CLASSIFICATION OF HYPERBOLICITY AND STABILITY PRESERVERS: THE MULTIVARIATE WEYL ALGEBRA CASE

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Abstract. A multivariate polynomial is *stable* if it is nonvanishing whenever all variables have positive imaginary parts. We characterize all finite order linear differential operators in the Weyl algebra $\mathcal{A}_n$ that preserve stability. An important tool that we develop in the process is the multivariate generalization of the notion of multiplier sequence. We give a complete description of all multivariate multiplier sequences as well as those of finite order. Next, we formulate and prove a natural analog of the Lax conjecture for real stable polynomials in two variables and use it to classify all finite order linear differential operators that preserve univariate hyperbolic polynomials by means of determinants and homogenized symbols. As a further consequence of our methods we establish symbol curve theorems and a duality theorem showing that a differential operator preserves stability if and only if its Fischer-Fock adjoint has the same property. These are vast generalizations of the classical Hermite-Poulain-Jensen theorem, Pólya's curve theorem and Schur-Maló-Szegö type composition theorems in the univariate case as well as natural multivariate extensions of the aforementioned theorems. We also give several other applications of our results and discuss further directions and open problems.

Contents

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Key words and phrases. Differential operators, Weyl algebra, symbol curves, hyperbolic polynomials, stable polynomials, multiplier sequences, Fischer-Fock duality, Lax conjecture.

1. Introduction and main results

In their 1914 Crelle paper [42] Pólya and Schur characterized all linear operators that are diagonal in the standard monomial basis of $\mathbb{C}[z]$ and preserve the set of polynomials with all real zeros. Polynomials of this type and linear transformations preserving them are of central interest in e.g. entire function theory [14, 29]: it is for instance well known that the Riemann Hypothesis is equivalent to saying that $\xi\big(\frac{1}{2}+it\big)$ may be approximated by real zero polynomials uniformly on compact sets, where ξ denotes Riemann's xi-function. Pólya-Schur's result generated a vast literature on this subject and related topics (see, e.g., *op. cit.* and references therein) but solutions to fundamental problems such as describing all linear preservers of the set of real zero polynomials – or, more generally, of the set of polynomials with all their zeros in a prescribed region $\Omega \subset \mathbb{C}$, cf. [14, 17] – are yet to be found.

This is the first in a series of papers [7, 8, 9] where we address the aforementioned questions as well as their analogs for multivariate polynomials and give various applications. In particular, in the present paper we describe all Weyl algebra operators on $\mathbb{C}[z_1,\dots,z_n]$ that preserve (real) stable polynomials and establish multivariate extensions of Pólya-Schur's theorem.

A nonzero univariate polynomial with real coefficients is called *hyperbolic* if all its zeros are real while a univariate polynomial $f(z)$ with complex coefficients is called *stable* if $f(z) \neq 0$ for all $z \in \mathbb{C}$ with $\mathfrak{Im}(z) > 0$. Hence a univariate real polynomial is stable if and only if it is hyperbolic. These classical concepts have several natural extensions to multivariate polynomials, see, e.g., four different definitions in [26]. Below we concentrate on the most general notion:

Definition 1.1. A polynomial $f \in \mathbb{C}[z_1,\dots,z_n]$ is *stable* if $f(z_1,\dots,z_n) \neq 0$ for all n-tuples $(z_1,\dots,z_n) \in \mathbb{C}^n$ with $\mathfrak{Im}(z_j) > 0$, $1 \le j \le n$. If in addition f has real coefficients it will be referred to as *real stable.*[1]

Thus f is stable (respectively, real stable) if and only if for all $\alpha \in \mathbb{R}^n$ and $v \in \mathbb{R}_+^n$ the univariate polynomial $f(\alpha + vt) \in \mathbb{C}[t]$ is stable (respectively, hyperbolic), see Lemma 2.1 in §2. In what follows we denote by $\mathcal{H}_n(\mathbb{C})$, respectively $\mathcal{H}_n(\mathbb{R})$, the set of stable, respectively real stable polynomials in n variables.

The notion of hyperbolicity in one variable has another classically known multivariate generalization. Namely, a homogeneous polynomial $p \in \mathbb{R}[z_1,\dots,z_n]$ is said to be *(Gårding) hyperbolic* with respect to a given vector $v \in \mathbb{R}^n$ if $p(v) \neq 0$ and for all vectors $\alpha \in \mathbb{R}^n$ the univariate polynomial $p(\alpha + vt) \in \mathbb{R}[t]$ has all real zeros. For background on (multivariate homogeneous) hyperbolic polynomials one may consult, e.g., [2, 20]. In §6.2 we prove the following result describing the relation between real stable and hyperbolic polynomials.

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[1] The existing terminology in this relatively new area makes an impression of being rather unstable. Very similar or even coinciding objects are called, for example, wide sense stable in [26], widest sense Hurwitz polynomials in [18], polynomials with the half-plane property (HPP) in [13], P-polynomials and POS-polynomials in [22, 23], etc. Other appropriate names are polynomials with the Lee-Yang or Lieb-Sokal property, cf. [28, 31]. The terminology adopted in this paper as well as in [6, 7, 8, 9] is inspired by Levin's book [29].

Proposition 1.1. *Let $f \in \mathbb{R}[z_1, \dots, z_n]$ be of degree d and let $f_H \in \mathbb{R}[z_1, \dots, z_{n+1}]$ be the (unique) homogeneous polynomial of degree d such that $f_H(z_1, \dots, z_n, 1) = f(z_1, \dots, z_n)$. Then $f \in \mathcal{H}_n(\mathbb{R})$ if and only if f_H is hyperbolic with respect to every vector $v \in \mathbb{R}^{n+1}$ such that $v_{n+1} = 0$ and $v_i > 0$, $1 \le i \le n$.*

It is worth mentioning that real stable multivariate polynomials appear already in Theorem 1 of the foundational article [20] by Gårding and that stable multivariate entire functions can be found in Chap. IX of Levin's book [29].

