NARAYANA NUMBERS AND SCHUR-SZEGŐ COMPOSITION

VLADIMIR P. KOSTOV, ANDREI MARTÍNEZ-FINKELSHTEIN, AND BORIS Z. SHAPIRO

Abstract. In the present paper we find a new interpretation of Narayana polynomials $N_n(x)$ which are the generating polynomials for the Narayana numbers $N_{n,k} = \frac{1}{n}C_n^{k-1}C_n^k$ where C_j^i stands for the usual binomial coefficient, i.e. $C_j^i = \frac{j!}{i!(j-i)!}$. They count Dyck paths of length n and with exactly k peaks, see e.g. [13] and they appeared recently in a number of different combinatorial situations, [5, 11, 16]. Strangely enough Narayana polynomials also occur as limits as $n \to \infty$ of the sequences of eigenpolynomials of the Schur-Szegő composition map sending $(n-1)$-tuples of polynomials of the form $(x+1)^{n-1}(x+a)$ to their Schur-Szegő product, see below. We present below a relation between Narayana polynomials and the classical Gegenbauer polynomials which implies, in particular, an explicit formula for the density and the distribution function of the asymptotic root-counting measure of the polynomial sequence $\{N_n(x)\}$.

1. Introduction

1.1. **The Narayana numbers, triangle and polynomials.** The *Narayana numbers* $N_{n,k}$, $1 \leq k \leq n$, defined in the abstract above were apparently introduced by G. Kreweras in [8]. One immediately observes that for any fixed k the Narayana numbers $N_{n,k}$ are given by a polynomial in n of degree $2k-2$ divisible by n. It is known that $N_{n,k}$ counts, in particular, the number of expressions containing n pairs of parentheses which are correctly matched and which contain exactly k distinct nestings and also the number of Dyck paths of length n with exactly k peaks. (Recall that a Dyck path is a staircase walk from $(0,0)$ to (n,n) that lies above but may touch the diagonal $y = x$.) Some other combinatorial interpretations of $N_{n,k}$ can be found in [13] and references therein.

The triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 6 & & 6 & & 1 & \\ 1 & & 10 & & 20 & & 10 & & 1 \end{array} \tag{1}$$

of Narayana numbers $N_{n,k}$ read by rows is called the *Narayana triangle*. Later we will also interpret this triangle as an infinite lower-triangular matrix taking its left side of ones as the first column and its right side of ones as the main diagonal, see (15).

The generating functions of the rows of the above triangle are called the *Narayana polynomials*. More exactly, following the standard convention (see [3]) one defines the n-th *Narayana polynomial* by the formula

2000 *Mathematics Subject Classification.* 12D10.

Key words and phrases. Schur-Szegő composition; composition factor; hyperbolic polynomial; self-reciprocal polynomial; reverted polynomial.

$$N_n(x) = \sum_{k=1}^{n} N_{n,k} x^k \ .$$

We will also need reduced Narayana polynomials $\widehat{N}_n(x)$ given by

$$\widehat{N}_n(x) = \sum_{k=1}^{n} N_{n,k} x^{k-1} \ \leftrightarrow \ N_n(x) = x\widehat{N}_n(x) \ .$$

In what follows we use the following notions. If P is a univariate polynomial of degree n, then its *reversion* or the *reverted polynomial* P^R is defined as $P^R(x) = x^n P(1/x)$. A polynomial P is called *self-reciprocal* if it coincides with its revertion up to a sign, i.e. $P = \pm P^R$. Hence for any self-reciprocal P if it vanishes at some x_0 then it vanishes at $1/x_0$ as well. A polynomial P is called *hyperbolic* if all its roots are real.

Remark 1. *Each polynomial $N_n(x)$ has a simple root at 0 and each $\widehat{N}_n(x)$ is self-reciprocal.*

The following simple 3-term recurrence relation satisfied by Narayana polynomials was found in [13, p. 2]:

$$(n+1)N_n(x) = (2n-1)(1+x)N_{n-1}(x) - (n-2)(x-1)^2 N_{n-2}(x), \tag{2}$$

with the initial conditions $N_1(x) = x,\ N_2(x) = x^2 + x$.

1.2. **Schur-Szegő composition.** The *Schur-Szegő composition (CSS)* of two degree n polynomials $P = \sum_{j=0}^{n} p_j x^j$ and $Q = \sum_{j=0}^{n} q_j x^j$ is defined by the formula:

$$P \ {*_n} \ Q = \sum_{j=0}^{n} p_j q_j x^j / C_n^j .$$

When the same P and Q are considered as polynomials of degree $n+k$ with vanishing k leading coefficients then in accordance with the above formula one gets:

$$P \ {*_{n+k}} \ Q = \sum_{j=0}^{n} p_j q_j x^j / C_{n+k}^j .$$

Extending these formulas one defines the composition of s polynomials by the formula:

$$P_1 \ {*_{n+k}} \ \cdots \ {*_{n+k}} \ P_s = \sum_{j=0}^{n} p_{1,j} \cdots p_{s,j} x^j / (C_{n+k}^j)^{s-1} \ .$$

(For more details on CSS see [9, 10].)

Our main goal below will be a further study of a certain linear inhomogeneous map Φ_n initially considered in [6, 2]. Namely, in these papers the first author of the present paper has shown the possibility to present every monic polynomial of degree n with complex coefficients and vanishing at (-1) in the form:

$$P = K_{a_1} \ {*_n} \ \cdots \ {*_n} \ K_{a_{n-1}} \tag{3}$$

where each *composition factor* K_{a_i} equals $(x+1)^{n-1}(x+a_i),\ a_i \in \mathbf{C}$. (For the sake of convenience, we set $K_\infty := (x+1)^{n-1}$.) Now we can introduce the map Φ_n.

