




Electrostatic Problems with a Rational Constraint and Degenerate Lamé Equations

Dimitar K. Dimitrov¹ · Boris Shapiro² 

Received: 6 October 2017 / Accepted: 3 December 2018 / Published online: 18 December 2018
© The Author(s) 2018

Abstract

In this note we extend the classical relation between the equilibrium configurations of unit movable point charges in a plane electrostatic field created by these charges together with some fixed point charges and the polynomial solutions of a corresponding Lamé differential equation. Namely, we find similar relation between the equilibrium configurations of unit movable charges subject to a certain type of rational or polynomial constraint and polynomial solutions of a corresponding degenerate Lamé equation, see details below. In particular, the standard linear differential equations satisfied by the classical Hermite and Laguerre polynomials belong to this class. Besides these two classical cases, we present a number of other examples including some relativistic orthogonal polynomials and linear differential equations satisfied by those.

Keywords Electrostatic equilibrium · Lamé differential equation

Mathematics Subject Classification (2010) Primary 31C10 · 33C45

To Heinrich Eduard Heine and Thomas Joannes Stieltjes whose mathematics continues to inspire after more than a century

Research supported by the Brazilian Science Foundations FAPESP under Grants 2016/09906-0 and 2017/02061-8 and CNPq under Grant 306136/2017-1.

✉ Boris Shapiro
shapiro@math.su.se

Dimitar K. Dimitrov
dimitar.k.dimitrov@unesp.br

¹ Departamento de Matemática Aplicada, IBILCE, Universidade Estadual Paulista, 15054-000 São José do Rio Preto, SP, Brazil

² Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden

1 Introduction

For a given configuration of $p + 1$ fixed point charges v_j located at the fixed points $a_j \in \mathbb{C}$ and n unit movable charges located at the variable points $x_k \in \mathbb{C}$, $k = 1, \dots, n$ respectively, the (logarithmic) energy of this configuration is given by

$$L(x_1, \dots, x_n) = - \sum_{k=1}^n \sum_{j=0}^p v_j \log |a_j - x_k| - \sum_{1 \leq i < k \leq n} \log |x_k - x_i|. \quad (1.1)$$

The standard electrostatic problem in this set-up is as follows.

Problem 1 Find/count all equilibrium configurations of movable charges, i.e., all the critical points of the energy function $L(x_1, \dots, x_n)$.

The above electrostatic problem has been initially studied by H. E. Heine [4] and T. J. Stieltjes [12] and is now commonly known as the *classical Heine-Stieltjes electrostatic problem*. Besides Heine and Stieltjes, the latter question has been considered by F. Klein [5], E. B. Van Vleck [16], G. Szegő [13], and G. Pólya [10], just to mention a few. A relatively recent survey of results on the classical Heine-Stieltjes problems can be found in [11]. The general case of arbitrary movable charges is much less studied due to the missing relation with linear ordinary differential equations, but the case when all movable charges are of the same sign was, in particular, considered by A. Varchenko in [14].

Theorem A (Stieltjes' theorem, [12]) *If all $p + 1$ fixed positive charges are placed on the real line, then for each of the $(n + p - 1)! / (n!(p - 1)!)$ possible placements of n unit movable charges in p finite intervals of the real axis bounded by the fixed charges, the classical Heine-Stieltjes problem possesses a unique solution.*

Heine's major result from [4] claims that in the situation with $p + 1$ fixed and n unit movable charges, the number of equilibrium configurations (assumed finite) can not exceed $(n + p - 1)! / (n!(p - 1)!)$, see Theorem B below. Therefore, the above Stieltjes' theorem describes all possible equilibrium configurations occurring under the assumptions of Theorem A.

The most essential observation of the Heine-Stieltjes theory is that in case of equal movable charges, each equilibrium configuration is described by a polynomial solution of a *Lamé differential equation*. Recall that a Lamé equation (in its algebraic form) is given by

$$A(x)y'' + 2B(x)y' + V(x)y = 0, \quad (1.2)$$

where $A(x) = (x - a_0) \cdots (x - a_p)$ and $B(x)$ is a polynomial of degree at most p such that

$$\frac{B(x)}{A(x)} = \sum_{j=0}^p \frac{v_j}{x - a_j} \quad (1.3)$$

and $V(x)$ is a polynomial of degree at most $p - 1$.

In this set-up the general Heine-Stieltjes electrostatic problem is equivalent to the following question about the corresponding Lamé equation:

Problem 2 Given polynomials $A(x)$ and $B(x)$ as above, and a positive integer n , find all possible polynomials $V(x)$ of degree at most $p - 1$ for which Eq. 1.2 has a polynomial solution y of degree n .

Heine’s original result was formulated in this language and it claims the following.

Theorem B (Heine [4], see also [11]) *If the coefficients of the polynomials $A(x)$ and $B(x)$ are algebraically independent numbers, i.e., they do not satisfy an algebraic equation with integer coefficients, then for any integer $n > 0$, there exist exactly $\binom{n+p-1}{n}$ polynomials $V(x)$ of degree $p - 1$ such that the Eq. 1.2 has a unique (up to a constant factor) polynomial solution y of degree n .*

Polynomial $V(x)$ solving Problem 2 is called a *Van Vleck polynomial* of the latter problem while the corresponding polynomial solution $y(x)$ is called the *Stieltjes polynomial* corresponding to $V(x)$.

The relation between Problem 1 and Problem 2 in case of equal movable charges is very straightforward. Namely, given the locations and values (a_j, v_j) , $j = 0, \dots, p$ of the fixed charges and the number n of the movable unit charges, every equilibrium configuration of the movable charges is exactly the set of all zeros of some Stieltjes polynomial $y(x)$ of degree n for Problem 2.

In the particular case of $p = 1, a_0 = -1, a_1 = 1, v_0 = (\beta + 1)/2$ and $v_1 = (\alpha + 1)/2$, this interpretation explains why the unique equilibrium position of n unit movable charges in the interval $(-1, 1)$ is determined by the fact that the set $\{x_k\}, k = 1, \dots, n$ coincides with the zero locus of the Jacobi polynomial $P_n^{(\alpha, \beta)}(x)$. (Recall that $\{P_n^{(\alpha, \beta)}(x)\}$ is the sequence of polynomials, orthogonal on $[-1, 1]$ with respect to the weight function $(1 - x)^\alpha(1 + x)^\beta$.)