Let $\mathcal{A}_n[\mathbb{C}]$ be the Weyl algebra of all finite order linear differential operators with polynomial coefficients on $\mathbb{C}[z_1, \dots, z_n]$. Recall the standard multi-index notation $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, where $z = (z_1, \dots, z_n)$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $\partial_i = \frac{\partial}{\partial z_i}$, $1 \le i \le n$. Then every operator $T \in \mathcal{A}_n[\mathbb{C}]$ may be (uniquely) represented as

$$T = \sum_{\alpha, \beta \in \mathbb{N}^n} a_{\alpha\beta} z^\alpha \partial^\beta, \tag{1.1}$$

where $a_{\alpha\beta} \in \mathbb{C}$ is nonzero only for a finite number of pairs (α, β). Let further $\mathcal{A}_n[\mathbb{R}]$ be the set of all $T \in \mathcal{A}_n[\mathbb{C}]$ with $a_{\alpha\beta} \in \mathbb{R}$ for all $\alpha, \beta \in \mathbb{N}^n$. A *nonzero* differential operator $T \in \mathcal{A}_n[\mathbb{C}]$ is called *stability preserving* if $T : \mathcal{H}_n(\mathbb{C}) \to \mathcal{H}_n(\mathbb{C}) \cup \{0\}$ and it is said to be *real stability preserving* if $T : \mathcal{H}_n(\mathbb{R}) \to \mathcal{H}_n(\mathbb{R}) \cup \{0\}$.

Given T of the form (1.1) define its *symbol* $F_T(z, w)$ to be the polynomial in $\mathbb{C}[z_1, \dots, z_n, w_1, \dots, w_n]$ given by $F_T(z, w) = \sum_{\alpha, \beta} a_{\alpha\beta} z^\alpha w^\beta$.

The first main results of this paper are the following characterizations of the multiplicative submonoids $\mathcal{A}_n(\mathbb{C}) \subset \mathcal{A}_n[\mathbb{C}]$ and $\mathcal{A}_n(\mathbb{R}) \subset \mathcal{A}_n[\mathbb{R}]$ consisting of all stability preservers and real stability preservers, respectively.

Theorem 1.2. *Let $T \in \mathcal{A}_n[\mathbb{C}]$. Then*

$$T \in \mathcal{A}_n(\mathbb{C}) \Leftrightarrow F_T(z, -w) \in \mathcal{H}_{2n}(\mathbb{C}).$$

Theorem 1.3. *Let $T \in \mathcal{A}_n[\mathbb{R}]$. Then*

$$T \in \mathcal{A}_n(\mathbb{R}) \Leftrightarrow F_T(z, -w) \in \mathcal{H}_{2n}(\mathbb{R}).$$

It is interesting to note that Theorems 1.2–1.3 essentially assert that *finite order stability (respectively, real stability) preservers in n variables are generated by stable (respectively, real stable) polynomials in $2n$ variables via the symbol map.* Geometric interpretations of these statements in terms of symbol "curves" are given in §4.6.

To prove the above theorems we need to generalize a large number of notions and results for univariate stable and hyperbolic polynomials to the multivariate case.

Let $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ and $\beta_1 \le \beta_2 \le \cdots \le \beta_m$ be the zeros (counted with multiplicities) of two given polynomials $f, g \in \mathcal{H}_1(\mathbb{R})$ with $\deg f = n$, $\deg g = m$. We say that these zeros *interlace* if they can be ordered so that either $\alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \cdots$ or $\beta_1 \le \alpha_1 \le \beta_2 \le \alpha_2 \le \cdots$, in which case one clearly must have $|n-m| \le 1$. Note that by our convention, the zeros of any two polynomials of degree 0 or 1 interlace. It is not difficult to show that if the zeros of f and g interlace then the *Wronskian* $W[f, g] := f'g - fg'$ is either nonnegative or nonpositive on the whole real axis $\mathbb{R}$, see, e.g., [44]. In the case when $W[f, g] \le 0$ we say that f and g are in *proper position*, denoted $f \ll g$.

For technical reasons we also say that the zeros of the polynomial 0 interlace the zeros of any (nonzero) hyperbolic polynomial and write $0 \ll f$ and $f \ll 0$. Note that if f, g are (nonzero) hyperbolic polynomials such that $f \ll g$ and $g \ll f$ then f and g must be constant multiples of each other, that is, $W[f, g] \equiv 0$.

The following theorem is a version of the classical Hermite-Biehler theorem [44].

Theorem 1.4 (Hermite-Biehler theorem). *Let* $h := f + ig \in \mathbb{C}[z]$, *where* $f, g \in \mathbb{R}[z]$. *Then* $h \in \mathcal{H}_1(\mathbb{C})$ *if and only if* $g \ll f$.

The Hermite-Biehler theorem gives an indication about how one should generalize the concept of interlacing to higher dimensions:

Definition 1.2. Two polynomials $f, g \in \mathbb{R}[z_1, \dots, z_n]$ are in *proper position*, denoted $f \ll g$, if $g + if \in \mathcal{H}_n(\mathbb{C})$.

Equivalently, f and g are in proper position if and only if for all $\alpha \in \mathbb{R}^n$ and $v \in \mathbb{R}^n_+$ the univariate polynomials $f(\alpha + vt), g(\alpha + vt) \in \mathbb{R}[t]$ are in proper position. It also follows that $f, g \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ whenever $f \ll g$, see Corollary 2.4 in §2.