Notation 2. *For any $P(x) := (x+1)(x^{n-1} + c_1 x^{n-2} + \cdots + c_{n-2}x + c_{n-1})$ and for $\nu = 1, \ldots, n-1$ set $\sigma_\nu := \sum_{1 \le j_1 < \cdots < j_\nu \le n-1} a_{j_1} \cdots a_{j_\nu}$, i.e. define σ_ν as the ν-th elementary symmetric function of the roots of the composition factors presenting $P(x)$. Finally, denote by Φ_n the mapping $(c_1, \ldots, c_{n-1}) \mapsto (\sigma_1, \ldots, \sigma_{n-1})$.*

Obviously, Φ_n is linear inhomogeneous. The following theorem was proven in [7].

Theorem 3. (1) *The mapping Φ_n has $n-1$ distinct real eigenvalues $\lambda_{1,n}=1$, $\lambda_{2,n}=\frac{n}{n-1}$, $\lambda_{3,n}=\frac{n^2}{(n-1)(n-2)}$, $\ldots$, $\lambda_{n-1,n}=\frac{n^{n-2}}{(n-1)!}$.*

(2) *The corresponding eigenvectors are monic polynomials of degree $n-1$ vanishing at (-1) and have the form: $(x+1)^{n-1}$, $x(x+1)^{n-2}$, $x(x+1)^{n-3}Q_{1,n}(x)$, $\ldots$, $x(x+1)Q_{n-3,n}(x)$ where $\deg Q_{j,n}(x)=j$, $j=1,\ldots,n-3$, $Q_{j,n}(-1)\neq 0$. The coefficients of each polynomial $Q_{j,n}(x)$ are rational numbers.*

(3) *Each $Q_{j,n}(x)$ is self-reciprocal. More exactly, $(Q_{j,n}(x))^R=(-1)^jQ_{j,n}(x)$.*

(4) *The roots of each $Q_{j,n}(x)$, $1\leq j\leq n-3$, are positive and distinct.*

(5) *For j odd (resp. for j even) one has $Q_{j,n}(1)=0$ (resp. $Q_{j,n}(1)\neq 0$). Additionally, the middle coefficient in $(x+1)^{n-j-2}Q_{j,n}(x)$ vanishes if n is even and j is odd.*

(6) *For any j fixed and $n\to\infty$ the sequence of polynomials $Q_{j,n}(x)$ converges coefficientwise to the monic polynomial $Q_j^*(x)$ of degree j which has rational coefficients, all roots positive, and satisfies the equality $(Q_j^*(x))^R=(-1)^jQ_j^*(x)$ and the condition $Q_j^*(1)=0$ for j odd.*

Remark 4. *In Theorem 3 we consider the action of Φ_n on the affine $(n-1)$-dimensional space of all monic polynomials of degree $n-1$. If we extend this action to the ambient linear space of all polynomials of degree at most $(n-1)$ then the extended Φ_n is still diagonalizable. It acquires one more eigenvalue and eigenvector respectively. Namely, the polynomial $(x+1)^{n-2}$ is this additional eigenvector of Φ_n with the eigenvalue* 1.

1.3. **Results.** Set $M_j(x)=(-1)^{j-1}xQ_{j-1}^*(-x)$. The following statement is valid.

Theorem 5. *For any positive integer j the polynomial $M_j(x)$ coincides with the Narayana polynomial $N_j(x)$.*

Our next result is as follows. Let

$$P_n^{(1,1)}(x)=2^{-n}(x-1)^n\sum_{m=0}^{n}C_{n+1}^mC_{n+1}^{n-m}\left(\frac{x+1}{x-1}\right)^m \tag{4}$$

denote the Jacobi (or more precisely, Gegenbauer) polynomial with standard notation, see e.g. [1, Ch. 22].

Proposition 6. *One has*

$$N_n(x)=\frac{x(x-1)^{n-1}}{n}P_{n-1}^{(1,1)}\left(\frac{x+1}{x-1}\right). \tag{5}$$

Surprisingly enough, this simple connection of Narayana polynomials with such a classical (and well studied) family of orthogonal polynomials has not been reported before. However, it simplifies greatly the proof of many algebraic properties of the Narayana polynomials established in the literature. Indeed, the mapping

$$x\mapsto\frac{x+1}{x-1} \tag{6}$$

is a bijection of the interval $(-1,1)$ and the negative semiaxis $(-\infty,0)$. In this context the "self-reciprocity" of $\widehat{N}_n$ (Remark 1) is just the well-known symmetry property of the Gegenbauer polynomials,

$$P_n^{(1,1)}(-x)=(-1)^nP_n^{(1,1)}(x),$$

and the recurrence (2) is a straightforward consequence of the three term recurrence relation for $P_n^{(1,1)}$, see e.g. [14]. We can get further results transplanting the known bounds, estimates and identities from $(-1,1)$ to $(-\infty,0)$ using (6). For instance, the "q-Log-convexity" of the Narayana polynomials established recently in [4] follows from the positivity of Turán determinants for the Gegenbauer polynomials, see e.g. [15].

