

FIRST STEPS TOWARDS TOTAL REALITY OF MEROMORPHIC FUNCTIONS

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ABSTRACT. It was earlier conjectured by the second and the third authors that any rational curve $\gamma : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^n$ such that the inverse images of all its flattening points lie on the real line $\mathbb{R}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^1$ is real algebraic up to a Möbius transformation of the image $\mathbb{C}\mathbb{P}^n$, see [1], [8] and [12]. (By a *flattening* point p on γ we mean a point at which the Frenet n -frame $(\gamma', \gamma'', \dots, \gamma^{(n)})$ is degenerate.) Below we extend this conjecture to the case of meromorphic functions on real algebraic curves of higher genera and settle it for meromorphic functions of degrees 2, 3 and several other cases.

To Victor Vassiliev on the occasion of his fiftieth birthday

1. INTRODUCTION

The above mentioned conjecture on total reality for rational curves was formulated by the authors in a private communication to F. Sottile in 1993 and attracted some attention due to its close connection with the problem of total reality in Schubert calculus. At the present moment it is supported by a large number of partial results and extensive numerical evidence, see [1],[2],[8],[15]-[19], and [20]. At the same time the only case of this conjecture which is completely settled is the case $n = 1$, i.e. the case of the usual rational functions. (The authors were recently informed by A. Eremenko and A. Gabrielov that they proved the above conjecture in the case of plane rational quintics.) Namely the main result of [1] is as follows.

Theorem 1 (Theorem 2 of [1]). *For any given $(2d - 2)$ -tuple of distinct real numbers there exist at least $\mathbf{Cat}_d = \frac{1}{d} \binom{2d-2}{d-1}$ real rational functions (considered up to a real Möbius transformation of the image $\mathbb{C}\mathbb{P}^1$) with these critical points.*

The above theorem together with the statement of L. Goldberg, see [4] claiming that for any $(2d - 2)$ -tuple of distinct complex numbers the number of complex rational functions (considered up to a complex Möbius transformation of the image $\mathbb{C}\mathbb{P}^1$) with these critical points is at most \mathbf{Cat}_d gives the proof in the case $n = 1$. The main idea of the proof of Theorem 1 is the explicit construction of such functions using the notion of *garden* which is the graph on the source $\mathbb{C}\mathbb{P}^1$ obtained as the inverse image of $\mathbb{R}\mathbb{P}^1$ under a real rational function, comp. [9]. The number of topologically different gardens for generic real rational functions of degree d with all real, simple and distinct critical points turned out to coincide with \mathbf{Cat}_d . The difficult part of the proof of Theorem 1 is then to show that for a given topological type of a garden there exists a real rational function with this garden and having $(2d - 2)$ prescribed real critical points.

The purpose of this short paper is to discuss a (conjectural) generalization of Theorem 1 to the case of the source curves of higher genera, i.e. to the case of

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meromorphic functions. Existence of real meromorphic functions with all real (and closely located) critical points on real curves of positive genus was recently proved by B. Osserman in [11].

We start with some standard notation.

Definition. A pair (\mathcal{C}, σ) consisting of a compact Riemann surface \mathcal{C} and its antiholomorphic involution σ is called a *real algebraic curve*. It is well-known that if \mathcal{C} is a compact Riemann surface of genus g then for any σ the set \mathcal{C}_σ (if nonempty) consists of at most $g + 1$ disjoint smooth closed non-selfintersecting loops called the *ovals* of \mathcal{C}_σ . The set $\mathcal{C}_\sigma \subset \mathcal{C}$ of all fixed points of σ is called the *real part* of (\mathcal{C}, σ) .

If (\mathcal{C}, σ) and (\mathcal{D}, τ) are real curves (varieties) and $f : \mathcal{C} \rightarrow \mathcal{D}$ a holomorphic map, then we shall use the notation \bar{f} for $\tau \circ f \circ \sigma$ which is another holomorphic map. The map f is *real* if $\bar{f} = f$.

Definition. Following the terminology of real algebraic geometry we call a real algebraic curve (\mathcal{C}, σ) with \mathcal{C} compact of genus g an *M-curve* if its \mathcal{C}_σ consists of exactly $g + 1$ ovals.

The main question we discuss below is as follows.

Problem 1. Given a meromorphic function $f : (\mathcal{C}, \sigma) \rightarrow \mathbb{CP}^1$ such that

- i) all its critical points and values are distinct;
- ii) all its critical points belong to \mathcal{C}_σ

is it true that that f becomes a real meromorphic function after a choice of a real structure of \mathbb{CP}^1 ?

Definition We say that the space of meromorphic functions of degree d on a genus g real algebraic curve (\mathcal{C}, σ) has *the total reality property* if Problem 1 has the affirmative answer for any meromorphic function from this space which satisfies the above assumptionis.