The next result is classical and often attributed to Obreschkoff [37].

Theorem 1.5 (Obreschkoff theorem). *Let* $f, g \in \mathbb{R}[z]$. *Then* $\alpha f + \beta g \in \mathcal{H}_1(\mathbb{R}) \cup \{0\}$ *for all* $\alpha, \beta \in \mathbb{R}$ *if and only if either* $f \ll g$, $g \ll f$ *or* $f = g \equiv 0$.

Let us first present a straightforward generalization of Obreschkoff's theorem to the multivariate case.

Theorem 1.6. *Let* $f, g \in \mathbb{R}[z_1, \dots, z_n]$. *Then* $\alpha f + \beta g \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ *for all* $\alpha, \beta \in \mathbb{R}$ *if and only if either* $f \ll g$, $g \ll f$ *or* $f = g \equiv 0$.

Recall that an infinite sequence of real numbers $\lambda : \mathbb{N} \to \mathbb{R}$ is called a *multiplier sequence (of the first kind)* if the associated linear operator T on $\mathbb{C}[z]$ defined by $T(z^n) = \lambda(n)z^n$, $n \in \mathbb{N}$, is a hyperbolicity preserver, i.e., $T : \mathcal{H}_1(\mathbb{R}) \to \mathcal{H}_1(\mathbb{R}) \cup \{0\}$. Any linear operator $T : \mathbb{C}[z_1, \dots, z_n] \to \mathbb{C}[z_1, \dots, z_n]$ can be represented as a formal power series in ∂ with polynomial coefficients. Indeed, this may be proved either by induction (see, e.g., [3] and [8]) or by invoking Peetre's abstract characterization of differential operators [39]. Note also that in general a multiplier sequence is represented by an infinite order differential operator with polynomial coefficients.

In their seminal paper [42] Pólya and Schur gave the following characterization of multiplier sequences of the first kind.

Theorem 1.7 (Pólya-Schur theorem). *Let* $\lambda : \mathbb{N} \to \mathbb{R}$ *be a sequence of real numbers and* $T : \mathbb{R}[z] \to \mathbb{R}[z]$ *be the corresponding (diagonal) linear operator. Define* $\Phi(z)$ *to be the formal power series*

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{\lambda(k)}{k!} z^k.$$

The following assertions are equivalent:

(i) λ *is a multiplier sequence,*

(ii) $\Phi(z)$ *defines an entire function which is the limit, uniformly on compact sets, of polynomials with only real zeros of the same sign,*

(iii) *Either* $\Phi(z)$ *or* $\Phi(-z)$ *is an entire function that can be written as*

$$\Phi(z) = Cz^n e^{az} \prod_{k=1}^{\infty} (1 + \alpha_k z),$$

where $n \in \mathbb{N}$, $C \in \mathbb{R}$, $a, \alpha_k \geq 0$ *for all* $k \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} \alpha_k < \infty$,

(iv) *For all nonnegative integers* n *the polynomial* $T[(1 + z)^n]$ *is hyperbolic with all zeros of the same sign.*

We introduce a natural higher dimensional analog of the notion of multiplier sequence and completely characterize all multivariate multiplier sequences as well as those that can be represented as finite order differential operators. For this we need the following notation. Given an integer $n \geq 1$ and $\alpha, \beta \in \mathbb{N}^n$ we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for $1 \leq i \leq n$. Let further $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$, $\alpha^\beta = \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $(\beta)_\alpha = 0$ if $\alpha \nleq \beta$ and $(\beta)_\alpha = \beta!/(\beta - \alpha)!$ otherwise.

Definition 1.3. A function $\lambda : \mathbb{N}^n \to \mathbb{R}$ is a (multivariate) *multiplier sequence* if the corresponding (diagonal) linear operator T defined by $T(z^\alpha) = \lambda(\alpha) z^\alpha$, $\alpha \in \mathbb{N}^n$, is a real stability preserver, that is, $T : \mathcal{H}_n(\mathbb{R}) \to \mathcal{H}_n(\mathbb{R}) \cup \{0\}$.

The following theorem completely describes multivariate multiplier sequences.

Theorem 1.8. *Consider an arbitrary map $\lambda : \mathbb{N}^n \to \mathbb{R}$. Then λ is a multivariate multiplier sequence if and only if there exist usual (univariate) multiplier sequences $\lambda_i : \mathbb{N} \to \mathbb{R}$, $1 \leq i \leq n$, such that*

$$\lambda(\alpha) = \lambda_1(\alpha_1)\lambda_2(\alpha_2) \cdots \lambda_n(\alpha_n), \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n,$$

and either all nonzero $\lambda(\alpha)$, $\alpha \in \mathbb{N}^n$, have the same sign or all nonzero $(-1)^{|\alpha|}\lambda(\alpha)$, $\alpha \in \mathbb{N}^n$, have the same sign.

We next characterize all multiplier sequences that are finite order differential operators, i.e., those whose symbols are (finite degree) polynomials:

Theorem 1.9. *Given a map $\lambda : \mathbb{N}^n \to \mathbb{R}$ let T be the corresponding (diagonal) linear operator. Then $T \in \mathcal{A}_n(\mathbb{R})$ if and only if T has a symbol $F_T(z, w)$ of the form*

$$F_T(z, w) = f_1(z_1 w_1) f_2(z_2 w_2) \cdots f_n(z_n w_n),$$

where $f_i(t)$, $1 \leq i \leq n$, are polynomials with all real and nonpositive zeros.