To describe asymptotic properties of the zero loci of the sequence $\{N_n(x)\}$ recall the following standard notions. Given a polynomial $P(x)$ of degree l define its *root-counting measure* $\mu_P = \frac{1}{l}\sum_{i=1}^{l}\delta(x-x_i)$ where $\{x_1,\dots,x_l\}$ is the set of all roots of $P(x)$ listed with possible repetitions (equal to the respective multiplicities) and $\delta(x-x_i)$ is the standard Dirac delta-function supported at x_i. Given a sequence $\{P_n(x)\}, \deg P_n(x)=n,\ n=1,2,\dots$ we call *asymptotic root-counting measure* of this sequence the weak limit $\mu=\lim_{n\to\infty}\mu_{P_n}$ (if it exists) understood in the sense of distribution theory.

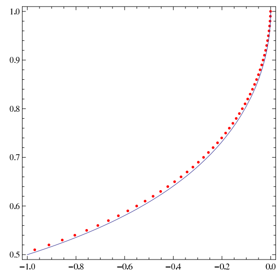

FIGURE 1. The theoretical distribution $\kappa(x)$ and the empirical distribution of roots of $N_{100}(x)$ on the interval $[-1,0]$.

Corollary 7. *(i) All roots of each $N_n(x)$ are real, simple, and non-positive. Roots of any two consecutive Narayana polynomials interlace.*
(ii) The density $\rho(x)$ and the distribution function $\kappa(x)$ of the asymptotic root-counting measure of the sequence $\{N_n(x)\}$ of the Narayana polynomials are given by:

$$\rho(x)=\frac{1}{\pi}\frac{1}{(1-x)\sqrt{-x}};\qquad \kappa(x)=1-\frac{2}{\pi}\arctan\sqrt{-x},\ x\le 0. \tag{7}$$

Remark 8. *Notice that self-reciprocity of $N_n(x)$ translates in the following (easily testable) property of $\rho(x)$:*

$$x^2\rho(x)=\rho\left(\frac{1}{x}\right).$$

Acknowledments. Research of the first author was partially supported by project 20682 of cooperation between CNRS and FAPESP "Zeros of algebraic polynomials". The second author is partially supported by the Ministry of Science and Innovation of Spain, project code MTM2008-06689-C02-01, and by Junta de Andaluca, grants FQM-229, FQM-481, and P06-FQM-01738. The third author expresses his gratitude to Laboratoire de Mathématiques, Université de Nice-Sophia-Antipolis for the financial support of his visit to Nice in January 2008.

2. PROOFS

2.1. **Preliminaries.** To prove Theorem 5 we will need a detailed study of the map Φ_n and, especially, of the equations defining its eigenvectors which in their turn give our polynomials $Q_{j,n}(x)$.

Notation 9. *Set $e_k(j) = \sigma_k(1, 2, \ldots, j)$ to be the value of the k-th symmetric function on the j-tuple of numbers $(1, 2, \ldots, j)$, i.e. $e_k(j) = \sum_{1\leq\nu_1<\cdots<\nu_k\leq j} \nu_1 \cdots \nu_k$, $k = 1, \ldots, j$. Denote by $\phi_k(j)$ the sum $1^k + 2^k + \cdots + j^k$.*

Remark 10. *The quantity $e_k(j)$ (resp. $\phi_k(j)$) is a polynomial in j of degree $2k$ (resp. of degree $k + 1$) divisible by $j(j + 1)$.*

Let $Q_{j,n}(x) := x^j + q_1x^{j-1} + \cdots + q_{j-1}x + (-1)^j$ be the polynomial introduced in Theorem 3 and set $l_j = (n - 1)\cdots(n - j)$. Then by (1) of Theorem 3 one has

$$\lambda_{j+2,n} = n^{j+1}/l_{j+1}.$$

(the coefficients q_ν depend also on j and n, but we prefer to avoid double indices.) By definition the polynomial $Q_{j,n}(x)$ satisfies the following relation:

$$x(x + 1)^{n-j-2}Q_{j,n}(x) =$$
$$= \lambda_{j+2,n}x(x+1)^{n-1} \underset{n}{*} (x + a_1)(x+1)^{n-1} \underset{n}{*} \cdots \underset{n}{*} (x + a_j)(x+1)^{n-1} \underset{n}{*} (x+1)^{n-1},$$

where $\{-a_1, \ldots, -a_j\}$ is the set of all roots of $Q_{j,n}(x)$. After multiplication of both sides of the latter relation by l_{j+1} one gets that the coefficient R_k of x^k, $k \geq 1$, in the right-hand side equals

$$R_k := n^{j+1}C_{n-1}^{k-1}(C_{n-1}^k a_1 + C_{n-1}^{k-1}) \cdots (C_{n-1}^k a_j + C_{n-1}^{k-1})C_{n-1}^k/(C_n^k)^{j+1} \ .$$

The corresponding coefficient L_k in the left-hand side equals

$$L_k := (n-1)\cdots(n-j-1)((-1)^jC_{n-j-2}^{k-1} + C_{n-j-2}^{k-2}q_{j-1} + \cdots + C_{n-j-2}^0 q_{j-k+1}). \quad (8)$$

Therefore one has $q_\nu = \sigma_\nu$ (see Notation 2) and, finally,

$$R_k = n^{j+1}C_{n-1}^{k-1}C_{n-1}^k((C_{n-1}^{k-1})^j + \sum_{\nu=1}^{j-1}(C_{n-1}^{k-1})^{j-\nu}(C_{n-1}^k)^\nu q_\nu + (-1)^j(C_{n-1}^k)^j)/(C_n^k)^{j+1}. \quad (9)$$

Thus the coefficients q_ν, $\nu = 1, \ldots, j - 1$ of $Q_{j,n}(x)$ solve the system of equations

$$(\Sigma) \ : \ \{ \ L_k = R_k \ , \ k = 1, \ldots, n - 1 \ \}. \quad (10)$$

Lemma 11. *The coefficients q_ν can be expanded in convergent series:*

$$q_\nu = q_\nu^{(0)} + \frac{q_\nu^{(1)}}{n-1} + \frac{q_\nu^{(2)}}{(n-1)^2} + \cdots \quad (11)$$

with respect to $\frac{1}{n-1}$, where the numbers $q_\nu^{(i)} \in \mathbf{R}$ are uniquely defined and independent of n.

