\mathbf{GL}_n(\mathbf{k})$ defined by the matrix M_σ. Just as before, the fact that a flag does not belong to the train of the initial (standard coordinate) flag is equivalent to the nonvanishing of the Sylvester polynomial of the matrix M_σ.

§3. Stratification of the manifold A_σ

3.1. The n-dimensional $\mathbf{k}$-torus $T_n = (\mathbf{k}\setminus 0)^n$ acts on the manifold A_σ. In the coordinate representation given in §2 this action can be described as expansions and contractions of basis vectors. The orbits of this action have the following convenient description.

Consider the mapping sending each flag from A_σ to its line. This line determines an n-dimensional vector of 0's and 1's whose i-th coordinate equals 0 if the line belongs t the subspace spanned by $e_{\sigma(1)},\dots,e_{\sigma(i-1)},e_{\sigma(i+1)},\dots,e_{\sigma(n)}$ and 1 otherwise. The transversality of all flags from A_σ to the given pair of flags (see Lemma 2.3) implies that two coordinates (coinciding if the hyperplanes of these two flags coincide) of this vector must equal 1. Clear that two flags determine the same 0-1-vector if and only if they both belong to the same orbit. Therefore the orbits are enumerated by 0-1-vectors having 1's at two prescribed places (possibly coinciding):

$$A_\sigma = \bigcup_{w\in W_\sigma} O_{w\sigma} \tag{1}$$

where

$$W_\sigma = \{w = (w_1,\dots,w_n) \in \{0,1\}^n : w_1 = w_{\sigma^{-1}(n)} = 1\},$$

$O_{w\sigma}$ is the orbit in A_σ corresponding to the vector w.

We shall prove now that $O_{w\sigma}$ is diffeomorphic to the product of several copies of $\mathbf{k}^*$ by the manifold A_π for a certain permutation $\pi\in\mathbf{S}_{n-1}$.

Let $\sigma\in\mathbf{S}_n$, $w\in W_\sigma$. To each pair (σ,w) we assign a permutation $\pi(\sigma,w)\in\mathbf{S}_{n-1}$ in the following way.

Given an arbitrary sequence $I=\{i_1,\dots,i_k\}$ denote by $R(I)$ the sequence obtained by the following process: put

$$r_1 = i_1, \qquad r_l = \max\{r_{l-1}, i_l\}, \quad 1 < l \le k,$$

and delete from each group of consecutive equal elements of the sequence $\{r_1,\dots,r_k\}$ all elements except the first one. Now let $I(w)$ be the ordered sequence of the numbers i such that $w_i = 1$. Put $J_\sigma(w) = \sigma^{-1}R(\sigma I(w))$; then a relation $1 = j_1 < j_2 < \dots < j_m = \sigma^{-1}(n)$ is obviously valid, m being the number of elements in $J_\sigma(w)$. Define $\pi(\sigma,w)$ by the formulas

$$\begin{aligned} \pi(\sigma,w)(i) &= \sigma(i+1) \quad \text{if } i\ne j_l-1, && 2\le l\le m,\\ \pi(\sigma,w)(j_l-1) &= \sigma(j_{l-1}), && 2\le l\le m. \end{aligned} \tag{2}$$

3.2. Lemma. *The manifold $O_{w\sigma}$ is diffeomorphic to the direct product of $A_{\pi(\sigma,w)}$ by $(\mathbf{k}^*)^{n(w)-1}$, where $n(w)$ is the number of nonzero entries in w.*

Proof. Note that $O_{w\sigma} \cong A_{w\sigma} \times (\mathbf{k}^*)^{n(w)-1}$, where $A_{w\sigma}$ is the Sylvester manifold of the matrix $M_{w\sigma}$ obtained from M_σ via replacing its first row by the vector $w\sigma = (w_{\sigma^{-1}(1)}, \dots, w_{\sigma^{-1}(n)})$. Let us prove that the Sylvester manifolds of $M_{w\sigma}$ and of $M_{\pi(\sigma,w)}$ are diffeomorphic. Indeed, put $\{r_1, \dots, r_m\} = R(\sigma I(w))$, $r_0 = 0$ and define the matrix L by the relations

$$l_{ii} = (-1)^{w_{\sigma(i)}}, \qquad 1 \le i \le n;$$
$$l_{ij} = \begin{cases} 1 & \text{if} \quad w_{\sigma(j)} = 1,\ r_{k-1} \le j \le r_k,\ k \in \{1, \dots, m\}; \\ 0 & \text{otherwise}; \end{cases} \qquad i \ne j.$$