The main goal of this note is to find the electrostatic interpretation of the zeros of polynomial solutions of Problem 2 for a more general class of Lamé equations when there is no restriction $\deg A(x) > \deg B(x)$. Namely, we say that a linear second order differential operator

$$\mathfrak{d} = A(x) \frac{d^2}{dx^2} + 2B(x) \frac{d}{dx}, \tag{1.4}$$

with polynomial coefficients $A(x)$ and $B(x)$ is a *non-degenerate Lamé operator* if $\deg A(x) > \deg B(x)$ and a *degenerate Lamé operator* otherwise. The *Fuchs index* $f_{\mathfrak{d}}$ of the operator \mathfrak{d} is, by definition, given by

$$f_{\mathfrak{d}} := \max(\deg A(x) - 2, \deg B(x) - 1).$$

For the Heine-Stieltjes problem to be well-defined, we consider below Lamé operators T with $f_{\mathfrak{d}} \geq 0$. For such operators, Problem 2 with Eq. 1.2 makes perfect sense. In other words, we are looking for Van Vleck polynomials $V(x)$ of degree at most $f_{\mathfrak{d}}$ such that Eq. 1.2 has a polynomial solution of a given degree n . We call (1.2) with a degenerate Lamé operator $\mathfrak{d} = A(x) \frac{d^2}{dx^2} + B(x) \frac{d}{dx}$ and $V(x)$ of degree at most $f_{\mathfrak{d}}$, a *degenerate Lamé equation*.

The most well-known examples of such equations are those satisfied by the Hermite and the Laguerre polynomials. Namely, the Hermite and the Laguerre polynomials are polynomial solutions of the second order differential equations:

$$y'' - 2x y' + 2n y = 0, \quad y(x) = H_n(x), \tag{1.5}$$

$$x y'' + (\alpha + 1 - x) y' + n y = 0, \quad y(x) = L_n^\alpha(x). \tag{1.6}$$

Obviously, Eqs. 1.5 and 1.6 are degenerate Lamé equations with Fuchs index 0. Due to this fact, the classical interpretation of the zeros of the Hermite and the Laguerre polynomials as coordinates of the critical points of the energy function (1.1) does not apply.

Nevertheless, as was already observed by G. Szegő [13], the zeros of the Hermite and the Laguerre polynomials still possess a nice electrostatic interpretation in terms of the minimum of the energy function (1.1) subject to certain polynomial constraints.

- Theorem C** ([13], Theorems 6.7.2 and 6.7.3) (i) *The zeros of the Hermite polynomial $H_n(c_2x)$, $c_2 = \sqrt{(n-1)/2M}$, $M > 0$, form an equilibrium configuration of n unit charges that obey the constraint $x_1^2 + \dots + x_n^2 \leq Mn$ without fixed charges. In particular, the zeros of $H_n(x)$ form an equilibrium configuration of n unit charges subject to the constraint $x_1^2 + \dots + x_n^2 \leq n(n-1)/2$.*
- (ii) *The zeros of the Laguerre polynomial $L_n^\alpha(c_1x)$, $c_1 = (n+\alpha)/K$, $K > 0$, form an equilibrium configuration of n unit movable charges that obey the additional constraint $x_1 + \dots + x_n \leq Kn$ in the presence of one fixed charge $v_0 = (\alpha+1)/2$ placed at the origin. In particular, the zeros of $L_n^{(\alpha)}(x)$ form an equilibrium configuration of n unit charges satisfying the constraint $x_1 + \dots + x_n \leq n(n+\alpha)$ in the electrostatic field created by them together with the above fixed charge.*

Remark 1 Due to homogeneity of the problem, one can easily conclude that it suffices to consider only the case of equality in the above constraints.

From the first glance it seems difficult to relate the electrostatic problems of Theorem C to the corresponding differential equation since e.g., in the case of the Hermite polynomials

$$B(x)/A(x) = -x$$

which has no non-trivial partial fraction decomposition.

In order to do this, we restrict ourselves to degenerate Lamé equations with all distinct roots of $A(x)$ and non-negative Fuchs index. Given such an equation, consider the classical electrostatic problem with all unit movable charges and assume that the positions $X = (x_1, \dots, x_n)$ of the movable charges are subject to an additional constraint $R(X) = 0$ of a special form. Namely, we say that a rational function $R(X)$ is *A-adjusted* if

$$R(X) := R(x_1, \dots, x_n) = \sum_{k=1}^n r(x_k), \tag{1.7}$$

where $r(x)$ is a fixed univariate rational function such that $D(x) := A(x)r'(x)$ is a polynomial. We will also call the univariate function $r(x)$ satisfying the latter condition *A-adjusted*. In particular, if $r(x)$ is an arbitrary polynomial, then Eq. 1.7 is automatically *A-adjusted*.

Given an *A-adjusted* rational function $R(X)$, set

$$\Omega = \{X \in \mathbb{C}^n : R(X) = 0\}$$

and denote by \mathcal{A} the hyperplane arrangement in \mathbb{C}^n with coordinates (x_1, \dots, x_n) consisting of all hyperplanes of the form $\{x_i = a_j\}$ and $\{x_i = x_\ell\}$ for $i = 1, \dots, n$, $j = 0, \dots, p$ and $i \neq \ell$. Here $\{a_0, \dots, a_p\}$ is the set of roots of $A(x)$ (assumed pairwise distinct).

Now consider the 1-parameter family of the Lamé differential equations of the form

$$A(x)y'' + (2B(x) - \rho D(x))y' + V(x)y = 0, \tag{1.8}$$

depending on a complex-valued parameter ρ . We call (1.8) the *parametric Lamé equation*.

Consider an arbitrary degenerate Lamé (1.2) with all distinct roots of $A(x)$. Given an *A-adjusted* rational function $R(X)$, set $q := \deg D(x)$.

Theorem 1 (Stieltjes’ theorem with an A -adjusted constraint) *In the above notation, the energy function $L(x)$ given by Eq. 1.1 and the parametric Lamé (1.8) satisfy the following:*

- Let $X^* = (x_1^*, \dots, x_n^*)$ be a vector lying in $\mathbb{C}^n \setminus \mathcal{A}$. Assume that X^* satisfies the constraint $R(X^*) = 0$ and is a critical point of the energy function $L(X)$. Then there exist a polynomial $V(x)$ of degree $\max\{p - 1, q - 1\}$ and a constant ρ^* such that

$$y(x) = (x - x_1^*) \cdots (x - x_n^*)$$

is a solution of the parametric Lamé (1.8) with $\rho = \rho^*$.