Remark 1.1. Note that Theorem 1.9 combined with well-known properties of univariate multiplier sequences (cf. Lemma 3.1 below) implies in particular that if $\lambda : \mathbb{N}^n \to \mathbb{R}$ is a finite order multivariate multiplier sequence then there exists $\gamma \in \mathbb{N}^n$ such that $\lambda(\alpha) = 0$ for $\alpha < \gamma$ and either $\lambda(\alpha) > 0$ for all $\alpha \geq \gamma$ or $\lambda(\alpha) < 0$ for all $\alpha \geq \gamma$. Note also that for $n = 1$ Theorem 1.9 gives an alternative description of finite order multiplier sequences that complements Pólya-Schur's Theorem 1.7.

Our next result is a vast generalization of the following classical theorem [44].

Theorem 1.10 (Hermite-Poulain-Jensen theorem)**.** *Let $p(z) = \sum_{k=0}^n a_k z^k \in \mathbb{R}[z] \setminus \{0\}$ and $T = \sum_{k=0}^n a_k \frac{d^k}{dz^k} \in \mathcal{A}_1[\mathbb{R}]$. Then $T \in \mathcal{A}_1(\mathbb{R})$ if and only if $p \in \mathcal{H}_1(\mathbb{R})$.*

The natural setting for our generalization is the *Fischer-Fock* (or *Bargmann-Segal*) *space* $\mathcal{F}_n$, which is the Hilbert space of holomorphic functions f on $\mathbb{C}^n$ such that

$$\|f\|^2 = \sum_{\alpha \in \mathbb{N}^n} \alpha! |a(\alpha)|^2 = \pi^{-n} \int |f(z)|^2 e^{-|z|^2} dz_1 \wedge \cdots \wedge dz_n < \infty.$$

Here $\sum a(\alpha) z^\alpha$ is the Taylor expansion of f. The inner product in $\mathcal{F}_n$ is given by

$$\langle f, g \rangle = \pi^{-n} \int f(z) \overline{g(z)} e^{-|z|^2} dz_1 \wedge \cdots \wedge dz_n \tag{1.2}$$

and one can easily check that monomials $\{z^\alpha/\sqrt{\alpha!}\}_{\alpha \in \mathbb{N}^n}$ form an orthonormal basis. From this it follows that for $1 \leq i \leq n$ one has

$$\langle \partial_i z^\alpha, z^\beta \rangle = \alpha! \delta_{\alpha = \beta + e_i} = \langle z^\alpha, z_i z^\beta \rangle,$$

where δ is the Kronecker delta and e_i denotes the i-th standard generator of the lattice $\mathbb{Z}^n$. Hence, if $T = \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha \partial^\beta \in \mathcal{A}_n[\mathbb{C}]$ then

$$\langle T(f), g\rangle = \sum_{\alpha,\beta} a_{\alpha\beta} \langle z^\alpha \partial^\beta f, g\rangle = \sum_{\alpha,\beta} a_{\alpha\beta} \langle f, z^\beta \partial^\alpha g\rangle = \langle f, \sum_{\alpha,\beta} \overline{a_{\beta\alpha}} z^\alpha \partial^\beta g\rangle.$$

Therefore, the (formal) Fischer-Fock dual (or adjoint) operator of T is given by $T^* = \sum_{\alpha,\beta} \overline{a_{\beta\alpha}} z^\alpha \partial^\beta$. Note that for $1 \le i \le n$ the dual of ∂_i is the operator given by multiplication with z_i and that diagonal operators (in the standard monomial basis) are self-dual. In particular, if T is a multiplier sequence then $T^* = T$.

Remark 1.2. As is well known, $\mathcal{F}_n$ is a reproducing kernel space. For this and further interesting properties of the Fischer-Fock space we refer to [4] and [34]. We should also point out that in $\mathcal{D}$-module theory and (microlocal) Fourier analysis one usually works with the inner product on $\mathcal{F}_n$ defined by $\langle f(z), g(z)\rangle_d = \langle f(iz), g(iz)\rangle$, the latter product being as in (1.2). Note that the dual operator of ∂_i with respect to the product $\langle \cdot, \cdot \rangle_d$ is the operator given by multiplication with $-z_i$.

In §4.6 we give a geometric interpretation (and proof) of the fact that the duality map with respect to the above scalar product preserves both $\mathcal{A}_n(\mathbb{C})$ and $\mathcal{A}_n(\mathbb{R})$. More precisely, from Theorems 1.2–1.3 we deduce the following natural property:

Theorem 1.11 ("Duality theorem"). *Let $T \in \mathcal{A}_n[\mathbb{C}]$. Then $T \in \mathcal{A}_n(\mathbb{C})$ if and only if $T^* \in \mathcal{A}_n(\mathbb{C})$. Similarly, if $T \in \mathcal{A}_n[\mathbb{R}]$ then $T \in \mathcal{A}_n(\mathbb{R})$ if and only if $T^* \in \mathcal{A}_n(\mathbb{R})$.*

We conclude this introduction with a series of examples of real stable polynomials and various applications of our results. Further interesting examples of multi-affine stable and real stable polynomials can be found in e.g. [10] and [13].

Proposition 1.12. *Let A_i, $1 \le i \le n$, be positive semidefinite $m \times m$ matrices and B be a complex Hermitian $m \times m$ matrix. Then the polynomial*

$$f(z_1, \dots, z_n) = \det\left(\sum_{i=1}^n z_i A_i + B\right) \tag{1.3}$$

is either real stable or identically zero.

A proof of Proposition 1.12 is given in §6. Using this result and Theorem 1.2 we obtain a multidimensional generalization of the Cauchy-Poincaré interlacing theorem, see Theorem 6.2 in §6.1.