Since L is a nondegenerate lower triangular matrix, the Sylvester polynomials for $M_{w\sigma}$ and $M_{w\sigma}L$ coincide up to the factor ± 1. The first row of $M_{w\sigma}L$ contains the unique unity: in the last column. Hence, the Sylvester polynomial for this matrix is equal to that for its submatrix obtained by deleting of the first row and of the last column; however, by linear changes of variables the latter can be reduced to $M_{\pi(\sigma,w)}$, q.e.d.

3.3. Call vectors $w^1, w^2 \in W_\sigma$ equivalent ($w^1 \sim w^2$) if $\pi(\sigma, w^1) = \pi(\sigma, w^2)$. Relations (2) imply

$$w^1 \sim w^2 \quad \text{iff} \quad J_\sigma(w^1) = J_\sigma(w^2).$$

Denote by $\tilde{W}_\sigma$ the quotient set $W_\sigma / \sim$ and by $\tilde{W}_\sigma(w)$ the equivalence class containing w.

The set $\{1, \dots, n\}$ can be decomposed in the disjoint union of the three subsets N_0, N_1 and N_{01}. Here N_0 contains all j such that $w'_j = 0$ for all $w' \in \tilde{W}_\sigma(w)$, N_1 contains all j such that $w'_j = 1$ for all $w' \in \tilde{W}_\sigma(w)$, N_{01} contains all other j's. The definition of J_σ implies that

$$i \in N_1 \quad \text{iff} \quad i \in J_\sigma(w);$$
$$i \in N_0 \quad \text{iff} \quad \exists l : j_l < i < j_{l+1},\ \sigma(i) > \sigma(j_{l+1}).$$

Define vectors $\underline{w}$ and $\overline{w}$ by the relations

$$\underline{w}_i = \begin{cases} 1 & \text{if} \quad i \in N_1, \\ 0 & \text{otherwise}, \end{cases} \quad \overline{w}_i = \begin{cases} 0 & \text{if} \quad i \in N_0, \\ 1 & \text{otherwise}, \end{cases} \quad 1 \le i \le n.$$

It is easy to see that $\tilde{W}_\sigma(w) = \{w' : \underline{w}_i \le w'_i \le \overline{w}_i,\ i = 1, \dots, n\}$.

For an arbitrary permutation τ denote by D_τ the set of all pairs (i, k) such that $i < k$ and $\tau(i) < \tau(k)$ and let d_τ be the cardinality of D_τ.

3.4. Lemma. *For arbitrary $\sigma \in \mathbf{S}_n$, $w \in W_\sigma$*

$$d_\sigma - d_{\pi(\sigma,w)} = n(\underline{w}) - 1 + 2(n - n(\overline{w})).$$

Proof. Let us define a mapping $\pi^* : D_\pi \to D_\sigma$ (since σ and w are fixed, we can and will omit dependence of π and of π^* on these parameters). To do this, decompose each of D_σ and D_π in 6 subsets.

Let $(i,k) \in D_\sigma$. Put

$$\begin{aligned}
(i,k) \in D_\sigma^1 &\quad \text{iff} \quad i \notin N_1, k \notin N_1,\\
(i,k) \in D_\sigma^2 &\quad \text{iff} \quad i \in N_1, k \in N_{01},\\
(i,k) \in D_\sigma^3 &\quad \text{iff} \quad i \in N_{01}, k \in N_1,\\
(i,k) \in D_\sigma^4 &\quad \text{iff} \quad i \in N_1, k \in N_0,\\
(i,k) \in D_\sigma^5 &\quad \text{iff} \quad i \in N_0, k \in N_1,\\
(i,k) \in D_\sigma^6 &\quad \text{iff} \quad i \in N_1, k \in N_1.
\end{aligned}$$

The corresponding partition of D_π is defined in the following way: let $(i,k) \in D_\pi$, then

$$(i,k) \in D_\pi^l \quad \text{iff} \quad (i+1,k+1) \in D_\sigma^l, \quad 1 \le l \le 6.$$

For $(i,k) \in D_\pi^1$ put $\pi^*(i,k) = (i+1,k+1)$. We have $i+1 < k+1$ while $\sigma(i+1) = \pi(i) < \pi(k) = \sigma(k+1)$ by (2), hence $(i+1,k+1) \in D_\sigma^1$. Evidently, π^* defines a bijection of D_π^1 and D_σ^1.