Notice that Problem 1 has the following modification. Since the number of critical points/values of a generic degree d meromorphic function from a genus g curve equals $2d - 2 + 2g$ one has that the dimension of the space of corresponding linear systems equals $2d - 2 - g$, i.e. one can arbitrarily assign the position of $2d - 2 - g$ critical points and find (in general, several) meromorphic functions with these critical points. For each such function the remaining $3g$ critical points will be uniquely determined.

Problem 2. Given a meromorphic function $f : (\mathcal{C}, \sigma) \rightarrow \mathbb{CP}^1$ of degree d such that

- i) all its critical points and values are distinct;
- ii) its $2d - 2 - g$ critical points belong to \mathcal{C}_σ

is it true that that f becomes a real meromorphic function after a choice of a real structure of \mathbb{CP}^1 ?

In the present note we prove the following results.

Theorem 2. *The space of meromorphic functions of any degree d which is a prime on any real curve (\mathcal{C}, σ) of genus g which additionally satisfies the inequality: $g > \frac{d^2 - 4d + 3}{3}$ has the total reality property.*

Corollary 1. *The total reality property holds for all meromorphic functions of degrees 2, 3. Moreover, if degree equals 2 then in case of positive genus it is sufficient that just one critical point is real.*

Relaxing the requirement that all critical points must be distinct we obtain the following corollary.

Corollary 2. *Any meromorphic function of a prime degree d on a real algebraic surface (\mathcal{C}, σ) , such that $g(\mathcal{C}) \geq (d-1)^2$ and all its (not necessary distinct) critical points belong to \mathcal{C}_σ becomes a real function after appropriate Möbius transformation of the image \mathbb{CP}^1 .*

The total reality property for degree 4 meromorphic functions is shown to be equivalent to the non-existence of a sextic in \mathbb{CP}^2 whose only singularities are: 7 real cusps and two complex conjugated nodes, such that the line connecting two nodes is tangent to the sextic in a real smooth point. Unfortunately, the known results about the realizability of singularities by real algebraic curves are not strong enough to cover this remaining case, comp. [13], [14].

On the other hand, we show that the answer to Problem 2 is negative. Namely,

Proposition 1. *There exists a real elliptic curve (\mathcal{C}, σ) with a nonempty real part \mathcal{C}_σ and a meromorphic function $f : \mathcal{C} \rightarrow \mathbb{CP}^1$ of degree 3 with 3 of its 6 critical points lying on \mathcal{C}_σ and which can not be made real by a Möbius transformation of the image \mathbb{CP}^1 .*

Remark 1. Note that (as it was pointed out to the second author by A. Eremenko) a further generalization of the Problem 1 to the case of maps between two real curves definitely has a negative answer. The counterexample can be found already for a map between two elliptic curves. By the Riemann-Hurwitz formula such a map has no ramification locus. If reality conjecture holds then any map $f : E_1 \rightarrow E_2$ from a real elliptic curve (E_1, σ) to an elliptic E_2 is real, i.e., there exists an involution $\tau : E_2 \rightarrow E_2$ such that $\bar{f} = \tau f \sigma = f$. However, it is evidently false. Namely, let $E_1 = E_2 = (\mathbb{C}/\mathbb{Z} + i\mathbb{Z}, \sigma)$ where σ is the standard complex conjugations. Let $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ be the linear map $z \mapsto (2+i)z$, and $\phi : E_1 \rightarrow E_2$ be degree 5 map induced by Φ . Existence of τ means, in particular, that push forward of σ is well defined. Put $\xi = \frac{i}{2+i} = \frac{1+2i}{5} = \Phi^{-1}(i)$. $\Phi(\bar{\xi}) = \frac{4-3i}{5} \notin \mathbb{Z} + i\mathbb{Z}$, which shows that the push forward of σ under ϕ is not well defined. Hence, there is no antiholomorphic involution on E_2 which makes ϕ into a real map.

The structure of the note is as follows. § 2 contains the proofs of Theorem 1, Corollary 2 and Proposition 1 while § 3 contains a number of remarks and open problems.

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2. PROOFS

We start our proofs with a characterization of real meromorphic functions on a real algebraic curve (\mathcal{C}, σ) .