- Let $V(x)$ be a polynomial satisfying the condition $\deg V \leq \max\{p - 1, q - 1\}$, and ρ^* be a constant for which the parametric Lamé (1.8) possesses a polynomial solution of the form $y(x) = (x - x_1^*) \cdots (x - x_n^*)$, such that $X^* = (x_1^*, \dots, x_n^*) \in \Omega \setminus \mathcal{A}$. Then

$$\left. \frac{\partial L(X)}{\partial x_k} \right|_{X^*} - \frac{\rho^*}{2} \left. \frac{\partial R(X)}{\partial x_k} \right|_{X^*} = 0, \quad k = 1, \dots, n.$$

Here, by definition, $\left. \frac{\partial R(X)}{\partial x_k} \right|_{X^*} := r'(x_k^*)$.

Remark 2 Additionally, if all fixed charges are positive and placed on the real line, and X^* is a point of a local minimum of $L(X)$, then the constant ρ^* must be positive.

Example 1 Observe that the above differential equations for the Hermite and Laguerre polynomials are nothing else but the parametric Lamé equations for the appropriate values of parameter ρ and they adequately describe the corresponding electrostatic problems. (The corresponding values of ρ are denoted by ρ^* .)

Namely, the Hermite polynomial $H_n(x)$ is a solution of the differential (1.5) which can be interpreted as a parametric Lamé (1.8) with $A(x) = 1$, $B(x) = 0$, $V(x) = 2n$, $D(x) = 2x$, and $\rho^* = 1$. The fact that $A(x) = 1$ and $B(x) = 0$ is equivalent to the absence of fixed charges. In this case, $R(X) = x_1^2 + \cdots + x_n^2 - (n - 1)/2n$ which implies that

$$D(x_k) := A(x_k) \frac{\partial R(X)}{\partial x_k} = A(x_k)r'(x_k) = 2x_k,$$

where $r(x) = x^2 - (n - 1)/2n^2$.

Analogously, the differential (1.6) satisfied by the Laguerre polynomial $L_n^{(\alpha)}(x)$ can be interpreted as a parametric Lamé (1.8) with $A(x) = x$, $B(x) = (\alpha + 1)/2$, $D(x) = 1$, and $\rho^* = 1$. Then the partial fraction decomposition $B(x)/A(x) = (\alpha + 1)/2x$ indicates the presence of one fixed charge $(\alpha + 1)/2$ at the origin. In this case, $R(X) = x_1 + \cdots + x_n - (n + \alpha)/n$ and

$$D(x_k) := A(x_k) \frac{\partial R(X)}{\partial x_k} = A(x_k)r'(x_k) = x_k,$$

where $r(x) = x - (n + \alpha)/n^2$.

Theorem 1 allows us to formulate the main result of this note which provides a general relation between degenerate Lamé equations and electrostatic problems in the presence of an A -adjusted rational constraint.

Theorem 2 *Let*

$$A(x)y'' + \tilde{B}(x)y' + V(x)y = 0, \tag{1.9}$$

be a degenerate Lamé equation, i.e., $\deg A(x) \leq \deg \tilde{B}(x)$ and $\deg V \leq f_{\mathfrak{d}}$ for $\mathfrak{d} = A(x)y'' + \tilde{B}(x)y'$. Assume that all roots of $A(x)$ are distinct and that $r(x)$ is an A -adjusted

univariate rational function such that $\deg(\tilde{B}(x) + \rho A(x)r'(x)) < \deg A(x)$ for at least one value of ρ . Set

$$\tilde{B}(x) := -\rho A(x)r'(x) + 2B(x), \quad R(X) = R(x_1, \dots, x_n) := \sum_{k=1}^n r(x_k),$$

where ρ is an arbitrary complex constant. Then there exists a value ρ^* of the constant ρ , for which the degenerate Lamé (1.9) coincides with the parametric Lamé equation corresponding to the electrostatic problem of Theorem 1 with fixed charges determined by the partial fraction decomposition of $B(x)/A(x)$ and a polynomial constraint of the form $R(X) = 0$.

Theorem 1, and especially Theorem 2, together with the classical relation between nondegenerate Lamé equations and electrostatics, reveal a rather general phenomenon. Namely, every Lamé differential (1.9) where $A(x)$ has distinct complex zeros is related to an electrostatic problem no matter what the degree of the polynomial $B(x)$ is, provided only that the Fuch index is nonnegative. Indeed, dividing $\tilde{B}(x)$ by $A(x)$ we obtain $\tilde{B}(x) = -\rho A(x)r'(x) + 2B(x)$. In the classical non-degenerate case when $\deg \tilde{B} < \deg A$, we have $r'(x) = 0$ and the partial fraction decomposition of $B(x)/A(x)$ determines the positions and the strengths of all fixed charges. In the degenerate case, the A -adjusted function $r(x)$, satisfying the assumptions of Theorem 2, is simply a primitive of the quotient $r'(x)$. In other words, $r(x)$ is a unique, up to an additive constant, polynomial with the above properties.

However, in some situations $r'(x)$ is a rational function and not just a polynomial, see Section 3.3 below. In many cases the value ρ^* of the constant ρ is uniquely determined. It is clear that $r(x)$ is determined up to an additive constant of integration, so that one needs to determine the value of the constant c , such that the zeros x_1^*, \dots, x_n^* of a polynomial solution $y(x)$ satisfy the constraint $\sum_{k=1}^n r(x_k^*) + c = 0$. Usually c is easily obtained either by comparing the coefficients of certain powers of x in the corresponding Lamé equation or when the Stieltjes polynomial is known explicitly, by explicit calculation of the corresponding quantities for the zeros of $y(x)$ via Viète’s relations. The partial fraction decomposition of $\tilde{B}(x)/A(x)$, where $\tilde{B}(x)$ is the remainder in the above presentation of $B(x)$, determines the fixed charges, if any.

2 Proofs

Consider the multi-valued analytic function

$$F(X) = \prod_{j=0}^p \prod_{i=1}^n (x_i - a_j)^{v_j} \prod_{1 \leq i < j \leq n} (x_j - x_i). \tag{2.10}$$

$F(X)$ is well-defined as a multi-valued function at least in $\mathbb{C}^n \setminus \mathcal{A}$ and also on some part of \mathcal{A} where it vanishes. (This function and its generalizations called the *master functions* were thoroughly studied by A. Varchenko and his coauthors in a large number of publications including [14, 15].) Although $F(x)$ is multi-valued (unless all v_j ’s are integers), its absolute value

$$H(X) := |F(X)| = \prod_{j=0}^p \prod_{i=1}^n |x_i - a_j|^{v_j} \prod_{1 \leq i < j \leq n} |x_j - x_i|.$$

is a uni-valued function in $\mathbb{C}^n \setminus \mathcal{A}$. Obviously, the energy function (1.1) satisfies the relation

$$L(X) = -\log H(x) = -\log |F(X)|$$

which implies that $L(X)$ is a well-defined pluriharmonic function in $\mathbb{C}^n \setminus \mathcal{A}$, see e.g. [3]. (If all ν_j 's are positive, then $L(x)$ is plurisuperharmonic function in \mathbb{C}^n .)