For $(i,k) \in D_\pi^2$ we have $i+1 = j_l$, $l > 1$ (recall that j_l is the l'th element in $J_\sigma(w)$). Put $\pi^*(i,k) = (j_{l-1},k+1)$. Then $j_{l-1} < j_l = i+1 < k+1$ and by (2) $\sigma(j_{l-1}) = \pi(j_l - 1) < \pi(k) = \sigma(k+1)$, hence $(j_{l-1},k+1) \in D_\sigma^2$. Since $(s,t) \in D_\sigma^2$ implies the existence of $s' \in N_1$ such that $s < s' < t$, we see that π^* defines a bijection of D_π^2 and D_σ^2.

For $(i,k) \in D_\pi^3$ we have $k+1 = j_l$, $l > 1$. Put $\pi^*(i,k) = (i+1,j_l)$. We have $i+1 < j_l$ while by (2) $\sigma(i+1) = \pi(i) < \pi(j_l - 1) = \sigma(j_{l-1}) < \sigma(j_l)$, hence $(i+1,j_l) \in D_\sigma^3$. Evidently, π^* defines a bijection of D_π^3 and D_σ^3.

For $(i,k) \in D_\pi^4$ we have $i+1 = j_l$, $l > 1$. Put $\pi^*(i,k) = (j_{l-1},k+1)$. As in the case of D_π^2 we obtain $(j_{l-1},k+1) \in D_\sigma^4$. Evidently, π^* is injective on D_π^4. Let $(s,t) \in D_\sigma^4$; from the definition of $J_\sigma(w)$ it follows that $(s,t) \in \pi^*(D_\pi^4)$ iff there exists s' such that $s < s' < t$. Since for each $t \in N_0$ there exists a unique $s \in N_1$ such that the pair (s,t) does not possess the above property, we see that the number of elements in D_σ^4 exceeds that in D_π^4 by $\operatorname{card} N_0 = n - n(\overline{w})$.

For $(i,k) \in D_\pi^5$ we have $k+1 = j_l$, $l > 1$. Put $\pi^*(i,k) = (i+1,j_{l-1})$. By (2) $\sigma(i+1) = \pi(i) < \pi(j_l - 1) = \sigma(j_{l-1})$. Let us prove that $i+1 < j_{l-1}$. Indeed, by the definition of N_0 the opposite inequality would imply $\sigma(J_{l-1}) < \sigma(i+1)$, i.e. $\pi(k) < \pi(i)$ which contradicts $(i,k) \in D_\pi^5$. Therefore $(i+1,j_{l-1}) \in D_\sigma^5$. Evidently, π^* is injective on D_π^5. Let $(s,t) \in D_\sigma^5$; from the definition of $J_\sigma(w)$ follows that $(s,t) \in \pi^*(D_\pi^5)$ iff $t \ne j_m$ (as before, m denotes the cardinality of $J_\sigma(w)$). Since for each $s \in N_0$ D_σ^5 contains the unique pair (s,j_m), we see that the number of elements in D_σ^5 exceeds that in D_π^5 by $\operatorname{card} N_0 = n - n(\overline{w})$.

For $(i,k) \in D_\pi^6$ we have $i+1 = j_l$, $k+1 = j_r$, $1 < l < r$. Put $\pi^*(i,k) = (j_{l-1},j_{r-1})$. We have $j_{l-1} < j_{r-1}$ while by (2) $\sigma(j_{l-1}) = \pi(j_l - 1) < \pi(j_r - 1) = \sigma(j_{r-1})$, hence $(j_{l-1},j_{r-1}) \in D_\sigma^6$. Evidently, π^* is injective on D_π^6. Let $(s,t) \in D_\sigma^6$; from the definition of $J_\sigma(w)$ follows that $(s,t) \in \pi^*(D_\pi^6)$ iff $t \ne j_m$. Since D_σ^6 contains the unique pair (s,j_m) for each $s \in N_1$, $s \ne j_m$, we see that the number of elements in D_σ^6 exceeds that in D_π^6 by $\operatorname{card} n_1 = n(\underline{w}) - 1$.