Proof: The coefficients q_ν solve system (Σ). They are uniquely defined because the polynomials $Q_{j,n}(x)$ are uniquely defined by the eigenvectors of the mapping Φ_n. They can be expanded in convergent series in $\frac{1}{n-1}$ because the same property holds for the coefficients of system (Σ). As q_ν are uniquely defined, thus $q_\nu^{(i)}$ are also uniquely defined. □

Remark 12. *We choose to expand q_ν as a series in $\frac{1}{n-1}$ (and not in $\frac{1}{n}$) because the eigenpolynomials of Φ_n (see (2) of Theorem 3) are all of degree $n-1$. Besides, numerical computations show that it is the factor $n - 1$ and not n which appears most often in the denominators of the eigenvectors of the mapping Φ_n.*

Proposition 13. *One has $q_1^{(0)} = (-1)^j q_{j-1}^{(0)} = -j(j + 1)/2$.*

Proof: For $k = 1$ one has

$$L_1 = (-1)^j (n-1)\cdots(n-j-1)\ ,\ R_1 = (n-1)(1+\sum_{\nu=1}^{j-1}(n-1)^\nu q_\nu + (-1)^j (n-1)^j)\ .$$

The equality $L_1 = R_1$ can be written in the form

$$(-1)^j (n-1)\cdots(n-j-1) = (-1)^j (n-1)^{j+1} + (n-1)^j q_{j-1}^{(0)} + o((n-1)^j)\ .$$

Hence $q_{j-1}^{(0)} + o(1) = (-1)^j((n-1)\cdots(n-j-1) - (n-1)^{j+1})/(n-1)^j$. Observe that

$$(n-1)\cdots(n-j-1) = (n-1)^{j+1} - (1+2+\cdots+j)(n-1)^j + o((n-1)^j)\ .$$

The quantity $q_{j-1}^{(0)}$ depends on j, but not on n. Therefore $(-1)^j q_{j-1}^{(0)} = -(1+2+\cdots+j) = -j(j+1)/2$. $\square$

The next statement is central.

Proposition 14. (1) *For each (ν, i) fixed the coefficient $q_\nu^{(i)}$ is given by a real polynomial in j of degree $2(\nu+i)$.*
(2) *For $i = 0$ this polynomial is divisible by $j(j+1)$.*

2.2. **Proof of Proposition 14 .** 1^0. To prove part (1) of the proposition we use induction on $\nu + i$. Proposition 13 constitutes the base of induction. The step of induction is explained in $2^0 - 3^0$.

Recall that the coefficients q_ν give the unique solution to system (Σ). From now on we assume that system (Σ) is infinite, i.e. $k = 1, 2, \ldots$. Substituting the expansions (11) of the coefficients q_ν in (Σ) we obtain a new system (denoted by (x)) with variables $q_\nu^{(i)}$, $\nu = 1, \ldots, j-1$, $i = 0, 1, \ldots$. After this substitution the equation $L_k = R_k$ of system (Σ) transforms into an equation of the form $\sum_{l=l_0}^{\infty} A_{k,l}/(n-1)^l = 0$ where the quantities $A_{k,l}$ are some linear inhomogeneous functions of the variables $q_\nu^{(i)}$. (Notice that $A_{k,l}$ depend on j but not on n.) The latter equation holds for all $n \in \mathbf{N}$ if and only if all $A_{k,l}$ vanish. (The equation $\{A_{k,l} = 0\}$ is denoted by $(A_{k,l})$.)

2^0. The solution to system (x) is unique (which follows from the uniqueness of the polynomials $Q_{j,n}(x)$ for every fixed n, see Theorem 3). This solution depends only on j. The self-reciprocity of $Q_{j,n}(x)$ implies that $q_\nu^{(i)} = (-1)^j q_{j-\nu}^{(i)}$.

In what follows we consider subsystems of system (x) of the form $\{(A_{k,l}), l = l_0, \ldots, l_1\}$, i.e. systems defined in accordance with the filtration of the space of Laurent series in $\frac{1}{n-1}$ by the degree of $\frac{1}{n-1}$. We set $l = s - j - k$.

Notation 15. *Denote by $\mathcal{I}_{a,b}$ the set of variables $\{q_a^{(0)},\ q_{a+1}^{(1)},\ \ldots,\ q_{a+b}^{(b)}\}$.*

To settle part (1) of Proposition 14 we need Lemmas 16 and 17 whose proofs are given after that of Proposition 14.