Although $F(X)$ is multi-valued, its critical points in $\mathbb{C}^n \setminus \mathcal{A}$ are given by a well-defined system of algebraic equations and are typically finitely many due to the fact that the ratio of any two branches is constant. (For degenerate cases, these critical points can form subvarieties of positive dimension.) The following general fact is straightforward.

Lemma 1 *Let $f_j : \mathbb{C}^k \rightarrow \mathbb{C}$, $j = 1, \dots, N$ be pairwise distinct linear polynomials and let (y_1, \dots, y_k) be coordinates in \mathbb{C}^k . For every j , denote by $H_j \subset \mathbb{C}^k$ the hyperplane given by $\{f_j = 0\}$ and set $T = \mathbb{C}^k \setminus \bigcup_{j=1}^N H_j$. Given a collection of complex numbers $\Lambda = \{\lambda_j\}_{j=1}^N$, define*

$$\Phi_\Lambda(y_1, \dots, y_k) = \prod_{j=1}^N f_j^{\lambda_j}.$$

(Φ_Λ is a multi-valued holomorphic function defined in T .) Then the system of equations defining the critical points of Φ_Λ in T is given by:

$$\sum_{j=1}^N \lambda_j \frac{\partial f_j}{\partial y_\ell} / f_j = 0, \quad \ell = 1, \dots, k.$$

(Note that by a critical point of a holomorphic function we mean a point where its complex gradient vanishes.) Similarly, for any algebraic hypersurface $Y \subset \mathbb{C}^k$, the critical points of the restriction of Φ_Λ to $Y \cap T$ are well-defined independently of its branch and can be found by using the complex version of the method of Lagrange multipliers.

Lemma 2 *In the notation of Lemma 1, let $Y \subset \mathbb{C}^k$ be an algebraic hypersurface given by $Q(y_1, \dots, y_k) = 0$. Then the critical points of the restriction of Φ_Λ to $Y \cap T$ are given by the condition that the gradient of the Lagrange function*

$$\mathcal{L}(y_1, \dots, y_k, \rho) = \Phi_\Lambda - \frac{\rho}{2} Q(y_1, \dots, y_k)$$

vanishes, where ρ is a complex parameter. The latter condition is given by the system of equations

$$\sum_{j=1}^N \lambda_j \frac{\partial f_j}{\partial y_\ell} / f_j = \frac{\rho}{2} \frac{\partial Q}{\partial y_\ell}, \quad \ell = 1, \dots, k, \quad \text{and} \quad Q(y_1, \dots, y_k) = 0. \quad (2.11)$$

More exactly, if $(y_1^*, \dots, y_k^*, \rho^*)$ solves (2.11), then (y_1^*, \dots, y_k^*) is a critical point of Φ_Λ restricted to $Y \cap T$.

Proof If Φ_Λ were a well-defined holomorphic function defined in T and Y were as above, then the claim of Lemma 2 would be an immediate consequence of the method of Lagrange multipliers.

In our situation, however, Φ_Λ is multi-valued, but the ratio of any two branches is a non-vanishing constant. This implies that at any point of T , the (complex) gradients of any two branches are proportional to each other with a non-vanishing constant of proportionality

independent of the point of consideration. Therefore, if at some point of T the complex gradient of some branch of Φ_Λ is proportional to that of Q , the same holds at the same point for any other branch of Φ_Λ with (possibly) different constant of proportionality. The latter observation means that although there are typically infinitely many solutions of Eq. 2.11 in the variables (y_1, \dots, y_k, ρ) , there are only finitely many projections of these solutions to the space T , obtained by forgetting the value of the Lagrange multiplier ρ . \square

Remark 3 Observe that Y has real codimension 2 and the equation of proportionality of complex gradients is, in fact, a system of two real equations; the real and the imaginary parts of the proportionality constant can be thought of as two real Lagrange multipliers corresponding to two constraints which express the vanishing of the real and imaginary parts of the polynomial Q defining the hypersurface Y . Additionally, the real part and imaginary parts of Φ_Λ are conjugated pluriharmonic functions and therefore have the same set of critical points.

Lemma 3 *Under the above assumptions, the set of critical points of $L(X)$ in $\mathbb{C}^n \setminus \mathcal{A}$ as well as the set of critical points of the restriction of $L(X)$ to any $Y \setminus \mathcal{A}$, where Y is an algebraic hypersurface in \mathbb{C}^n coincides with those of $F(X)$.*

Notice that in Lemma 3, the meaning of a *critical point* of the real-valued function $L(X)$ and the meaning of a *critical point* of the multi-valued holomorphic function $F(X)$ are different. In the former case we require that the *real* gradient of $L(X)$ with respect to the real and imaginary parts of the complex coordinates vanishes while in the latter case we require that the complex gradient of $F(X)$ vanishes. Lemma 3 holds due to the fact that $L(X)$ is a pluriharmonic function closely related to $F(X)$.

Proof As a warm-up exercise let us prove that the sets of critical points of $L(X)$ and $F(X)$ in $\mathbb{C}^n \setminus \mathcal{A}$ coincide. Notice that $F(X)$ is non-vanishing in $\mathbb{C}^n \setminus \mathcal{A}$ which means that no branch of $F(X)$ vanishes there. As was mentioned above,

$$L(X) = -\log |F(X)| = -\operatorname{Re} (\operatorname{Log} F(X)),$$

where $\operatorname{Log} F(X)$ is a multi-valued holomorphic logarithm function in $\mathbb{C}^n \setminus \mathcal{A}$ which is well-defined due to the fact that $F(X)$ is non-vanishing. Similarly to the case of $F(X)$, vanishing of the complex gradient of $\operatorname{Log} F(X)$ is given by a well-defined system of algebraic equations which coincides with that for $F(X)$.

Let us now express the real gradient of $L(X)$ using the complex gradient of $\operatorname{Log} F(X)$. Consider the decomposition of the complex variable x_k into its real and imaginary parts, i.e. $x_k = u_k + I \cdot v_k$, where $I = \sqrt{-1}$.