Therefore

$$d_\sigma - d_\pi = \sum_{l=1}^{6} (\operatorname{card} D_\sigma^l - \operatorname{card} D_\pi^l) = n(\underline{w}) - 1 + 2(n - n(\overline{w})),$$

q.e.d.

§4. Euler characteristics of stratified manifolds for cohomology with compact supports

4.1. It is known (see e.g. [M]) that the theory of cohomology with compact supports is "the theory of a single space". It means that if $Y \subset X$ is a closed subset of a locally compact topological space X then for $U = X \setminus Y$ one has $H_c^i(X,Y) \equiv H_c^i(U)$ and the exact sequence of the pair can be written as

$$\cdots \to H_c^k(U) \to H_c^k(X) \to H_c^k(Y) \to H_c^{k+1}(U) \to \dots \tag{3}$$

thus implying

$$\chi_c(X) = \chi_c(U) + \chi_c(Y). \tag{4}$$

The following proposition was mentioned by many authors (see e.g. [G] and especially [V]).

4.2. Lemma. *Suppose X is an n-dimensional manifold represented as a finite disjoint union of open manifolds X_i: $X = \cup_i X_i$. Then*

$$\chi_c(X) = \sum_i \chi_c(X_i).$$

Proof. Consider the filtration $X = X^n \supset X^{n-1} \supset X^{n-2} \supset \cdots \supset X^0 = \emptyset$, where X^l is the union of all X_i's whose dimension does not exceed l and apply (4) consequently to the pairs (X^n, X^{n-1}), (X^{n-1}, X^{n-2}) and so on.

It turns out that a similar proposition is true for χ_c^{pq}'s of quasiprojective manifolds. (From now on in this section we shall follow mainly [D].)

4.3. Theorem. *Suppose X is a complex quasiprojective manifold represented as a finite disjoint union of quasiprojective manifolds X_i: $X = \cup_i X_i$. Then*

$$\chi_c^{pq}(X) = \sum_i \chi_c^{pq}(X_i).$$

The following property of the exact sequence (3) for quasiprojective manifolds immediately implies the Theorem.

4.4. Lemma. *Let $Y \subset X$ be a closed quasiprojective submanifold of a quasiprojective manifold X and $U = X \setminus Y$. Then the exact sequence (3) is an exact sequence of Hodge structures.*

Proof. Choose a compactification $\bar{X} \supset X$ and denote by X' the complement to X in $\bar{X}$, by $\bar{Y}$—the closure of Y in $\bar{X}$ and by Y'—the complement to Y in $\bar{Y}$. Then sequence (3) is reduced to

$$\cdots \to H^k(\bar{X}, X' \cup \bar{Y}) \to H^k(\bar{X}, X') \to H^k(\bar{Y}, Y') \to H^{k+1}(\bar{X}, X' \cup \bar{Y}) \to \dots .$$

By the excision isomorphism, the third group is isomorphic to $H^k(X' \cup \bar{Y}, X')$, thus (3) is reduced to the exact sequence of the triple $(\bar{X}, X' \cup \bar{Y}, X')$. However, the exact sequence of a triple is an exact sequence of Hodge structures; to prove this it is sufficient to check that Hodge structures are respected by the connecting homomorphism. The latter fact is implied by the similar property of the exact sequence of a pair.

4.5. To prove Theorem 4.3 it suffices to consider the case $X = X_1 \cup X_2$. If both X_1 and X_2 are closed the Theorem is implied immediately by Lemma 4.4. In general, let $\bar{X}_1$ be the closure of X_1 in X and $C = \bar{X}_1 \cap X_2$. Then X_1 is open in $\bar{X}_1$ (since X_1 is quasiprojective), $\bar{X}_1$ is closed in X and C is closed in X_2. Applying Lemma 4.4 three times we obtain

$$\begin{aligned}\chi_c^{pq}(X) &= \chi_c^{pq}(\bar{X}_1) + \chi_c^{pq}(X_2 \setminus C) = \\ &= \chi_c^{pq}(X_1) + \chi_c^{pq}(C) + \chi_c^{pq}(X_2 \setminus C) = \\ &= \chi_c^{pq}(X_1) + \chi_c^{pq}(X_2),\end{aligned}$$

which completes the proof.

§5. Main results

5.1. Theorem. *In the real case*

$$\chi_\sigma = \sum_{w \in W_\sigma} (-1)^{n-n(w)} 2^{n(w)-1} \chi_{\pi(\sigma,w)},$$

where $n(w)$ as before is the number of unit entries in w.