Lemma 16. *The linear inhomogeneous form $A_{k,s-j-k}$ depends only on the variables in the set $\mathcal{J}_{j-s,s-1} := \mathcal{I}_{j-s,s-1} \cup \mathcal{I}_{j-s+1,s-2} \cup \cdots \cup \mathcal{I}_{j-1,0}$.*

3^0. Suppose that the variables belonging to the set $\mathcal{J}_{j-s+1,s-2}$ are already determined. (For $s = 2$ one has $\mathcal{J}_{j-s+1,s-2} = \mathcal{I}_{j-s+1,s-2} = \mathcal{I}_{j-1,0} = \{q_{j-1}^{(0)}\}$; see Proposition 13.) The system of s linear equations $(A) := \{(A_{k,s-j-k}), k = 1, \ldots, s\}$ is a system with s unknown variables, namely, those in the set $\mathcal{I}_{j-s,s-1}$. This system has a unique solution (which follows from the existence and uniqueness of the polynomials $Q_{j,n}(x)$, see Theorem 3). Hence the variables in the set $\mathcal{I}_{j-s,s-1}$ are uniquely defined.

Lemma 17. *The solution to system* (A) *is an* s*-vector consisting of real polynomials in* j *of degree* $2s$*.*

This concludes the proof of the step of induction in part (1) of Proposition 14.

4^0. For $\nu = 1$ part (2) of the proposition follows from Proposition 13. When solving the linear system (x) we express the variables in the set $\mathcal{I}_{j-s,s-1}$ as affine functions of the ones in the set $\mathcal{J}_{j-s+1,s-2}$. Suppose that all variables in that set are shown to be polynomials divisible by $j(j+1)$. Then the variables in the set $\mathcal{I}_{j-s,s-1}$ will be divisible by $j(j+1)$ if and only if this is the case of the constant terms of system (A) (we call them CTs for short).

The CTs are the coefficients of $(n-1)^{s-j-k}$ of the expression $(-1)^j U_1 - U_2 - (-1)^j U_3$ where

$$U_1 = (n-1)\cdots(n-j-1)C_{n-j-2}^{k-1}\ ,\ U_2 = n^{j+1}C_{n-1}^{k-1}C_{n-1}^{k}(C_{n-1}^{k-1})^j\ ,$$
$$U_3 = n^{j+1}C_{n-1}^{k-1}C_{n-1}^{k}(C_{n-1}^{k})^j/(C_n^k)^{j+1}\ ,$$

see (8) and (9). In this difference the product U_2 is irrelevant. Indeed, the highest power of $(n-1)$ multiplying any of the variables $q_\nu^{(0)}$ in R_k is higher than the highest power of $(n-1)$ in U_2, see (9).

Set $(n-1)\cdots(n-j-1) = (n-1)^{j+1}+V$. Hence $U_1 = ((n-1)^{j+1}+V)C_{n-j-2}^{k-1}$. By Remark 10 the quantity V is a polynomial divisible by $j(j+1)$. Therefore for $j=0$ one has $U_1 = (n-1)C_{n-2}^{k-1} = nC_{n-1}^{k-1}C_{n-1}^{k}/C_n^k = U_3$, and for $j=-1$ one has $U_1 = C_{n-1}^{k-1} = U_3$. Hence the CTs are divisible by $j(j+1)$. This completes the proof of Proposition 14. Now we settle Lemmas 16 and 17.

Proof of Lemma 16: 1^0. Using the equalities $C_{n-1}^{k-1}/C_n^k = k/n$ and $C_{n-1}^{k}/C_n^k = (n-k)/n$, one can present the equality $L_k = R_k$ (see (8) and (9)) in the form

$$(n-1)\cdots(n-j-1)((-1)^jC_{n-j-2}^{k-1} + C_{n-j-2}^{k-2}q_{j-1} + \cdots + C_{n-j-2}^{0}q_{j-k+1}) =$$
$$= (n-k)C_{n-1}^{k-1}(k^j + \sum_{\nu=1}^{j-1}(n-k)^\nu k^{j-\nu}q_\nu + (-1)^j(n-k)^j). \tag{12}$$

Replace in (12) the quantities q_ν by their expansions (11). Consider the right-hand side R_k of (12) as a Laurent series in $\frac{1}{n-1}$. Observe that if the integer k is bounded, then the following relations hold:

$$(n-k)C_{n-1}^{k-1} = \frac{(n-1)^k}{k!} + O((n-1)^{k-1})\ ,\ (n-k)^\nu k^{j-\nu} = k^{j-\nu}((n-1)^\nu + O((n-1)^{\nu-1}).$$

We use the last equality for $\nu = j-s$. The coefficient of $(n-1)^{j-s+k}$ in R_k is of the form:

$$\frac{1}{k!}(k^s q_{j-s}^{(0)} + k^{s-1}q_{j-s+1}^{(1)} + \cdots + kq_{j-1}^{(s-1)} + \mathcal{H} + r),$$

where $\mathcal{H}$ is a linear form in the variables $q_\mu^{(m)}$ with $\mu - m > j-s$ and r is a real number. Hence the form $\mathcal{H}$ contains only variables belonging to the union $\mathcal{J}_{j-s+1,s-2}$ (because $\mu \leq j-1$, $m \geq 0$) while the linear form $(1/k!)(k^s q_{j-s}^{(0)} + k^{s-1}q_{j-s+1}^{(1)} + \cdots + kq_{j-1}^{(s-1)})$ depends only on the variables in the set $\mathcal{I}_{j-s,s-1}$. Hence R_k depends only on the variables in the set $\mathcal{J}_{j-s,s-1}$.