Recall also that due to the Cauchy-Riemann equations, for any (locally) holomorphic function $W(x_1, \dots, x_n) = U(x_1, \dots, x_n) + I \cdot V(x_1, \dots, x_n)$, one has

$$\frac{\partial W}{\partial x_k} = 2 \frac{\partial U}{\partial x_k} = 2I \frac{\partial V}{\partial x_k}, \quad \frac{\partial W}{\partial \bar{x}_k} = \frac{\partial U}{\partial \bar{x}_k} = \frac{\partial V}{\partial \bar{x}_k} = 0, \quad k = 1, \dots, n.$$

The real gradient $\operatorname{grad}_{\mathbb{R}} L$ is given by

$$\operatorname{grad}_{\mathbb{R}} L = \left(\frac{\partial L}{\partial u_1}, \frac{\partial L}{\partial v_1}, \dots, \frac{\partial L}{\partial u_n}, \frac{\partial L}{\partial v_n} \right) = -\operatorname{grad}_{\mathbb{R}} \operatorname{Re} (\operatorname{Log} F(X)).$$

Denoting $G := -\operatorname{Log} F(X)$ and using the latter relations, we get

$$\operatorname{grad}_{\mathbb{R}} L = \left(-\operatorname{Re} \left(\frac{\partial G}{\partial x_1} \right), \operatorname{Im} \left(\frac{\partial G}{\partial x_1} \right), \dots, -\operatorname{Re} \left(\frac{\partial G}{\partial x_n} \right), \operatorname{Im} \left(\frac{\partial G}{\partial x_n} \right) \right), \quad (2.12)$$

see [6], p. 3. Observe that

$$\operatorname{Re} \left(\frac{\partial G}{\partial x_k} \right) + I \cdot \operatorname{Im} \left(\frac{\partial G}{\partial x_k} \right) = \frac{\partial G}{\partial x_k} = -2 \frac{\partial L}{\partial x_k},$$

implying that

$$\operatorname{Re} \left(\frac{\partial G}{\partial x_k} \right) - I \cdot \operatorname{Im} \left(\frac{\partial G}{\partial x_k} \right) = \overline{\frac{\partial G}{\partial x_k}} = -2 \overline{\frac{\partial L}{\partial x_k}}.$$

Thus $\operatorname{grad}_{\mathbb{R}} L$ interpreted as a complex vector in coordinates (x_1, \dots, x_n) is given by

$$\operatorname{grad}_{\mathbb{R}} L = - \left(\overline{\frac{\partial G}{\partial x_1}}, \dots, \overline{\frac{\partial G}{\partial x_n}} \right). \tag{2.13}$$

Therefore, $\operatorname{grad}_{\mathbb{R}} L = 0$ if and only if $\operatorname{grad}_{\mathbb{C}} \operatorname{Log} F(X) = 0$, which is equivalent to $\operatorname{grad}_{\mathbb{C}} F(X) = 0$.

Let us finally discuss the situation when one restricts $L(X)$ to an algebraic hypersurface Y . If $p \in Y$ is a singular point of Y , then both $L(X)$ and $\operatorname{Log} F(X)$ have critical points at p . If $p \in Y$ is a nonsingular point, then introduce new complex coordinates $(\tilde{x}_1, \dots, \tilde{x}_{n-1}, \tilde{x}_n)$ adjusted to the tangent plane to Y at p . Namely, the origin with respect these coordinates is placed at p , the hyperplane spanned by $(\tilde{x}_1, \dots, \tilde{x}_{n-1})$ coincides with the tangent plane to Y at p , and \tilde{x}_n spans the same line as $\operatorname{grad}_{\mathbb{C}} Q(p)$, where Q is the polynomial defining Y . Formulas (2.12) and Eq. 2.13 are valid with respect to the new coordinate system as well. The condition that $L(X)|_Y$ has a critical point at p means that $\operatorname{grad}_{\mathbb{R}} L(p)$ lies in the complex line spanned by \tilde{x}_n which is equivalent to the vanishing of $2n - 2$ real quantities $-\operatorname{Re} \left(\frac{\partial G(0)}{\partial \tilde{x}_1} \right), \operatorname{Im} \left(\frac{\partial G(0)}{\partial \tilde{x}_1} \right), \dots, -\operatorname{Re} \left(\frac{\partial G(0)}{\partial \tilde{x}_{n-1}} \right), \operatorname{Im} \left(\frac{\partial G(0)}{\partial \tilde{x}_{n-1}} \right)$. Analogously, the condition that $G(X)|_Y$ has a critical point at p means that $\operatorname{grad}_{\mathbb{C}} G(p)$ spans in the complex line spanned by \tilde{x}_n which is equivalent to the vanishing of $n - 1$ complex quantities $\frac{\partial G(0)}{\partial \tilde{x}_1}, \dots, \frac{\partial G(0)}{\partial \tilde{x}_{n-1}}$. But vanishing of the former $(2n - 2)$ real quantities is obviously equivalent to the vanishing of the latter $(n - 1)$ complex quantities. \square

Proof of Theorem 1 We want to find all the critical points of $L(X)$ subject to the restriction $R(X) = 0$, that is for X lying in $\Omega \setminus \mathcal{A}$. (Recall that $L(X)$ is defined in $\mathbb{C}^n \setminus \mathcal{A}$ and Ω is the hypersurface in \mathbb{C}^n given by $R(X) = 0$.) Using Lemma 3, we need to find the critical points of the multi-valued holomorphic function $F(X)$ restricted to Ω . System (2.11) of Lemma 2 provides the corresponding equations defining the critical points. In the particular case of $F(X)$ and $R(X)$ as above, this system is composed by the equations

$$\sum_{j=0}^p \frac{v_j}{x_k - a_j} + \sum_{i \neq k} \frac{1}{x_k - x_i} = \frac{\rho}{2} r'(x_k), \quad k = 1, \dots, n, \tag{2.14}$$

and $R(X) = r(x_1) + \dots + r(x_n) = 0$.

This system contains $n + 1$ equations in the $(n + 1)$ variables x_1, \dots, x_n and ρ . Abusing our notation, assume for the moment that $X = (x_1, \dots, x_n)$ is not the set of variables for \mathbb{C}^n , but some concrete complex vector in $\mathbb{C}^n \setminus \mathcal{A}$ solving the system (2.14). Introducing the polynomials $y(x) = \prod_{j=1}^n (x - x_j)$, $y_k(x) = y(x)/(x - x_k)$ and taking into account the obvious relations $y_k(x_k) = y'(x_k)$, $2y'_k(x_k) = y''(x_k)$, we conclude that the first n equations of the system (2.14) are equivalent to the system given by

$$A(x_k)y''(x_k) + (2B(x_k) - \rho D(x_k))y'(x_k) = 0, \quad k = 1, \dots, n. \tag{2.15}$$

Indeed, multiplying the k -th equation of Eq. 2.14 by $A(x_k)y_k(x_k)$, we obtain exactly the k -th equation of Eq. 2.15.