Proof. From (1) and Lemma 4.2 it follows that

$$\chi_c(A_\sigma) = \sum_{w \in W_\sigma} \chi_c(O_{w\sigma}). \tag{5}$$

Lemma 2.3 and the explicit description of the orbits $O_{w\sigma}$ (see section 3.1) imply $\dim O_{w\sigma} = \frac{n(n-1)}{2} - (n - n(w))$. Hence by Lemma 3.2.

$$\begin{aligned}\chi_c(O_{w\sigma}) &= \sum_k (-1)^k h_c^k(O_{w\sigma}) = \sum_k (-1)^k h^{n(n-1)/2-n+n(w)-k}(O_{w\sigma}) = \\ &= \sum_k (-1)^k h^0\left((\mathbf{R}^*)^{n(w)-1}\right) h^{n(n-1)/2-n+n(w)-k}(A_{\pi(\sigma,w)}) = \\ &= 2^{n(w)-1} \sum_k (-1)^k h_c^{k+1-n(w)}(A_{\pi(\sigma,w)}) = (-2)^{n(w)-1} \chi_c(A_{\pi(\sigma,w)}) = \\ &= (-1)^{(n-1)(n-2)/2} (-2)^{n(w)-1} \chi_{\pi(\sigma,w)}.\end{aligned}$$

From the above relation and formula (5) follows that

$$\chi_\sigma = (-1)^{n(n-1)/2} \chi_c(A_\sigma) = \sum_{w \in W_\sigma} (-1)^{n-n(w)} 2^{n(w)-1} \chi_{\pi(\sigma,w)},$$

q.e.d.

5.2. To handle the complex case we need the following technical proposition. Suppose X is an arbitrary complex manifold; denote

$$P_X(t) = \sum_i \chi^{ii}(X) t^i,$$

$$h_k^{ij}(X) = \dim \mathrm{Gr}_F^i \mathrm{Gr}_{i+j}^W H^k(X).$$

Lemma. *Let $X = U \times Y$ and the following assumptions are true:*
1) $\chi^{ij}(Y) = 0$ *for* $i \neq j$;
2) $h_k^{ij}(U) = 0$ *for* $i \neq j$.

Then

$$P_X(t) = P_U(t)P_Y(t), \tag{6}$$

$$\chi^{ij}(X) = 0 \quad for \quad i \neq j. \tag{7}$$

Proof. Evidently, assumption (2) implies that

$$h_k^{ij}(X) = \sum_{p,q,l} h_l^{pq}(U)h_{k-l}^{i-p,j-q}(Y) = \sum_{p,l} h_l^{pp}(U)h_{k-l}^{i-p,j-p}(Y).$$

Hence

$$\chi^{ij}(X) = \sum_k (-1)^k h_k^{ij}(X) = \sum_{p,l} (-1)^l h_l^{pp}(U) \sum_k (-1)^{k-l} h_{k-l}^{i-p,j-p}(Y) =$$
$$= \sum_{p,l} (-1)^l h_l^{pp}(U)\chi^{i-p,j-p}(Y) = \sum_p \chi^{pp}(U)\chi^{i-p,j-p}(Y).$$

From this relation we see that (7) follows immediately from assumption (1). Now,

$$P_X(t) = \sum_i \sum_p \chi^{pp}(U)\chi^{i-p,i-p}(Y)t^i = \left(\sum_p \chi^{pp}(U)t^p\right)\left(\sum_j \chi^{jj}(Y)t^j\right) =$$
$$= P_U(t)P_Y(t),$$

which coincides with (6).

5.3. Theorem. *Put $P_\sigma(t) \equiv P_{A_\sigma}(t)$, then*

$$P_\sigma(t) = \sum_{w \in W_\sigma} t^{n-n(w)}(1-t)^{n(w)-1}P_{\pi(\sigma,w)}(t), \tag{8}$$

$$\chi_\sigma^{ij} = 0 \quad for \quad i \neq j, \tag{9}$$

where $n(w)$ is the same that in Theorem 5.1.