The Lax conjecture for (Gårding) hyperbolic polynomials in three variables has recently been settled by Lewis, Parillo and Ramana [30]. Their proof relies on the results of Helton and Vinnikov [25]. In §6.2 we prove the following converse to Proposition 1.12 in the case $n = 2$ and thus establish a natural analog of the Lax conjecture for real stable polynomials in two variables.

Theorem 1.13. *Any real stable polynomial in two variables x, y can be written as $\pm \det(xA + yB + C)$ where A and B are positive semidefinite matrices and C is a symmetric matrix of the same order.*

Remark 1.3. A characterization of real stable polynomials in an arbitrary number of variables has recently been obtained in [10], see Theorem 7.1 in §7.

Combining Theorem 1.13 with Theorem 1.3 we get two new descriptions of finite order linear preservers of hyperbolicity (i.e., univariate real stability), namely a determinantal characterization and one in terms of homogenized operator symbols, see Theorems 6.8 and 6.9 in §6.3.

Further applications of our results include multivariate Schur-Maló-Szegö composition formulas and closure properties under the Weyl product of (real) stable polynomials (§4.5), a unified treatment of Pólya type "curve theorems" as well as multivariate extensions (§4.6), and necessary and sufficient criteria for strict stability and strict real stability preservers (§5). Finally, in §7 we discuss a number of related results and open problems.

Brief excursion around the literature. The study of univariate stable polynomials was initiated by Hermite in the 1860's and continued by Laguerre, Maxwell, Routh, Hurwitz and many others in the second half of the XIX-th century. The contributions of the classical period are well summarized in [19, 43]. Important results on stability of entire functions were obtained in the mid XX-th century by e.g. Krein, Pontryagin, Chebotarev, Levin [29]. Modern achievements in this area can be found in [38] and references therein. Much less seems to be known concerning multidimensional stability. In control theory one can name a series of papers by Kharitonov *et al* [26] with numerous references to the earlier literature on this topic. Another origin of interest to multivariate stable polynomials comes from an unexpected direction, namely the Lee-Yang theorem on ferromagnetic Ising models and its generalizations by Heilmann, Lieb and Sokal, see [24, 28, 31]. Combinatorial theory provides yet another rich source of stable polynomials as multivariate spanning tree polynomials and generating polynomials for various classes of matroids turn out to be stable (cf., e.g., [10, 13, 47]). Multivariate stable polynomials were recently used in [22, 23] to generalize and reprove in a unified manner a number of classical conjectures, including the van der Waerden and Schrijver-Valiant conjectures, and in [9] to solve some long-standing conjectures of Johnson and Bapat in matrix theory. Further recent contributions include [8], where a complete classification of linear preservers of univariate polynomials with all real zeros – and, more generally, of univariate polynomials with all zeros in a closed circular domain or on the boundary of such a domain – has been obtained, thus solving an old open problem going back to Laguerre [27] and Pólya-Schur [42] (see also the discussion in §7). To close this brief survey let us mention that real stable polynomials have also found remarkable applications in probability theory and interacting particle systems. Indeed, these polynomials were recently used in [7] to develop a theory of negative dependence for the class of strongly Rayleigh probability measures, which contains several important examples such as uniform random spanning tree measures, determinantal measures (for contractions), balls-and-bins measures and distributions for symmetric exclusion processes.

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2. BASIC PROPERTIES AND GENERALIZED OBRESCHKOFF SATZ

The following criterion for (real) stability is an easy consequence of the definitions and the fact that if $p(t) \in \mathbb{R}[t]$ then $p(t) \in \mathcal{H}_1(\mathbb{R}) \Leftrightarrow p(-t) \in \mathcal{H}_1(\mathbb{R})$.

Lemma 2.1. *Let* $f \in \mathbb{C}[z_1, \dots, z_n]$*. Then* $f \in \mathcal{H}_n(\mathbb{C})$ *if and only if* $f(\alpha + vt) \in \mathcal{H}_1(\mathbb{C})$ *for all* $\alpha \in \mathbb{R}^n$ *and* $v \in \mathbb{R}^n_+$*. Moreover, if* $f \in \mathbb{R}[z_1, \dots, z_n]$ *then* $f \in \mathcal{H}_n(\mathbb{R})$ *if and only if* $f(\alpha + vt) \in \mathcal{H}_1(\mathbb{R})$ *for all* $\alpha \in \mathbb{R}^n$ *and* $\pm v \in \mathbb{R}^n_+$*.*

The next lemma extends the Hermite-Biehler theorem to the multivariate case and provides a useful alternative description of the proper position/"interlacing" property for multivariate polynomials.

Lemma 2.2. *Let* $f, g \in \mathbb{R}[z_1, \dots, z_n]$ *and* z_{n+1} *be a new indeterminate. Then* $f \ll g$ *if and only if* $g + z_{n+1} f \in \mathcal{H}_{n+1}(\mathbb{R})$*. Moreover, if* $f \in \mathcal{H}_n(\mathbb{R})$ *then* $f \ll g$ *if and only if*

$$\mathfrak{Im}\left(\frac{g(z)}{f(z)}\right) \geq 0$$

whenever $\mathfrak{Im}(z) > 0$*.*

Proof. The "if" direction is obvious. Suppose that $f \ll g$ and that $z_{n+1} = a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}_+$. Then by Lemma 2.1 we have that $f(\alpha+vt) \ll g(\alpha+vt)$ for all $\alpha \in \mathbb{R}^n$ and $v \in \mathbb{R}^n_+$. By Obreschkoff's theorem the zeros of $g(\alpha+vt)+af(\alpha+vt)$ and $bf(\alpha + vt)$ interlace (both cannot be identically zero). Moreover,

$$W(bf(\alpha + vt), g(\alpha + vt) + af(\alpha + vt)) = bW(f(\alpha + vt)), g(\alpha + vt)).$$