Under our assumptions,

$$D(x_k) := A(x_k) \frac{\partial R(X)}{\partial x_k} = A(x_k)r'(x_k)$$

is a polynomial in x_k of degree q . Since the first term in Eq. 2.15 is a polynomial of degree $n + p - 1$ and the second one is a polynomial of degree $n + q - 1$, then by the fundamental theorem of algebra, there exists a polynomial $V(x)$, of degree $\max\{p - 1, q - 1\}$, such that

$$A(x)y''(x) + (2B(x) - \rho D(x))y'(x) + V(x)y = 0,$$

where $y(x) = \prod_{j=1}^n (x - x_j)$. □

Notice that the condition that $R(X)$ is a rational symmetric function of a special form was imposed to guarantee that the parametric Lamé equation admits a polynomial solution.

Proof of Theorem 2 Indeed, observe that the above conditions show that the degenerate Lamé (1.9) takes the form

$$A(x)y'' + (2B(x) - \rho A(x)r'(x))y' + V(x)y = 0.$$

According to the second statement of Theorem 1, the existence of a pair $(V(x), y(x))$ yields that $X^* = (x_1^*, \dots, x_n^*)$ is a critical point of the restriction of the energy function $L(X)$ to the hypersurface given by $R(X) = 0$. Here $V(x)$ is a Van Vleck polynomial satisfying the condition $\deg V \leq \max\{\deg A - 2, \deg q - 1\}$ and $y(x) = (x - x_1^*) \cdots (x - x_n^*)$ is a Stieltjes polynomial which satisfies the latter differential equation with $X^* = (x_1^*, \dots, x_n^*)$ obeying the restriction $R(X^*) = 0$. □

3 Examples

3.1 Hermite Polynomials in Disguise

A straightforward change of variables in the differential (1.5) satisfied by the Hermite polynomials $H_n(x)$ implies that for any $m \in \mathbb{N}$, the polynomial $y(x) = y_{m,n}(x) := H_n(x^m)$ solves the degenerate Lamé differential equation

$$x y''(x) - (2m x^{2m} + m - 1) y'(x) + 2m^2 n x^{2m-1} y(x) = 0.$$

For any fixed m , the zeros of $H_n(x^m)$ are located at the intersections of the $2m$ rays emanating from the origin with the slopes $e^{i\pi j/m}$, $j = 0, \dots, 2m - 1$ and $[n/2]$ circles with the radii $(h_k)^{1/m}$, $k = 1, \dots, [n/2]$, where h_k are the positive zeros of $H_n(x)$. When n is odd, there is an additional zero of multiplicity m at the origin.

Theorem 2 implies that the coordinates of these mn zeros form a critical point of the logarithmic energy of the electrostatic field generated by the moving charges together with the negative charge $-(m - 1)/2$ at the origin, where the mn movable charges x_k are subject to the constraint

$$\sum_{k=1}^{mn} x_k^{2m} = \frac{mn(n - 1)}{2}.$$

The constant $mn(n - 1)/2$ is determined by the fact that

$$\sum_{k=1}^{mn} x_k^{2m} = m \sum_{k=1}^n h_k^2 = \frac{mn(n - 1)}{2}.$$

3.2 Laguerre Polynomials in Disguise

A procedure similar to that in the previous example shows that for any $m \in \mathbb{N}$, the polynomial $y(x) = y_{m,n}(x) := L_n^\alpha(x^m)$ solves the differential equation

$$x y''(x) + (1 + \alpha m - m x^m) y'(x) + m^2 n x^{m-1} y(x) = 0.$$

For any fixed m , the zeros of $L_n^\alpha(x^m)$ are located at the intersections of the m rays emanating from the origin with the slopes $e^{2\pi i/j}$, $j = 0, \dots, m - 1$ and the n circles with the radii $(\ell_k)^{1/m}$, $k = 1, \dots, n$, where ℓ_k are the zeros of $L_n^\alpha(x)$.

Theorem 2 implies that the coordinates of these mn zeros form a critical point of the logarithmic energy of the electrostatic field generated by the moving charges together with the charge $(1 + \alpha m)/2$ at the origin, where the mn movable charges x_k are subject to the constraint

$$\sum_{k=1}^{mn} x_k^m = mn(n + \alpha).$$

3.3 Laguerre Polynomials and Electrostatic Problem with a Rational Constraint

Substituting $x \mapsto x + 1/x$ in $L_n^\alpha(x)$, we conclude that the polynomial of degree $2n$

$$Y(x) = x^n L_n^\alpha(x + 1/x)$$

solves the differential equation

$$A(x) y''(x) + B(x) y'(x) + V(x)y(x) = 0,$$

with

$$A(x) = x^6 - x^2,$$

$$B(x) = -x^6 + x^2 + (a + 1 - 2n)x^5 + x^4 - 2(a + 2)x^3 + (a + 2n - 1)x - 1,$$

$$V(x) = 2nx^5 + n(n - a)x^4 - 4nx^3 + 2n(a + 2)x^2 + 2nx - n(n + a).$$

Now take $r(x) = x + 1/x$. Then $A(x)r'(x) = (x^2 + 1)(x^2 - 1)^2$ and $B(x) = -A(x)r'(x) + 2\tilde{B}(x)$, where $2\tilde{B}(x) = (a + 1 - 2n)x^5 - 2(a + 2)x^3 + (a + 2n - 1)x$.

The partial fraction decomposition of $\tilde{B}(x)/A(x)$ is given by

$$\frac{\tilde{B}(x)}{A(x)} = \left(-n + \frac{1 - a}{2}\right) \frac{1}{x} - \frac{1}{2} \left(\frac{1}{x - 1} + \frac{1}{x + 1}\right) + \frac{a + 1}{2} \left(\frac{1}{x - i} + \frac{1}{x + i}\right).$$

Theorems 1 and 2 imply that the $2n$ zeros of $x^n L_n^\alpha(x + 1/x)$ are the coordinates of a critical point of the logarithmic energy of the electrostatic field generated by the movable charges together with the following five fixed charges. One charge equal to $-n + (1 - a)/2$ is placed at the origin; two charges equal to $-1/2$ are placed at ± 1 , and two charges $(a + 1)/2$ are placed at $\pm i$.