Proposition 2. *If (\mathcal{C}, σ) is a proper irreducible real curve and $f : \mathcal{C} \rightarrow \mathbb{CP}^1$ a non-constant holomorphic map (with \mathbb{CP}^1 provided with its standard real structure), then f is real for some real structure on \mathbb{CP}^1 precisely when there is a Möbius transformation $\varphi : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ such that $\bar{f} = \varphi \circ f$.*

Proof. Any real structure on \mathbb{CP}^1 is of the form $\tau \circ \phi$ for a complex Möbius transformation ϕ and τ the standard real structure with $\bar{\phi} \circ \phi = \text{id}$ and conversely any such ϕ gives a real structure. If f is real for such a structure we have

$f = \tau \circ \phi \circ f \circ \sigma$, i.e., $\bar{f} = \varphi \circ f$ for $\varphi = \tau \circ \phi^{-1} \circ \tau$. Conversely, if $\bar{f} = \varphi \circ f$, then $f = \bar{\bar{f}} = \overline{\varphi \circ f} = \bar{\varphi} \circ \bar{f} = \bar{\varphi} \circ \varphi \circ f$ and as f is surjective we get $\bar{\varphi} \circ \varphi = \text{id}$. That means that $\phi := \tau \circ \varphi^{-1} \circ \tau$, then ϕ defines a real structure on \mathbb{CP}^1 and by construction f is real for that structure and the fixed one on \mathcal{C} . \square

We recall that up to a real isomorphism there are only two real structures on \mathbb{CP}^1 , the standard one and the one on an isotropic real quadric in \mathbb{CP}^2 . The latter is distinguished from the former by not having any real points.

Assume now that (\mathcal{C}, σ) is a proper irreducible real curve and $f : \mathcal{C} \rightarrow \mathbb{CP}^1$ a non-constant holomorphic map. It defines the holomorphic map

$$\mathcal{C} \xrightarrow{(f, \bar{f})} \mathbb{CP}^1 \times \mathbb{CP}^1$$

and if $\mathbb{CP}^1 \times \mathbb{CP}^1$ is given the real structure that takes (x, y) to $(\tau(y), \tau(x))$, which we shall call the *involutive real structure*, then it is clearly a real map.

Proposition 3.

- (1) *The image \mathcal{D} of the curve \mathcal{C} under the map (f, \bar{f}) is of type (δ, δ) for some positive integer δ and if ∂ is the degree of the map $\mathcal{C} \rightarrow \mathcal{D}$ we have that $d = \delta\partial$, where d is the degree of the original f .*
- (2) *The function f is real for some real structure on \mathbb{CP}^1 precisely when $\delta = 1$.*
- (3) *Assume that \mathcal{C} is smooth and all the critical points of f are real. Then all the critical points of $g : \tilde{\mathcal{D}} \rightarrow \mathbb{CP}^1$, the composite of the normalization map $\tilde{\mathcal{D}} \rightarrow \mathcal{D}$ and the restriction of the projection of $\mathbb{CP}^1 \times \mathbb{CP}^1$ has all its critical points real.*

Proof. The image under the real map (f, \bar{f}) of \mathcal{C} is a real curve so that \mathcal{D} is a real curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ (with its involutive real structure). Any such curve is of type (δ, δ) as the real structure permutes the two degrees. The rest of (1) follows by using the multiplicativity of degrees for the maps $f : \mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathbb{CP}^1$, where the last map is projection on the first factor.

As for (2) assume first that f can be made real for some real structure on \mathbb{CP}^1 . By Proposition 2 there is a Möbius transformation φ such that $\bar{f} = \varphi \circ f$ but that in turn means that (f, \bar{f}) maps \mathcal{C} into the graph of φ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and that graph is hence equal to \mathcal{D} and is thus of type $(1, 1)$. Conversely, assume that \mathcal{D} is of type $(1, 1)$. Then it is a graph of an isomorphism φ from \mathbb{CP}^1 to \mathbb{CP}^1 and by construction $\bar{f} = \varphi \circ f$ so we conclude by another application of Proposition 2.

Finally, for (3) we have that the map $\mathcal{C} \rightarrow \mathcal{D}$ factors as a, necessarily real, map $h : \mathcal{C} \rightarrow \tilde{\mathcal{D}}$ and then $f = g \circ h$. If $pt \in \tilde{\mathcal{D}}$ is a critical point, then all points of $h^{-1}(pt)$ are critical for f and hence by assumption real. As h is real this implies that pt is also real. \square

Part (2) of the above Proposition gives another reformulation of the total reality property for meromorphic functions.

Corollary 3. *If a degree d function $f : (\mathcal{C}, \sigma) \rightarrow \mathbb{CP}^1$ is real for some real structure on \mathbb{CP}^1 then the map $\mathcal{C} \xrightarrow{(f, \bar{f})} \mathcal{D} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ must have degree d as well.*

Remark 2. Notice that without the requirement of reality of f the degree of $\mathcal{C} \xrightarrow{(f, \bar{f})} \mathcal{D}$ can be any factor of d .

By a *cusp* we mean a curve singularity of multiplicity 2 and whose tangent cone is a double line. It has the local form $y^2 = x^k$ for some integer $k \geq 3$ where k is an invariant which we shall call its *type*. A cusp of type k gives a contribution of $\lceil (k-1)/2 \rceil$ to the arithmetic genus of a curve. A cusp of type 3 will be called *ordinary*.