Thus $bf(\alpha + vt) \ll g(\alpha + vt) + af(\alpha + vt)$ for all $\alpha \in \mathbb{R}^n$ and $v \in \mathbb{R}^n_+$, which by Lemma 2.1 gives $g + (a + ib)f \in \mathcal{H}_n(\mathbb{C})$. But $g + z_{n+1}f$ clearly has real coefficients so $g + z_{n+1}f \in \mathcal{H}_{n+1}(\mathbb{R})$. The final statement of the lemma is a simple consequence of the above arguments. □

Lemma 2.3. *Suppose that* $f_j \in \mathbb{C}[z_1, \dots, z_n]$*,* $j \in \mathbb{N}$*, are nonvanishing in an open set* $U \subseteq \mathbb{C}^n$ *and that* f *is the limit, uniformly on compact sets, of the sequence* $\{f_j\}_{j\in\mathbb{N}}$*. Then* f *is either nonvanishing in* U *or it is identically equal to* 0*.*

Proof. The lemma follows from the multivariate version of Hurwitz' theorem on the continuity of zeros of analytic functions, see, e.g., [13] and [9]. □

Let $f(z_1, \dots, z_n) \in \mathcal{H}_n(\mathbb{C})$, $\alpha \in \mathbb{R}$ and $\lambda > 0$. Then $f(\alpha + \lambda z_1, \dots, z_n) \in \mathcal{H}_n(\mathbb{C})$. By letting $\lambda \to 0$ we have by Lemma 2.3 that $f(\alpha, z_2, \dots, z_n) \in \mathcal{H}_{n-1}(\mathbb{C}) \cup \{0\}$.

Corollary 2.4. *For each* $n \in \mathbb{N}$ *one has*

$$\mathcal{H}_n(\mathbb{C}) = \{g + if : f, g \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}, f \ll g\}.$$

Proof. The only novel part is that $f, g \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ whenever $f \ll g$. This follows from Lemma 2.2 and Lemma 2.3 when we let z_{n+1} tend to 0 and ∞, respectively. □

We are ready to prove our multivariate Obreschkoff theorem.

Proof of Theorem 1.6. Suppose that $f \ll g$. By Corollary 2.4 we have $g \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ so we can normalize and set $\beta = 1$. By Lemma 2.2 we have $g + z_{n+1}f \in \mathcal{H}_{n+1}(\mathbb{R}) \subset \mathcal{H}_{n+1}(\mathbb{C})$, so by letting $z_{n+1} = i + \alpha$ with $\alpha \in \mathbb{R}$ we have $g + \alpha f + if \in \mathcal{H}_n(\mathbb{C})$, i.e., $f \ll g + \alpha f$. From Corollary 2.4 again it follows that $g + \alpha f \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$, as was to be shown.

To prove the converse statement suppose that we do not have $f = g \equiv 0$. If $\alpha f + \beta g \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ for all $\alpha, \beta \in \mathbb{R}$ then by Lemma 2.1 and Obreschkoff's theorem for all $\gamma \in \mathbb{R}^n$ and $v \in \mathbb{R}^n_+$ we have either $f(\gamma + vt) \ll g(\gamma + vt)$ or $f(\gamma+vt) \gg g(\gamma+vt)$. If both instances occur for different vectors, i.e., $f(\gamma_1+v_1t) \ll g(\gamma_1+v_1t)$ and $f(\gamma_2+v_2t) \gg g(\gamma_2+v_2t)$ for some $\gamma_1, \gamma_2 \in \mathbb{R}^n$ and $v_1, v_2 \in \mathbb{R}^n_+$, then by continuity arguments there exists $\tau \in [0,1]$ such that $f(\gamma_\tau + v_\tau t) \ll g(\gamma_\tau + v_\tau t)$ and $f(\gamma_\tau + v_\tau t) \gg g(\gamma_\tau + v_\tau t)$, where $\gamma_\tau := \tau\gamma_1 + (1-\tau)\gamma_2 \in \mathbb{R}^n$ and $v_\tau := \tau v_1 + (1-\tau)v_2 \in \mathbb{R}^n_+$. This means that $f(\gamma_\tau + v_\tau t)$ and $g(\gamma_\tau + v_\tau t)$ are constant multiples of each other, say $f(\gamma_\tau + v_\tau t) = \lambda g(\gamma_\tau + v_\tau t)$ for some $\lambda \in \mathbb{R}$. By hypothesis we have $h := f - \lambda g \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ and $h(\gamma_\tau + v_\tau t) \equiv 0$, in particular $h(\gamma_\tau + iv_\tau) = 0$. Since $v_\tau \in \mathbb{R}^n_+$ it follows that $h \equiv 0$ and $f = \lambda g$. Consequently, if both instances occur we have $f \ll g$ for trivial reasons. Thus we may assume that only one of them occurs. But then the conclusion follows from Lemma 2.1. □

Define

$$\mathcal{H}_n(\mathbb{C})^- = \{f \in \mathbb{C}[z_1, \dots, z_n] : f(z_1, , \dots, z_n) \neq 0 \text{ if } \mathfrak{Im}(z_i) < 0 \text{ for } 1 \le i \le n\}.$$

Clearly, $f(z) \in \mathcal{H}_n(\mathbb{C}) \Leftrightarrow f(-z) \in \mathcal{H}_n(\mathbb{C})^-$. Hence by Corollary 2.4 and Lemma 2.1 we have $f := h + ig \in \mathcal{H}_n(\mathbb{C})^-$ with $h, g \in \mathbb{R}[z_1, \dots, z_n]$ if and only if $h \ll g$.