The $2n$ unit movable charges obey the constraint

$$\sum_{k=1}^n (x_k + 1/x_k) = \frac{n((\alpha + 1)(n + \alpha) + 1)}{\alpha + 1}.$$

Since the relation $x_k + 1/x_k = \ell_k$ associates each zero ℓ_k of $L_n^\alpha(x)$ to two zeros of $x^n L_n^\alpha(x + 1/x)$, we conclude that the critical points are either positive reals or belong to the semicircle $\{x \in \mathbb{C} : |x| = 1, \Re(x) > 0\}$.

3.4 Schrödinger-Type Equations

Now we consider another type of analogs of the equation satisfied by the Hermite polynomials. We say that Eq. 1.4 is of *Schrödinger-type* if $A(x)$ is a non-vanishing constant (which we can always assume equals to 1). In this case, there are no fixed charges and the system of equations defining the equilibrium (usually called the Bethe ansatz) is given by

$$\sum_{j \neq k} \frac{1}{x_k - x_j} = -B(x_k), \quad k = 1, \dots, n. \quad (3.16)$$

Denoting by $r(x)$ a primitive function of $B(x)$, we observe that Eq. 3.16 determines critical points of the Vandermonde function

$$Vd(x_1, \dots, x_n) := \prod_{1 \leq i < j \leq n} |x_i - x_j|$$

on the hypersurface \mathcal{H} given by the equation

$$R(x_1, \dots, x_n) := r(x_1) + r(x_2) + \dots + r(x_n) = C$$

for an appropriate constant C . (Notice that the critical points of the Vandemonde function and its generalizations to several types of hypersurfaces have been studied in [7–9].)

Some interesting examples of Schrödinger-type equations are satisfied by the Laguerre polynomials of certain degrees with special values of parameter α . More precisely, for a given $m \in \mathbb{N}$, consider the equation

$$y''(x) - (m+1)x^m y'(x) + m(m+1)nx^{m-1}y(x) = 0. \quad (3.17)$$

One expects that eventual polynomial solutions of Eq. 3.17 must have degree mn . Observe that a basis of linearly independent solutions of Eq. 3.17 is given by

$${}_1F_1(-mn/(m+1), 1 - 1/(m+1), x^{m+1})$$

and

$$x \cdot {}_1F_1((1-mn)/(m+1), 1 + 1/(m+1), x^{m+1}),$$

where ${}_1F_1(a, b, x)$ is the basic hypergeometric function given by

$${}_1F_1(a, b, x) := \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{x^j}{j!}.$$

Here $(t)_j$ is the standard Pochhammer symbol defined by $(t)_0 := 1$ and $(t)_j := t(t+1) \dots (t+j-1)$, j being a positive integer.

It is clear that ${}_1F_1(a, b, x)$ reduces to a polynomial if and only if a is a nonpositive integer, i.e., $a = -N$ with $N \in \mathbb{N}$. Moreover for $b > 0$, these polynomials coincide with the Laguerre polynomials since $L_N^\alpha(x)$ is a constant multiple of ${}_1F_1(-N, \alpha+1, x)$. Therefore, Eq. 3.17 has a polynomial solution if and only if either $(1-mn)/(m+1)$ or $-mn/(m+1)$ is a negative integer $-N$. Moreover, in these cases $xL_N^{1/(m+1)}(x^{m+1})$ and $L_N^{1/(m+1)}(x^{m+1})$ are respective polynomial solutions.

Let us first consider the case when $(1-mn)/(m+1) = -N$. It is not difficult to observe that for a fixed $m \in \mathbb{N}$, the pairs $(n, N) = (dm+m-1, dm-1)$ satisfy the above relation for any nonnegative integer d . Therefore for every $d \in \mathbb{N}$, the polynomials $xL_{dm-1}^{1/(m+1)}(x^{m+1})$ are solutions of Eq. 3.17. Similar reasoning yields that the relation $mn/(m+1) = N$, with fixed m is satisfied by the pairs $(n, N) = ((m+1)d, md)$, implying that $L_{dm}^{1/(m+1)}(x^{m+1})$ are also solutions of Eq. 3.17 for every $d \in \mathbb{N}$.

Applying Theorem 2 to Eq. 3.17 we conclude the following. Since $A(x) \equiv 1$, there are no fixed charges. There are Nm movable charges x_k in the complex plane which obey the constraint

$$\sum_{k=1}^{mn} x_k^{m+1} = C,$$

where $C = (m + 1)N(N + 1)/(m + 1)$. In one of the situations $N = md - 1$, or equivalently $N = (mn - 1)/(m + 1)$ and in the other case $N = md - 1 = mn/(m + 1)$. In the first case, one charge is at the origin and the remaining $nm - 1 = N(m + 1) = (m + 1)(md - 1)$ are placed on the $m + 1$ rays $e^{2j\pi i/(m+1)}$, $j = 0, \dots, m$. In the second case, there are only $nm = N(m + 1) = dm(m + 1)$ charges placed on the latter $m + 1$ rays.

3.5 Relativistic Hermite Polynomials

Our last example illustrates how polynomial solutions of a non-degenerate Lamé equation depending on a parameter become polynomial solutions of a degenerate Lamé equation in case of the zeros of the so-called relativistic Hermite polynomials $H_n^N(x)$, see [2]. Namely, for any positive number $N > 0$, define $H_n^N(x)$ by the Rodrigues formula

$$H_n^N(x) := (-1)^n \left(1 + \frac{x^2}{N}\right)^{N+n} \left(\frac{d}{dx}\right)^n \frac{1}{(1 + x^2/N)^N}.$$

One can show that for $N > 1/2$, polynomials $H_n^N(x)$ are orthogonal with respect to the following varying weight:

$$\int_{-\infty}^{\infty} H_m^N(x)H_n^N(x) \frac{dx}{(1 + x^2/N)^{N+1+(m+n)/2}} = c_n \delta_{mn}.$$

Additionally, $H_n^N(x)$ is the unique polynomial solution of the non-degenerate Lamé equation

$$(N + x^2) y'' - 2(N + n - 1)x y' + n(2N + n - 1)y = 0.$$

The partial fraction decomposition of $B(z)/A(z)$ is given by

$$\frac{B(x)}{A(x)} = -\frac{N + n - 1}{2} \left(\frac{1}{x - i\sqrt{N}} + \frac{1}{x + i\sqrt{N}} \right).$$

Placing two fixed equal negative charges $-(N + n - 1)/2$ at $\pm i\sqrt{N}$ and n unit movable charges on the real line, we obtain after a straightforward calculation, that the zeros of $H_n^N(x)$ coincide with the unique equilibrium configuration of n movable unit charges where the energy attains its minimum. In fact, this minimum is global.