2^0. Consider now the left-hand side L_k of (12). One can write

$$B(n,j) := (n-1)\cdots(n-j-1) = (n-1)^{j+1}\left(1 - \frac{e_1(j)}{n-1} + \frac{e_2(j)}{(n-1)^2} - \cdots\right),$$

see Notation 9. For each $\nu = j-k+1,\ldots,j$ the product $B(n,j)C_{n-j-2}^{\nu-j+k-1}$ is a polynomial in the variable $(n-1)$ of degree $\nu+k$. More precisely,

$$B(n,j)C_{n-j-2}^{\nu-j+k-1} = \frac{(n-1)^{\nu+k}}{(\nu-j+k-1)!}\left(1-\frac{e_1(\nu+k-1)}{n-1}+\frac{e_2(\nu+k-1)}{(n-1)^2}-\cdots\right). \tag{13}$$

Therefore, the coefficient of $(n-1)^{j-s+k}$ in the term $B(n,j)C_{n-j-2}^{\nu-j+k-1}q_\nu$ of L_k is of the form

$$\frac{1}{(\nu-j+k-1)!}(q_\nu^{(\nu-j+s)}-q_\nu^{(\nu-j+s-1)}e_1(\nu+k-1)+\cdots+(-1)^{\nu-j+s}q_\nu^{(0)}e_{\nu-j+s}(\nu+k-1)). \tag{14}$$

The index ν takes the values $j-k+1,\ldots,j-1$, see (12). Hence L_k is also a linear inhomogeneous form of the variables in the set $\mathcal{J}_{j-s,s-1}$. $\square$

Proof of Lemma 17: 1^0. Consider equation $(A_{k,s-j-k})$. Recall that its unknown variables are the ones in the set $\mathcal{I}_{j-s,s-1}$. Present this equation in the form $\alpha_1 q_{j-s}^{(0)}+\cdots+\alpha_s q_{j-1}^{(s-1)}=\beta+\mathcal{G}$ where the term β depends on j but not on the variables $q_\nu^{(i)}$ and $\mathcal{G}$ is a linear form in the variables $q_\nu^{(i)}$ from the set $\mathcal{J}_{j-s+1,s-2}$.

2^0. The quantity β is obtained by adding the terms $(n-1)^{j-s+k}$ from the Laurent series of the expressions : $A:=(n-1)\cdots(n-j-1)(-1)^jC_{n-j-2}^{k-1}$, $D:=-(n-k)C_{n-1}^{k-1}k^j$ and $W:=C_{n-1}^{k-1}(-1)^j(n-k)^{j+1}$ in equation (12). The coefficient of $(n-1)^{j-s+k}$ in A equals $(-1)^{j+s}e_s(j+k-1)/(k-1)!$ (see equation (13) with $\nu=j$) which is a degree $2s$ polynomial in j, see Lemma 10. Its coefficient in W is a polynomial in j of degree s. Indeed,

$$W=(-1)^jC_{n-1}^{k-1}((n-1)-(k-1))^{j+1}=(-1)^jC_{n-1}^{k-1}\sum_{\gamma=0}^{j+1}C_{j+1}^{\gamma}(n-1)^\gamma(k-1)^{j+1-\gamma},$$

where $(-1)^jC_{n-1}^{k-1}$ is a polynomial in $(n-1)$ of degree $k-1$ and $C_{j+1}^{j-s+1}=C_{j+1}^s$ is a polynomial in j of degree s. As $k\leq s<j$, there is no term $(n-1)^{j-s+k}$ in D. Hence β is a polynomial in j of degree $2s$.

3^0. The linear form $\mathcal{G}$ is obtained from certain expressions in both sides of equality (12). First, one considers the terms $B(n,j)C_{n-j-2}^{\nu-j+k-1}q_\nu$ in L_k and their coefficients of $(n-1)^{j-s+k}$ given by formula (14). Recall that (see Lemma 16) the index ν takes values $\geq j-s$. Set $\theta:=j-\nu$. Hence $\theta\leq s$. In formula (14) the term $\sigma:=q_\nu^{(\delta)}e_{\nu-j+s-\delta}=q_\nu^{(\delta)}e_{s-\theta-\delta}$ is the product of the degree $2(s-\theta-\delta)$ polynomial $e_{s-\theta-\delta}$ in the variable j (see Remark 10) and of $\pm q_{j-\nu}^{(\delta)}=\pm q_\theta^{(\delta)}$ which is a polynomial of degree $2(\theta+\delta)$ by inductive assumption. Thus σ (and, hence, the whole contribution of L_k to the linear form $\mathcal{G}$) is a polynomial in j of degree $2s$.

4^0. Secondly, consider in R_k the product

$$S:=C_{n-1}^{k-1}(n-k)^{\nu+1}k^{j-\nu}q_\nu=C_{n-1}^{k-1}k^\theta((n-1)+(1-k))^{\nu+1}\sum_{\eta=0}^{\infty}q_\nu^{(\eta)}(n-1)^{-\eta}.$$

The quantity $q_\nu^{(\eta)}=\pm q_\theta^{(\eta)}$ is a polynomial in j of degree $2(\theta+\eta)$.

Our goal now is to show that *the coefficient of* $(n-1)^{k+\nu-r}$ *in the product* $C_{n-1}^{k-1}(n-k)^{\nu+1}$ *is a polynomial in* j *of degree* r. (We prove this statement in 5^0 below.) This implies that the coefficient of $(n-1)^{j-s+k}$ in S is a finite sum of polynomials in j of degrees $\tau:=2(\theta+\eta)+r$. To obtain a term $(n-1)^{j-s+k}$ we multiply the terms $(n-1)^{k+\nu-r}$ and $(n-1)^{-\eta}$. In other words, one has $(k+\nu-$

$r) - \eta = j - s + k$, i.e. $\nu = j + r + \eta - s$ and $\tau = 2(j - \nu + \eta) + r = 2s - r < 2s$. Thus the contribution of R_k to the linear form $\mathcal{G}$ is a polynomial in j of degree $< 2s$. The lemma is proved.