If \mathcal{C} is a curve and p_1, \dots, p_k are smooth points on it then we let $\pi : \mathcal{C} \rightarrow \mathcal{C}(p_1, \dots, p_k)$ be the finite map which is a homeomorphism and for which $\mathcal{O}_{\mathcal{C}(p_1, \dots, p_k)} \rightarrow \pi_* \mathcal{O}_{\mathcal{C}}$ is an isomorphism outside of $\{p_1, \dots, p_k\}$ with $\mathcal{O}_{\mathcal{C}(p_1, \dots, p_k), \pi(p_i)} \rightarrow \mathcal{O}_{\mathcal{C}, p_i}$ having image the inverse image of \mathbb{C} in $\mathcal{O}_{\mathcal{C}, p_i} / \mathfrak{m}_{p_i}^2$. In other words, $\mathcal{C}(p_1, \dots, p_k)$ has ordinary cusps at points $\pi(p_i)$.

Then π has the following two (obvious) properties:

Lemma 1.

- (1) *A holomorphic map $f : \mathcal{C} \rightarrow X$ which is not an immersion at all the points p_1, \dots, p_k factors through π .*
- (2) *If \mathcal{C} is proper, then the arithmetic genus of $\mathcal{C}(p_1, \dots, p_k)$ is k plus the arithmetic genus of \mathcal{C} .*

Proposition 4. *Assume that (\mathcal{C}, σ) is a smooth and proper real curve and let $f : \mathcal{C} \rightarrow \mathbb{CP}^1$ be a holomorphic map of degree d . If there are k real points p_1, \dots, p_k on \mathcal{C} which are critical points for f and if (f, \bar{f}) gives a map of degree 1 from \mathcal{C} to its image \mathcal{D} in $\mathbb{CP}^1 \times \mathbb{CP}^1$, then $g(\mathcal{C}) + k \leq (d-1)^2$. If $g(\mathcal{C}) + k = (d-1)^2$, then the map $h : \mathcal{C} \rightarrow \mathcal{D}$ factors to give an isomorphism $\mathcal{C}(p_1, \dots, p_k) \xrightarrow{\sim} \mathcal{D}$.*

Proof. As p_i is real it is a critical point also for \bar{f} and hence for (f, \bar{f}) . This implies by the first property for $\mathcal{C} \rightarrow \mathcal{C}(p_1, \dots, p_k)$ that the map $\mathcal{C} \rightarrow \mathcal{D}$ factors as $\mathcal{C} \rightarrow \mathcal{C}(p_1, \dots, p_k) \rightarrow \mathcal{D}$ and hence the arithmetic genus of $\mathcal{C}(p_1, \dots, p_k)$, which is $g(\mathcal{C}) + k$ by the second property of $\mathcal{C}(p_1, \dots, p_k)$, is less than or equal to the arithmetic genus of \mathcal{D} , which by the adjunction formula is equal to $(d-1)^2$. If we have equality then their genera are equal and hence the map $\mathcal{C}(p_1, \dots, p_k) \rightarrow \mathcal{D}$ is an isomorphism. \square

Now we are ready to start the proof of Theorem 2. It is now a simple corollary of Proposition 4. Indeed, if we assume that all the critical points of a generic meromorphic function $f : (\mathcal{C}, \sigma) \rightarrow \mathbb{CP}^1$ are real then k in the above Proposition equals $2d - 2 + 2g(\mathcal{C})$. Under the assumption $g(\mathcal{C}) > \frac{d^2 - 4d + 3}{3}$ one gets $g(\mathcal{C}) + k = 2d - 2 + 3g(\mathcal{C}) > (d-1)^2$. Thus, the case when \mathcal{C} maps birationally to \mathcal{D} is impossible by the above Proposition. Since d is prime the only other possible case is when the degree of the map $\mathcal{C} \rightarrow \mathcal{D}$ equals d and therefore, the degree of the map $\mathcal{D} \rightarrow \mathbb{CP}^1$ equals 1 which by (2) of Proposition 3 gives the the total reality property holds. \square

We start with the proof of Corollary 1 for the case of degree 2, i.e. in the hyper-elliptic situation. Then we will reprove and strengthen the same result in a different way. But the simple classical argument below has an independent interest.