Let us briefly discuss how this equilibrium configuration depends on the positive parameter N . One can deduce the following.

- When N is a small positive number, then the fixed negative charges are located close to the origin and their total strength equals $-(n - 1)$. Therefore, they attract all movable charges to the origin.
- When N grows, then all movable charges move away from the origin because the force of attraction of the fixed negative charges decreases. This can be proved rigorously via a refinement of Sturm’s comparison theorem obtained in [1].
- When $N \rightarrow +\infty$, the force of attraction of the fixed negative charges decreases because they move away from the real line, but at the same time their strength increases in such a way that the location of each movable charge has a limit, see Fig. 1.

DIMITAR K. DIMITROV AND BORIS SHAPIRO

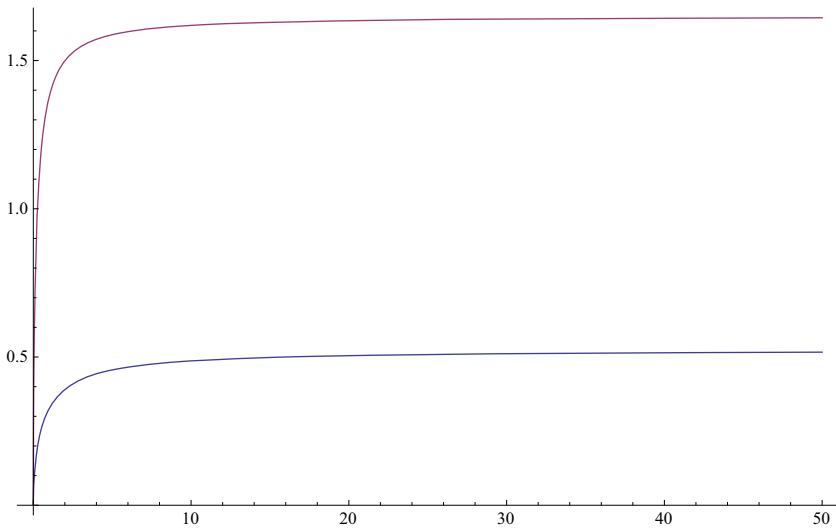


Fig. 1 Dependence of the positions of the two positive zeros of $H_4^N(x)$ on N . (The remaining two zeros of $H_4^N(x)$ are negative)

The conclusion based on the above observations is as follows. Since $H_n^N(x)$ converges locally uniformly to the Hermite polynomial $H_n(x)$ as N goes to infinity, the zeros of $H_n^N(x)$ converge to those of $H_n(x)$. Therefore, when N increases, the negative charges at $\pm i\sqrt{N}$ increase in absolute value. The corresponding equilibrium configuration formed by the coordinates of the zeros of $H_n^N(x)$ is such that the latter zeros move monotonically to those of $H_n(x)$. What is special about this phenomenon is that the influence of the increasing negative charges at $\pm i\sqrt{N}$ do not “disappear at infinity” as one can suspect, but it transforms into a constraint.

Acknowledgements The authors are sincerely grateful to the anonymous referees for their careful reading of the initial version of the manuscript and helpful suggestions. The second author is grateful to Universidade Estadual Paulista for the hospitality and excellent research atmosphere during his visit to São José do Rio Preto in July 2017.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

1. Dimitrov, D.K.: A late report on interlacing of zeros of polynomials. In: Nikolov, G., Uluchev, R. (eds.) *Constructive Theory of Functions*, Sozopol 2010: in Memory of Borislav Bojanov, pp. 69–79. Marin Drinov Academic Publishing House, Sofia (2012)
2. Gawronski, W., Van Assche, W.: Strong asymptotics for relativistic Hermite polynomials, *Rocky Mountain J. Math.* **33**, 489–524 (2003)
3. Gunning, R.C., Hugo, R.: *Analytic Functions of Several Complex Variables*, Prentice-Hall Series in Modern Analysis, pp. xiv+317. Prentice-Hall, Englewood Cliffs (1965)
4. Heine, E.: *Handbuch der Kugelfunctionen*. G Reimer Verlag, Berlin (1878). vol. 1, pp. 472–479
5. Klein, F.: Über die Nullstellen von den Polynomen und den Potenzreihen, Göttingen, pp. 211–218 (1894)
6. Korevaar, J., Wiegerinck, J.: Several complex variables, version November 18, 2011, p. 260. <https://staff.fnwi.uva.nl/j.j.o.o.wiegerinck/edu/scv/scv1.pdf>
7. Lundengrd, K., Österberg, J., Silvestrov, S.: Optimization of the determinant of the Vandermonde matrix and related matrices. *AIP Conference Proceedings* **1637**(1), 627–636 (2014)
8. Lundengrd, K., Österberg, J., Silvestrov, S.: Extreme points of the Vandermonde determinant on the sphere and some limits involving the generalized Vandermonde determinant, arXiv:1312.6193
9. Lundengrd, K.: Generalized Vandermonde matrices and determinants in electromagnetic compatibility. Mälardalen University Press Licentiate Theses **253**, 157 (2017)
10. Pólya, G.: Sur un théorème de Stieltjes. *C. R. Acad. Sci. Paris* **155**, 767–769 (1912)
11. Shapiro, B.: Algebro-geometric aspects of Heine-Stieltjes theory. *J. London Math. Soc.* **83**, 36–56 (2011)
12. Stieltjes, T.J.: Sur certains polynomes qui vérifient une équation différentielle linéaire du second ordre et sur la theorie des fonctions de Lamé. *Acta Math.* **6**(1), 321–326 (1885)
13. Szegő, G.: *Orthogonal Polynomials*, 4th edn., Amer. Math. Soc. Coll. Publ., vol. 23, Providence RI (1975)
14. Varchenko, A.: *Multidimensional Hypergeometric Functions the Representation Theory of Lie Algebras and Quantum Groups*. World Scientific Publishing Company
15. Varchenko, A.: Critical points of the product of powers of linear functions and families of bases of singular vectors. *Compos. Math.* **97**, 385–401 (1995)
16. Van Vleck, E.B.: On the polynomials of Stieltjes. *Bull. Amer. Math. Soc.* **4**, 426–438 (1898)