Proposition 2.5. *For any $n \in \mathbb{N}$ the following holds:*

$$\mathcal{H}_n(\mathbb{C}) \cap \mathcal{H}_n(\mathbb{C})^- = \mathbb{C}\mathcal{H}_n(\mathbb{R}) := \{cf : c \in \mathbb{C}, f \in \mathcal{H}_n(\mathbb{R})\}.$$

Proof. Suppose that $h = g+if \in \mathcal{H}_n(\mathbb{C}) \cap \mathcal{H}_n(\mathbb{C})^-$. By Corollary 2.4 we have $f \ll g$ and $g \ll f$. Hence for all $\alpha \in \mathbb{R}^n$ and $v \in \mathbb{R}^n_+$ we also have $f(\alpha + vt) \ll g(\alpha + vt)$ and $g(\alpha + vt) \ll f(\alpha + vt)$. This means that $f(\alpha + vt)$ and $g(\alpha + vt)$ are constant multiples of each other, say $f(\alpha+vt) = \lambda g(\alpha+vt)$. By the multivariate Obreschkoff theorem we have that $f - \lambda g \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$. Since $(f - \lambda g)(\alpha + vi) = 0$ we must have $f - \lambda g \equiv 0$, i.e., $h = (1 + i\lambda)g \in \mathbb{C}\mathcal{H}_n(\mathbb{R})$. □

Lemma 2.6. *Let $f \in \mathcal{H}_n(\mathbb{R})$. Then the sets*

$$\{g \in \mathcal{H}_n(\mathbb{R}) : f \ll g\} \text{ and } \{g \in \mathcal{H}_n(\mathbb{R}) : f \gg g\}$$

are nonnegative cones, i.e., they are closed under nonnegative linear combinations.

Proof. Let $f \in \mathcal{H}_n(\mathbb{R})$ and suppose that $f \ll g$ and $f \ll h$. Then by Lemma 2.2 we have that $\mathfrak{Im}(g(z)/f(z)) \ge 0$ and $\mathfrak{Im}(h(z)/f(z)) \ge 0$ whenever $\mathfrak{Im}(z) > 0$. Hence if $\lambda, \mu \ge 0$ we have $\mathfrak{Im}\left((\lambda g(z) + \mu h(z))/f(z)\right) \ge 0$ whenever $\mathfrak{Im}(z) > 0$. By Lemma 2.2 again we have $f \ll \lambda g + \mu h$. The other assertion follows similarly. □

3. Classifications of multivariate multiplier sequences and finite order ones

3.1. **Univariate and multivariate multiplier sequences.** Let us first recall a few well-known properties of (usual) univariate multiplier sequences, see, e.g., [14].

Lemma 3.1. *Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a multiplier sequence. If $0 \le i \le j \le k$ are such that $\lambda(i)\lambda(k) \neq 0$ then $\lambda(j) \neq 0$. Furthermore, either*

- (i) *all nonzero $\lambda(i)$ have the same sign, or*
- (ii) *all nonzero entries of the sequence $\{(-1)^i\lambda(i)\}_{i\ge0}$ have the same sign.*

In what follows we denote the standard basis in $\mathbb{R}^n$ by $\{e_k\}_{k\in\mathbb{N}}$.

Remark 3.1. Suppose that $\alpha \in \mathbb{N}^n$, $1 \leq k \leq n$, $f(z_k) := \sum_{i=0}^{N} a_i z_k^i \in \mathcal{H}_1(\mathbb{R})$ and assume that λ is a multivariate multiplier sequence. Then

$$T(z^\alpha f(z_k)) = z^\alpha \sum_{i=0}^{N} \lambda(\alpha + ie_k) a_i z_k^i \in \mathcal{H}_n(\mathbb{R}).$$

Hence $\mathbb{N} \ni i \mapsto \lambda(\alpha + ie_k)$ is a usual (univariate) multiplier sequence.

The proofs of our characterizations of multivariate multiplier sequences and those of finite order build on a series of statements that we proceed to describe.

Lemma 3.2. *Let* $f(z_1, z_2) = a_{00} + a_{01}z_2 + a_{10}z_1 + a_{11}z_1z_2 \in \mathbb{R}[z_1, z_2] \setminus \{0\}$. *Then* $f \in \mathcal{H}_2(\mathbb{R})$ *if and only if* $\det(a_{ij}) \leq 0$.

Proof. Let $\alpha \in \mathbb{R}$ and denote by $A = (a_{ij})$ the matrix of coefficients of $f(z_1, z_2)$. Clearly, $f(z_1, z_2) \in \mathcal{H}_2(\mathbb{R})$ if and only if $f(z_1 + \alpha, z_2) \in \mathcal{H}_2(\mathbb{R})$ and $f(z_1, z_2 + \alpha) \in \mathcal{H}_2(\mathbb{R})$. We get the matrix corresponding to $f(z_1+\alpha, z_2)$ by adding α times the last row of A to the first row of A, and we get the matrix corresponding to $f(z_1, z_2+\alpha)$ by adding α times the last column of A to the first column of A. Since the determinant is preserved under such row and column operations we can assume that A has one of the following forms:

$$\begin{pmatrix} a_{00} & 0 \\ 0 & a_{11} \end{pmatrix}, \quad \begin{pmatrix} 0 & a_{01} \\ a_{10} & 0 \end{pmatrix}, \quad \begin{pmatrix} a_{00} & a_{01} \\ 0 & 0 \end{pmatrix}.$$

Obviously, these matrices correspond to a polynomial $f(z_1, z_2) \in \mathcal{H}_2(\mathbb{R})$ if and only if $\det(a_{ij}) \leq 0$. ☐