Proof. Consider a degree 2 meromorphic function $f : (\mathcal{C}, \sigma) \rightarrow \mathbb{CP}^1$. It defines the holomorphic involution on \mathcal{C} which we abusing notation denote by f as well. Since \mathcal{C} is smooth then by Riemann-Hurwitz formula the involution f has exactly $2g + 2$ distinct fixed points on \mathcal{C} which are the critical points of the function f . Take the map $\tilde{f} = \sigma \cdot f \cdot \sigma$. Then \tilde{f} is also a holomorphic involution on \mathcal{C} . It is known that on a given hyper-elliptic curve of genus $g > 1$ there exist exactly one holomorphic involution, see [3], sect. 2.3. Therefore, $f = \tilde{f}$, or, equivalently $f \cdot \sigma = \sigma \cdot f$. Therefore, the antiholomorphic involution σ on \mathcal{C} induces the antiholomorphic involution $\bar{\sigma}$ on the quotient $\mathcal{C}/f \simeq \mathbb{CP}^1$. Moreover, the real part \mathcal{C}_σ is projected by f on the real part $\mathbb{RP}^1 \subset \mathbb{CP}^1$ w.r.t the antiholomorphic involution $\bar{\sigma}$ of the image $\mathcal{C}/f \simeq \mathbb{CP}^1$. Thus the images of all the critical points of f are real as well, i.e. belong to the above \mathbb{RP}^1 . This implies that our original curve \mathcal{C} is realized as the Riemann surface of the plane algebraic curve given by the equation: $y^2 = (x - a_1) \dots (x - a_{2g+2})$ with all a_i 's real and the function f coincides with the projection $(y, x) \rightarrow x$ where (y, x) satisfies the above equation. Notice that in

this argument we also used the fact that any antiholomorphic involution on \mathbb{CP}^1 possessing a fixed point is conjugate to the standard complex conjugation on \mathbb{CP}^1 , see Proposition 2. The fact that (\mathcal{C}, σ) is an M -curve follows from the consideration of the above equation with all real a_i 's. In the case $g = 1$ we also get the second holomorphic involution $\tilde{f} = \sigma \cdot f \cdot \sigma$. Notice that \tilde{f} and f have the same set of 4 fixed points. It is well-known that for any point p on \mathcal{C} of genus $g = 1$ there exists and unique holomorphic involution on \mathcal{C} having p as its fixed point. Therefore, $\tilde{f} = f$ and the rest of the argument applies. \square

Let us now apply Theorem 2 to prove Corollary 1 in its full generality.

Case $d = 2$. Suppose that the degree d of the map $f : (\mathcal{C}, \sigma) \rightarrow \mathbb{CP}^1$ is equal to 2. That only leaves two possibilities: The first is that the map $\mathcal{C} \rightarrow \mathcal{D}$ has degree 2 and then by Proposition 3 f is real for some real structure (\mathbb{CP}^1, τ) on \mathbb{CP}^1 . In particular, if the set \mathcal{C}_σ of real points is nonempty then (\mathbb{CP}^1, τ) has the same property which means that it is equivalent to the standard real structure. The second is that the map $\mathcal{C} \rightarrow \mathcal{D}$ is birational and then by Proposition 4 we get $g(\mathcal{C}) + k \leq 1^2 = 1$, where k is the number of real critical points of f . In particular if $g(\mathcal{C}) > 0$ then there are no real critical points. Thus a hyper-elliptic map from a real curve (\mathcal{C}, σ) is real if one of its critical points is real.

Case $d = 3$. In this case again we have only two possibilities; either f is real for a real structure on \mathbb{CP}^1 or $\mathcal{C} \rightarrow \mathcal{D}$ is birational in which case we have $g(\mathcal{C}) + k \leq 2^2 = 4$. The case $g(\mathcal{C}) = 0$ was settled in [1]. Recall that the total number of critical points equals $2d - 2 + 2g(\mathcal{C})$. But if $g(\mathcal{C}) > 0$ then $2 \cdot 3 - 2 + 3g > 4$ and this case of Theorem 1 is settled. Analogously to the case $d = 2$ a function f with the degree $d = 3$ is real if it has more than $\max(4 - g(\mathcal{C}), 1)$ real critical points. \square

Next we prove Corollary 2.

Proof. Notice that since d is prime again it is enough to eliminate the case $\deg h = 1$. Then as in proof of the main theorem the arithmetic genus of $\mathcal{C}(p_1, \dots, p_k)$ must not exceed the arithmetic genus of \mathcal{D} . The arithmetic genus of \mathcal{D} equals $(d - 1)^2$ whereas the arithmetic genus of $\mathcal{C}(p_1, \dots, p_k)$ is the sum of $g(\mathcal{C})$ and a contribution from the singular points p_1, \dots, p_k . The latter is at least $k \geq \frac{2d+2g(\mathcal{C})-2}{d-1} \geq 2$. Hence, the hypothesis $g(\mathcal{C}) \geq (d - 1)^2$ contradicts the assumption $\deg h = 1$. \square