Proof. Denote $P_{w\sigma}(t) \equiv P_{O_{w\sigma}}(t)$. Lemmas 3.2 and 5.2 imply

$$P_{w\sigma}(t) = P_{(\mathbf{C}^*)^{n(w)-1}}P_{\pi(\sigma,w)}(t) = (1-t)^{n(w)-1}P_{\pi(\sigma,w)}(t). \tag{10}$$

For an arbitrary complex manifold X define a polynomial

$$P_X^c(t) = \sum_i t^i \sum_k \dim \mathrm{Gr}_F^i \mathrm{Gr}_{2i}^W H_c^k(X)$$

and extend on this the abbreviated notions P_σ^c and $P_{w\sigma}^c$. Evidently,

$$P_X^c(t) = t^d P_X(1/t), \tag{11}$$

where d is the complex dimension of X. According to Theorem 4.3 relation (1) implies

$$P_\sigma^c(t) = \sum_{w \in W_\sigma} P_{w\sigma}^c(t).$$

Together with (11) this implies

$$t^{n(n-1)/2}P_\sigma(1/t) = \sum_{w \in W_\sigma} t^{n(n-1)/2-n+n(w)}P_{w\sigma}(1/t).$$

Introducing (10) in the above relation and redenoting $\frac{1}{t}$ by t we obtain (8).

5.4. Corollary. *In the complex case* $\chi_\sigma = 0$.

Proof. Evidently, $\chi_\sigma = \sum_{i,j} \chi_\sigma^{ij}$. Together with (8), (9) this implies $\chi_\sigma = \sum_i \chi_\sigma^{ii} = P_\sigma(1) = 0$.

5.5. A connection between χ_σ for the real case (ad hoc denote it by $\chi_\sigma^{\mathbf{R}}$) and χ_σ^{ii} is given by the following proposition.

Corollary. $\chi_\sigma^{\mathbf{R}} = \sum_i |\chi_\sigma^{ii}|$.

Proof. Follows immediately from Theorem 5.1, formula (8) and the relation $\sum_i |\chi_\sigma^{ii}| = P_\sigma(-1)$.

5.6. Theorem. *For any* $\sigma \in W_\sigma$

$$\deg P_\sigma = d_\sigma, \qquad \chi_\sigma^{d_\sigma d_\sigma} = (-1)^{d_\sigma}, \tag{12}$$

$$\chi_\sigma^{ii} = (-1)^{d_\sigma} \chi_\sigma^{d_\sigma - i, d_\sigma - i}, \quad 0 \le i \le d_\sigma. \tag{13}$$

Proof. For an arbitrary $\tilde{w} \in \tilde{W}_\sigma$ put

$$Q_{\tilde{w}}(t) = \sum_{w \in \tilde{w}} t^{n-n(w)} (1-t)^{n(w)-1} P_{\pi(\sigma,w)}(t).$$

Then (8) can be rewritten as

$$P_\sigma(t) = \sum_{\tilde{w} \in \tilde{W}_\sigma} Q_{\tilde{w}}(t) \tag{14}$$

while

$$\begin{aligned} Q_{\tilde{w}}(t) &= \sum_{i=0}^{n(\overline{w})-n(\underline{w})} \binom{n(\overline{w}) - n(\underline{w})}{i} t^{n-n(\overline{w})+i} (1-t)^{n(\overline{w})-i-1} P_{\pi(\sigma,w)}(t) = \\ &= t^{n-n(\overline{w})} (1-t)^{n(\underline{w})-1} P_{\pi(\sigma,w)}(t) \end{aligned} \tag{15}$$

with w satisfying $\tilde{W}_\sigma(w) = \tilde{w}$.

Suppose that (12) is already proved for permutations from $\mathbf{S}_{n-1}$ (the basis of the induction is trivial). Then

$$\deg Q_{\tilde{w}} = n - n(\overline{w}) + n(\underline{w}) - 1 + d_{\pi(\sigma,w)}.$$

By Lemma 3.4 this relation implies $\deg Q_{\tilde{w}} = d_\sigma - n + n(\overline{w})$. This means that only one polynomial in the right hand side of (14) has degree d_σ (the corresponding w consists of all 1's), while the degrees of all other polynomials are strictly less then d_σ. Together with (15) this implies (12).

Now note that by (12) relation (13) is equivalent to

$$P_\sigma(t) = (-t)^{d_\sigma} P_\sigma(1/t).$$

Suppose that this relation is already proved for permutations from $\mathbf{S}_{n-1}$. From (15) and Lemma 3.4 we see that

$$Q_{\tilde{w}}(1/t) = (-t)^{-d_\sigma} Q_{\tilde{w}}(t).$$

Introducing this into (14) we obtain the required relation.