5^0. Proof of the latter statement. One has $(n-k)^{\nu+1} = \sum_{r=0}^{\nu+1} C_{\nu+1}^r (1-k)^r (n-1)^{\nu+1-r}$ and $C_{\nu+1}^r$ is a degree r polynomial in ν, and therefore, also in the variable $j-\theta$ and thus in the variable j as well. The binomial coefficient C_{n-1}^{k-1} is a polynomial in $(n-1)$ of degree $(k-1)$. Thus $C_{n-1}^{k-1}(n-k)^{\nu+1}$ is of the form $\sum_{r=0}^{k+\nu} d_r (n-1)^{k+\nu-r}$ where d_r is a polynomial in j of degree r. □

Now we finally return to our main results formulated in the introduction.

2.3. **Proof of Theorem 5.** 1^0. Consider the lower triangular matrix $\mathcal{M}$ whose j-th row contains the coefficients of the polynomial $M_j(x)$ (starting with the coefficient of the linear term) followed by zeros. Let us turn the Narayana triangle (1) into an infinite lower triangular matrix of the form

$$\mathcal{N} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 6 & 6 & 1 & 0 & \cdots \\ 1 & 10 & 20 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{15}$$

Theorem 5 claims that the matrices $\mathcal{M}$ and $\mathcal{N}$ coincide. Denote by $\mathcal{M}_l$ and $\mathcal{N}_l$ their l-th columns and by $\mathcal{M}^l$ and $\mathcal{N}^l$ their l-th diagonals (i.e. the sets of entries in positions $(r, r+l-1)$, $r = 1, 2, \ldots$ in $\mathcal{M}$ and $\mathcal{N}$ respectively).

The polynomials $M_j(x)$ are monic and self-reciprocal with positive coefficients by definition. Therefore, one has $\mathcal{M}_1 = \mathcal{N}_1$, $\mathcal{M}^1 = \mathcal{N}^1$.

2^0. Suppose that the first m columns (and, hence, the first m diagonals as well) of $\mathcal{M}$ coincide with the first m columns (respectively, diagonals) of $\mathcal{N}$. The first m entries of $\mathcal{M}_{m+1}$ and of $\mathcal{N}_{m+1}$ vanish. Their next m entries belong to the first m diagonals, hence, they coincide as well.

3^0. The entries of $\mathcal{M}_{m+1}$ and the ones of $\mathcal{N}_{m+1}$ (denoted respectively by $\mathcal{M}_{m+1,j}$ and $\mathcal{N}_{m+1,j}$) are the values of polynomials R_M^{m+1} and R_N^{m+1} in j of the same degree $2m$. For $\mathcal{M}_{m+1,j}$ this follows from Proposition 14, and for $\mathcal{N}_{m+1,j}$ it follows from the next formula for the Narayana numbers:

$$N_{j,m+1} = \frac{1}{j} C_j^m C_j^{m+1} = \frac{(j-1)(j-2)\cdots(j-m+1)}{m!} \frac{j(j-1)\cdots(j-m)}{(m+1)!}. \tag{16}$$

For $j = 1, \ldots, 2m$ one has $R_M^{m+1}(j) = R_N^{m+1}(j)$, see 2^0. For $j = 0$ one has $N_{j,m+1} = \mathcal{N}_{m+1,j} = 0 = \mathcal{M}_{m+1,j}$. The first two equalities follow from formula (16), the last one can be deduced from part (2) of Proposition 14. This proposition implies that $\mathcal{M}_{m+1,j}$ is divisible by $j(j-1)$ (recall that $M_j(x) = (-1)^{j-1} x Q_{j-1}^*(-x)$). Hence $R_M^{m+1}(x) = R_N^{m+1}(x)$. □

3. Relation between Narayana and Gegenbauer polynomials

In the notation of the introduction the following statement holds.

Lemma 18. *One has*

$$\widehat{N}_n(x) = {}_2F_1\left(1-n, -n; 2; x\right), \tag{17}$$

where ${}_2F_1\left(a, b; c; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{k!(c)_k}$ *is the standard hypergeometric function.*

Proof. Direct summation. □

This gives us the clue of the possible connection of $N_n(x)$ with the "classical" hypergeometric polynomials. Namely, straightforward computation shows that

$$\widehat{N}_n(x) = \frac{1}{n}\,(x-1)^{n-1} P_{n-1}^{(1,1)}\left(\frac{x+1}{x-1}\right). \tag{18}$$

where $P_n^{(1,1)}(x)$ defined in (4) is the standard Gegenbauer polynomial. Formula (18) is just a different form of Proposition 6 which is now settled.

Using (17) it yields an alternative expression for $\widehat{N}_n$:

$$\widehat{N}_n(x) = x^{n-1}\,{}_2F_1\left(1-n, -n; 2; 1/x\right).$$

Furthermore, if $-1 < \xi_1^{(n-1)} < \cdots < \xi_{n-1}^{(n-1)} < 1$ are the zeros of the Gegenbauer polynomial $P_n^{(1,1)}$, then with account of (5), the zeros of N_n are $0 = \zeta_0^{(n)} > \zeta_1^{(n)} > \cdots > \zeta_{n-1}^{(n)}$,

$$\zeta_k^{(n)} = \begin{cases} 0, & k = 0, \\ \dfrac{\xi_k^{(n-1)}+1}{\xi_1^{(n-1)}-1}, & k = 1, \ldots, n-1. \end{cases}$$

Since (6) is a bijection, the interlacing of the roots of two consecutive Narayana polynomials, see (i) of Corollary 7 is just the translation of the analogous property of the zeros of orthogonal polynomials, see e.g. [14, Ch. IV].