The case $d = 4$ can be reduced to the problem of existence for some special plane sextics. Indeed, we have three possibilities; \mathcal{D} has degree $(1, 1)$, $(2, 2)$, or $(4, 4)$. In the first case f can be made real. In the second case, by Proposition 3, the projection on the first factor will give a map from the normalization $\tilde{\mathcal{D}}$ of \mathcal{D} . The arithmetic genus $p_a(\mathcal{D}) = 1$, and the geometric genus $g(\tilde{\mathcal{D}})$ of the normalization $\tilde{\mathcal{D}}$ does not exceed 1. Let $\tilde{h} : \mathcal{C} \rightarrow \tilde{\mathcal{D}}$ be the lift of $h : \mathcal{C} \rightarrow \mathcal{D}$. Note that if $p_i \in \mathcal{C}$ is a critical point of f then either its image $h(p_i)$ is a cusp of \mathcal{D} or p_i is a ramification point of \tilde{h} . The ramification divisor $R(\tilde{h}) = 2g(\mathcal{C}) + 2 - 4g(\tilde{\mathcal{D}})$. The number of cusps of \mathcal{D} does not exceed 1, whereas the number of distinct critical points of f is $2g(\mathcal{C}) + 6$. Note also that any preimage of a cusp of \mathcal{D} must be a critical point. Since $\deg h = 2$, we must have $\frac{1}{2} \left(2g(\mathcal{C}) + 6 - \left(2g(\mathcal{C}) + 2 - 4g(\tilde{\mathcal{D}}) \right) \right) \leq 1$ which is impossible.

We are hence left with the case when \mathcal{D} has degree $(4, 4)$. The only case when $2 \cdot 4 - 2 + 3g(\mathcal{C}) \leq 9$ for $g(\mathcal{C}) > 0$ is the case of $g(\mathcal{C}) = 1$. If all the critical points p_1, \dots, p_8 of $f : \mathcal{C} \rightarrow \mathbb{CP}^1$ are real, then we get a birational map $\mathcal{C}(p_1, \dots, p_8) \rightarrow \mathcal{D}$ and as then both $\mathcal{C}(p_1, \dots, p_8)$ and \mathcal{D} have arithmetic genus 9, this map is an isomorphism. Hence \mathcal{D} is a curve with 8 ordinary real cusps and no other singularities. Projecting from a cusp p gives us a real plane curve \mathcal{D}' of degree 6.

Apart from the 7 surviving real cusps, \mathcal{D}' will have a pair of complex conjugate singularities obtained by contracting the strict transforms of the two lines through p . These strict transforms will have intersection number two with the strict transform of the curve and hence the singularities will be ordinary nodes (when they intersect in two points) or cusps (when they are tangent). The line through these two points will then be a tangent to \mathcal{D}' . Conversely, if we have an irreducible plane curve of degree 6 with 7 real ordinary cusps and two complex conjugate ordinary nodes or cusps such that the line through them is tangent to the curve we can go backwards and get a curve of degree (4, 4) with the desired properties.

Note also that if one composes the map from the plane blown up at two points to a quadric with the projection onto one of the rulings one obtains the projection from one of the two blown up points. Hence, given a plane curve \mathcal{D}' of degree 6 as above, the map f is obtained by composing the normalization map $\mathcal{C} \rightarrow \mathcal{D}'$ with the projection from one of the non-real singularities.

Remark 3. As a curiosity, if one has a, not necessarily real, irreducible plane curve \mathcal{D}' of degree 6 with 7 real ordinary cusps and one tangent at a smooth real point that intersects \mathcal{D}' in two complex conjugate singular points, then the curve is real. Indeed, if not it is distinct from its complex conjugate $\overline{\mathcal{D}'}$ and by the Bezout theorem the total intersection multiplicity $\mathcal{D}' \cdot \overline{\mathcal{D}'}$ is 36, whereas the nine singular points of \mathcal{D}' are also singular points of $\overline{\mathcal{D}'}$ giving a total contribution of at least $9 \cdot 4 = 36$ and the smooth real point gives an extra contribution of at least 1 (at least 2 actually as its tangent is common to both curves).

Consider the existence problem for real algebraic curves in $\mathbb{C}\mathbb{P}^2$ of degree 6 with the following singularities: 7 real cusps and either 2 complex conjugate cusps or nodes with additional property that the line through those is tangent to the real part at a smooth point. The situation with 7 real and 2 complex conjugate cusps is easy to reject, comp. [7]. Namely, any complex degree 6 genus 1 curve has at most 9 cusps. If it is additionally real and has exactly 9 cusps then it is dual to a nonsingular real curve of degree 3 with exactly 3 real inflection points, see e.g. [21]. But then the original curve has exactly 3 (and no more) real cusps.