§6. Calculations for low dimensions

Theorems 5.1 and 5.3 enable us to calculate χ_σ and P_σ for any $\mathbf{S}_n$ consecutively by n. The results for $n = 1, 2, 3, 4$ are displayed below.

n	σ	χ_σ	P_σ
1	1	1	1
2	(1,2)	2	$1-t$
	(2,1)	1	1
3	$(1,2,3)$	6	$1-2t+2t^2-t^3$
	$(1,3,2)$, $(2,1,3)$	4	$1-2t+t^2$
	$(2,3,1)$, $(3,1,2)$	2	$1-t$
	$(3,2,1)$	1	1
4	$(1,2,3,4)$	20	$1-3t+4t^2-4t^3+4t^4-3t^5+t^6$
	$(1,3,2,4)$	18	$1-3t+5t^2-5t^3+3t^4-t^5$
	$(1,2,4,3)$, $(2,1,3,4)$	16	$1-3t+4t^2-4t^3+3t^4-t^5$
	$(2,1,4,3)$, $(2,3,1,4)$, $(3,1,2,4)$, $(1,3,4,2)$, (1,4,2,3)	12	$1-3t+4t^2-3t^3+t^4$
	$(3,2,1,4)$, $(1,4,3,2)$, $(2,4,1,3)$, $(3,1,4,2)$	8	$1-3t+3t^2-t^3$
	$(2,3,4,1)$, $(4,1,2,3)$	6	$1-2t+2t^2-t^3$
	$(3,4,1,2)$, $(2,4,3,1)$, $(3,2,4,1)$, $(4,1,3,2)$, (4,2,1,3)	4	$1-2t+t^2$
	$(4,2,3,1)$, $(3,4,2,1)$, (4,3,1,2)	2	$1-t$
	$(4,3,2,1)$	1	1

Moreover, $\chi_{(1,2,3,4,5)} = 52$, $\chi_{(1,2,3,4,5,6)} = 104$.

The natural conjecture that for n fixed the maximal value ofχ_σ is achieved at $\sigma = (1, 2, \dots, n)$ fails. Indeed, already for $n = 5$ one has $\chi_{(1,3,2,4,5)} = \chi_{(1,2,4,3,5)} = 56$, $\chi_{(1,3,4,2,5)} = 60$ (the latter value is maximal for $\mathbf{S}_5$).

It would be interesting to study the topology of A_σ. For $n = 1, 2, 3$ in the real case all A_σ are disconnected unions of cells. The same is apparently true for $n = 4$ thus adding one more number—52—to the list of the numbers of connected components of the set of all flags transversal to a given transversal pair of flags (see [A]). For $n > 4$ the topology of A_σ can be nontrivial.

Note that to each permutation σ from $\mathbf{S}_n$ one can assign the permutation $\hat\sigma = (\sigma, n+1)$ from $\mathbf{S}_{n+1}$. Evidently, $A_{\hat\sigma}$ is homotopically equivalent to A_σ (in fact $A_{\hat\sigma}$ is a cylinder over A_σ). Hence, on the set of all possible permutations one obtains the generalized Bruhat ordering, whose maximal element is "the inverse permutation of all positive integers". Does cohomology of the corresponding A_σ stabilize? If so, find the stable cohomology ring.

It would be also interesting to find the cohomology and the mixed Hodge structure of A_σ in the complex case. We managed to obtain the answer for $n \leq 4$. In all this cases the corresponding mixed Hodge structure is pure, namely, $h_i^{ii}(A_\sigma) = |\chi_\sigma^{ii}|$ while all the other h_k^{ij} vanish. It is tempting to prove that the same is true for all A_σ, but most likely it is a low dimensional effect. It is apparently easy to prove (decomposing A_σ in the disjoint union of quasiprojective manifolds each diffeomorphic to the product of the certain numbers of copies of $\mathbf{C}$'s and $\mathbf{C}^*$'s) that in the mixed Hodge structure of A_σ the h_k^{ij} always vanish for $i \neq j$. On the other hand, the analogs of A_σ for Grassmann manifolds turn out to be isomorphic to $\mathbf{GL}_n(\mathbf{C})$ while the mixed Hodge structure of these fails to be pure.

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