Formula (4) allows us to establish several properties of the zeros of N_n: just transplant the known bounds, estimates and identities from $(-1,1)$ to $(-\infty,0)$ using (6). For instance, we can easily prove (ii) of Corollary 7, i.e. get the asymptotic zero distribution of N_n. Observe that the equilibrium measure

$$\frac{1}{\pi}\,\frac{dx}{\sqrt{1-x^2}} \text{ on } (-1,1)$$

is transformed by (6) into the following measure on $(-\infty,0)$:

$$d\mu(t) \stackrel{\text{def}}{=} \frac{1}{\pi}\,\frac{dt}{(1-t)\sqrt{-t}}.$$

Any function $f \in C(-\infty,0)$ can be written as

$$f(t) = F\left(\frac{t+1}{t-1}\right), \qquad F \in C(-1,1).$$

We have

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\zeta_k^{(n)}\right) = \frac{1}{n}\left(f(0) + \sum_{k=1}^{n-1} f\left(\frac{\xi_k^{(n-1)}+1}{\xi_k^{(n-1)}-1}\right)\right) = \frac{1}{n}\left(F(-1) + \sum_{k=1}^{n-1} F\left(\xi_k^{(n-1)}\right)\right)$$
$$\longrightarrow \frac{1}{\pi}\int_{-1}^{1} F(x)\frac{dx}{\sqrt{1-x^2}} = \frac{1}{\pi}\int_{-\infty}^{0} f(t)\frac{dt}{(1-t)\sqrt{-t}} = \int_{-\infty}^{0} f(t)\,d\mu(t),$$

as $n \to \infty$ which proves Corollary 7.

Further trivial result is the varying orthogonality of $\widehat{N}_n$,

$$\int_{-\infty}^{0} t^k \widehat{N}_n(t)\frac{t\,dt}{(1-t)^{2n-1}} = 0, \qquad k = 0, 1, \ldots, n-2,$$

which is another way to prove interlacing and weak convergence. □

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover Publ., New York, 1972.

[2] S. Alkhatib, V.P. Kostov, The Schur-Szegő composition of real polynomials of degree 2, Revista Matemática Complutense, to appear.

[3] P. Barry, On integer-sequence-based constructions of generalized Pascal triangles, J. Integer Sequences, vol. 9 (2006), article 06.2.4.

[4] W. Y. C. Chen, L. X. W. Wang and A. L. B. Yang, Schur Positivity and the q-Log-convexity of the Narayana Polynomials, preprint arxiv:0806.1561.

[5] T. Doslic, D. Syrtan, and D. Veljan, Enumerative aspects of secondary structures, Discr. Math., 285 (2004) 67–82.

[6] V. P. Kostov, The Schur-Szegő composition for hyperbolic polynomials, C.R.A.S. Sér. I 345/9 (2007) 483-488.

[7] V. P. Kostov, Eigenvectors in the context of the Schur-Szegő composition of polynomials, Mathematica Balkanica, 22(1-2) (2008), 155–173.

[8] G. Kreweras, Sur les éventails de segments, Cahiers du Bureau Universitaire de Recherche Opérationelle, Institut de Statistique, Université de Paris, #15 (1970) 3–41.

[9] V. Prasolov, *Polynomials*. Translated from the 2001 Russian second edition by Dimitry Leites. Algorithms and Computation in Mathematics, 11. Springer-Verlag, Berlin, 2004, xiv+301 pp. ISBN: 3-540-40714-6.

[10] Q.I. Rahman, G. Schmeisser, *Analytic Theory of Polynomials*, London Math. Soc. Monogr. (N.S.), vol. 26, Oxford Univ. Press, New York, NY, 2002.

[11] A. Sapounakis, I. Tasoulas, and P. Tsikouras, Counting strings in Dyck paths, Discr. Math., 307 (2007) 2909–2924.

[12] H. Stahl, W. Totik, *General orthogonal polynomials*, Encyclopedia of Mathematics, Vol. 43, Cambridge Univ. Press, Cambridge, 1992

[13] R.A. Sulanke, The Narayana distribution. Special issue on lattice path combinatorics and applications (Vienna, 1998). J. Statist. Plann. Inference 101(1-2), (2002) 311–326.

[14] G. Szegő. *Orthogonal Polynomials*, volume 23 of *Amer. Math. Soc. Colloq. Publ.* Amer. Math. Soc., Providence, RI, fourth edition, 1975.

[15] R. Szwarc, Positivity of Turán determinants for orthogonal polynomials, preprint arxiv:0710.3389.

[16] F. Yano, H. Yoshida, Some set partitions statistics in non-crossing partitions and generating functions, Discr. Math. 307 (2007) 3147–3160.

Université de Nice, Laboratoire de Mathématiques, Parc Valrose, 06108 Nice Cedex 2, France

E-mail address: kostov@math.unice.fr

Departamento de Estadística y Matemática Aplicada, Universidad de Almería 04120 Almería, Spain

E-mail address: andrei@ual.es

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden

E-mail address: shapiro@math.su.se