Therefore in order to settle the total reality conjecture for all degree 4 meromorphic functions it remains to prove or disprove the existence of a sextic in $\mathbb{C}\mathbb{P}^2$ with 7 real cusps and 2 complex conjugate nodes such that the line connecting two nodes is tangent to the sextic at a smooth real point.

Finally, let us settle Proposition 1. Suppose that $d = 3$, then again we have only two possibilities; either f is real for a real structure on $\mathbb{C}\mathbb{P}^1$ or $\mathcal{C} \rightarrow \mathcal{D}$ is birational in which case we have $g(\mathcal{C}) + k \leq 2^2 = 4$. Assume further that $g(\mathcal{C}) = 1$ and $k = 3$ so that $\mathcal{D} \subset \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ is a real curve of type (3, 3) with three real cusps. Given an arbitrary Möbius transformation φ , the holomorphic map $\Phi: (x, y) \mapsto (\varphi(x), \overline{\varphi(y)})$ is a real automorphism of $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ with involutive real structure. Projection on the first factor identifies the real points of $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ (still with its involutive structure) with $\mathbb{C}\mathbb{P}^1$ and under this identification Φ acts on the real points as φ acts on $\mathbb{C}\mathbb{P}^1$. This means that we may assume that the three cusps are $(0, 0)$, $(1, 1)$ and (∞, ∞) .

Now, the complete linear system of type (1, 1) embeds $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ as a quadric Q in $\mathbb{C}\mathbb{P}^3$. This system has a real structure with respect to the involutive real structure and realizes $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ as a quadric of signature $(+1, -1, -1, -1)$. It has the property that there are real points on it but the two rulings on it are not defined over the reals, instead through each real point on it there are two complex conjugate lines on Q passing through it. Projecting Q from a real point p gives a map from Q with p blown up to the plane which gives an isomorphism from Q

with p blown up and the strict transform of the two lines passing through it blown down, taking the exceptional curve to a line through the two blown down curves. Conversely, given two complex conjugate points q and \bar{q} in the projective plane, the linear system of quadrics passing through them gives a map from the plane with q and \bar{q} blown up onto a quadric in \mathbb{CP}^3 which contracts exactly the line through q and \bar{q} .

Now, assume that $\mathcal{D} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is a real curve of type $(3, 3)$ with 3 real ordinary (i.e., of type 3) cusps. We now project from one of the cusps p . The strict transform $\tilde{\mathcal{D}}$ of \mathcal{D} in the blowing up Q of $\mathbb{CP}^1 \times \mathbb{CP}^1$ will then meet the exceptional curve in one real point. The strict transform of the two lines through p , E_1 and E_2 are complex conjugate in Q and meet $\tilde{\mathcal{D}}$ transversally in one point each (which are each other's complex conjugate). Mapping to the projective plane gives a curve \mathcal{D}' of degree 4 with two real ordinary cusps and no other singular points as well as one smooth real point whose tangent intersects \mathcal{D}' in the point and two complex conjugate points. Conversely, suppose \mathcal{D}' is a plane curve of degree 6 with two real ordinary real cusps, no other singularities, and a smooth real point whose tangent intersects the curve in the point and two complex conjugate points. Blowing up those two complex conjugate points and blowing down the tangent gives a curve on $\mathbb{CP}^1 \times \mathbb{CP}^1$ with three ordinary real cusps.

We will show that there exist curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ real in the involutive real structure, having degree $(3, 3)$ and 3 real ordinary cusps and no other singularities. Above we explained that this is equivalent to constructing a plane curve \mathcal{D}' of degree 4 with two real ordinary real cusps, no other singularities, and a smooth real point whose tangent intersects the curve in that point and two complex conjugate points. Indeed, blowing up those two complex conjugate points and blowing down the tangent gives a curve on $\mathbb{CP}^1 \times \mathbb{CP}^1$ with three ordinary real cusps.

Proposition 5. *The space of degree 4 plane curves with two ordinary cusps and no other singularities is a smooth subvariety of the space of all quartics. In the real locus of this space, the conditions that the two cusps are both real and that there is a smooth point whose tangent intersects the curve in the point and two complex conjugated points is open and nonempty.*

Proof. For the non-real part it will be enough to show that for a given curve $\mathcal{D} = \{f = 0\}$ in the space, the map from the linear space of quartics to the product of the tangent spaces to mini-versal deformations of the two singularities is surjective. That product can be identified with the product of the ‘‘Milnor spaces’’ $\mathcal{O}_{D,p}/(f_x, f_y, f_z)$ at the two singular points. As the points are ordinary cusps we have that $\mathfrak{m}_{\mathbb{CP}^2,p}^2 \subset (f_x, f_y, f_z)$ but it is clear that the quartics fill out $\mathcal{O}_{\mathbb{CP}^2,p}/\mathfrak{m}_p^2 \oplus \mathcal{O}_{\mathbb{CP}^2,q}/\mathfrak{m}_q^2$ for any two points of p and q of \mathbb{CP}^2 .

For the real part, the openness is clear and hence it is enough to show that it is nonempty. It is easily verified that $f = xz^3 - yz^3 + xyz^2 + x^2y^2$ has ordinary cusps at $(1 : 0 : 0)$ and $(0 : 1 : 0)$ and no other singularities. $(0 : 0 : 1)$ lies on the curve and its tangent is given by $x = y$ which intersects the curve in $(0 : 0 : 1)$, $(i : i : 1)$, and $(-i : -i : 1)$.

The latter statement implies that there exist real curves of bidegree $(3, 3)$ with exactly 3 real ordinary cusps and no other singularities which settles Proposition 1. \square

A more explicit example of a degree 3 function from a real elliptic curve with 3 real critical points but which can not be made real is presented below.

Example. It is well known that any degree 3 meromorphic function on an elliptic curve \mathcal{C} is realized in \mathbb{CP}^2 by the standard equation $y^2 = P_3(x)$ where $P_3(x)$ is a cubic polynomial can be represented as the composition of the group shift of

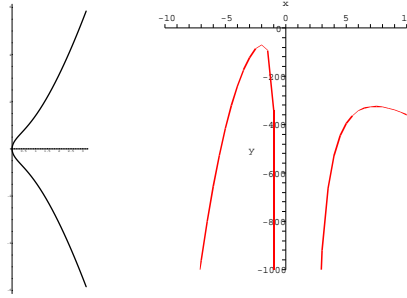


FIGURE 1. The curve $y^2 = x^3 + x$ and the graph of the determinant of the three tangent lines at $P_0 + P$, $P_1 + P$, $P_2 + P$ computed in Maple.

the whole \mathcal{C} by some fixed point on it with the projection from some point on $\mathbb{C}\mathbb{P}^2$, see e.g. [1]. The critical points on $\mathcal{C} \subset \mathbb{C}\mathbb{P}^2$ for the projection from some point $pt \in \mathbb{C}\mathbb{P}^2$ are the points where the pencil of lines through pt is tangent to \mathcal{C} . Thus in order to prove that a given triple (P_0, P_1, P_2) of real points on a given real elliptic curve $\mathcal{C} \subset \mathbb{C}\mathbb{P}^2$ can not serve as the set of critical points of a real degree 3 function we have to show that for any choice of a fourth real point $P \in \mathcal{C}$ the (real) tangent lines to \mathcal{C} at the points $P_0 + P, P_1 + P, P_2 + P$ never meet at the same (real) point on $\mathbb{C}\mathbb{P}^2$. Recall that (up to a sign change) the addition of two point A and B on a real elliptic $\mathcal{C} \subset \mathbb{C}\mathbb{P}^2$ can be interpreted as the third intersection point of the line \overline{AB} with \mathcal{C} .

Consider the real elliptic curve (\mathcal{C}, σ) given by the equation $y^2 = x^3 + x$ with 3 real points P_0, P_1, P_2 , where P_0 is "almost" infinite point $(100.35, 1005.3)$, P_1 is the point slightly above the origin $(0.05, 0.224)$, and, finally, P_2 is the inflection point $(0.4, -0.67)$.

Then we claim that for any point P on the real part \mathcal{C}_σ there is no real function whose inflection points coincide with $P_0 + P, P_1 + P, P_2 + P$. Indeed as we explained above the existence of such a function would mean that tangents to \mathcal{C}_σ would intersect at the same point of $\mathbb{R}\mathbb{P}^2$, or the corresponding dual points in $(\mathbb{R}\mathbb{P}^2)^*$ are collinear. However, computing the determinant of these three dual points we see that the obtained function of P is non-vanishing, see Fig. 1.

3. REMARKS AND PROBLEMS

I. Analogously to the total reality property for rational curves one can ask a similar question for projective curves of any genus, namely

Problem 3. Given a real algebraic curve (\mathcal{C}, σ) with compact \mathcal{C} and nonempty real part \mathcal{C}_σ and a complex algebraic map $\Psi : \mathcal{C} \rightarrow \mathbb{C}\mathbb{P}^n$ such that the inverse images of all the flattening points of $\Psi(\mathcal{C})$ lie on the real part $\mathcal{C}_\sigma \subset \mathcal{C}$ is it true that Ψ is a real algebraic up to a Möbius transformation of the image $\mathbb{C}\mathbb{P}^n$?

(Recent private communication from Gabriellov claims that this is now proven for the plane rational quintics.)

II. In the recent [2] the authors found another generalization of the conjecture on total reality in case of the usual rational functions.

Problem 4. Extend the results of [2] to the case of meromorphic functions on curves of higher genera.

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