Lemma 3.3. *Let* $\lambda : \mathbb{N}^n \to \mathbb{R}$, $n \geq 2$, *be a multivariate multiplier sequence and let* $\gamma \in \mathbb{N}^n$ *and* $1 \leq i < j \leq n$. *Then*

$$\lambda(\gamma)\lambda(\gamma + e_i + e_j) = \lambda(\gamma + e_i)\lambda(\gamma + e_j).$$

Proof. Without loss of generality we may assume that $i = j - 1 = 1$. Let $f(z) = z^\gamma g(z) \in \mathcal{H}_n(\mathbb{R})$, where $g(z_1, z_2) = a_{00} + a_{01}z_2 + a_{10}z_1 + a_{11}z_1z_2 \in \mathcal{H}_2(\mathbb{R})$. It follows that

$$\lambda(\gamma)a_{00} + \lambda(\gamma + e_2)a_{01}z_2 + \lambda(\gamma + e_1)a_{10}z_1 + \lambda(\gamma + e_1 + e_2)a_{11}z_1z_2 \in \mathcal{H}_2(\mathbb{R}) \cup \{0\}.$$

By choosing A as

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},$$

respectively, we get by Lemma 3.2 that $\lambda(\gamma)\lambda(\gamma+e_1+e_2) \leq \lambda(\gamma+e_1)\lambda(\gamma+e_2)$ and $\lambda(\gamma)\lambda(\gamma + e_1 + e_2) \geq \lambda(\gamma + e_1)\lambda(\gamma + e_2)$, respectively, which proves the lemma. ☐

Given $\alpha, \beta \in \mathbb{N}^n$ with $\alpha \leq \beta$ set $[\alpha, \beta] := \{\gamma \in \mathbb{N}^n : \alpha \leq \gamma \leq \beta\}$.

Lemma 3.4. *Let* $\lambda : \mathbb{N}^n \to \mathbb{R}$ *be a multivariate multiplier sequence. If* $\alpha, \beta \in \mathbb{N}^n$ *are such that* $\lambda(\alpha)\lambda(\beta) \neq 0$ *and* $\gamma \in [\alpha, \beta]$ *then* $\lambda(\gamma) \neq 0$.

Proof. We use induction on $\ell = |\beta| - |\alpha|$, the length of the interval $[\alpha, \beta]$. The cases $\ell = 0$ and $\ell = 1$ are clear. By Remark 3.1 and Lemma 3.1 the result is true in the univariate case. So we may assume that α and β differ in more than one coordinate, i.e., that $\alpha + e_1, \alpha + e_2 \in [\alpha, \beta]$.

If there exists an atom $\alpha+e_i \in [\alpha,\beta]$ such that $\lambda(\alpha+e_i)\neq 0$ then by induction we have that λ is nonzero in $[\alpha+e_i,\beta]$. If $\alpha+e_j$ is another atom then $\alpha+e_i+e_j \in [\alpha+e_i,\beta]$ so $\lambda(\alpha+e_i+e_j)\neq 0$. Lemma 3.3 then gives that $\lambda(\alpha+e_j)\neq 0$. Thus, by induction, λ is nonzero in $[\alpha+e_j,\beta]$ for all $\alpha+e_j\in[\alpha,\beta]$ and we are done.

In order to get a contradiction we may assume by the above that $\lambda(\alpha+e_i)=0$ for all $\alpha+e_i\in[\alpha,\beta]$. Let $\gamma\in(\alpha,\beta]$ be a minimal element such that $\lambda(\gamma)\neq 0$. If T is the (diagonal) linear operator associated to λ then

$$T(z^\alpha(1+z)^{\gamma-\alpha}) = \lambda(\alpha)z^\alpha + \lambda(\gamma)z^\gamma \in \mathcal{H}_n(\mathbb{R}).$$

By Lemma 3.3 we have that $\lambda(\alpha+e_i+e_j)=0$ for all atoms $\alpha+e_i,\alpha+e_j\in[\alpha,\beta]$. By Remark 3.1 and Lemma 3.1 we also have $\lambda(\alpha+me_i)=0$ for all

$m\geq 1$ and atoms $\alpha+e_i\in[\alpha,\beta]$. It follows that $|\gamma|-|\alpha|\geq 3$. Now, if we set $z_i=t$ for all i then by the above we obtain that the polynomial

$$\lambda(\alpha)t^{|\alpha|}+\lambda(\gamma)t^{|\gamma|}$$

is hyperbolic in t, which is a contradiction since $|\gamma|-|\alpha|\geq 3$ and $\lambda(\alpha)\lambda(\gamma)\neq 0$. □

Lemma 3.5. *Let $\lambda:\mathbb{N}^n\to\mathbb{R}$ be a multivariate multiplier sequence and suppose that $\lambda(\alpha)\neq 0$. Then*

$$\lambda(\beta)=\frac{\lambda(\alpha+(\beta_1-\alpha_1)e_1)\lambda(\alpha+(\beta_2-\alpha_2)e_2)\cdots\lambda(\alpha+(\beta_n-\alpha_n)e_n)}{\lambda(\alpha)^{n-1}}$$

for all $\beta\geq\alpha$.

Proof. Let $\tilde{\lambda}(\beta)$ be the expression in the right-hand side of the identity stated above. The proof that $\tilde{\lambda}(\gamma)=\lambda(\gamma)$ for all $\gamma\geq\alpha$ is again by induction on $\ell=|\gamma|-|\alpha|$. One easily checks that $\lambda(\alpha+me_i)=\tilde{\lambda}(\alpha+me_i)$ for all $m\geq 0$. Hence we may assume that $\gamma=\beta+e_i+e_j$ and that the proposed formula holds for $\beta+e_i$, $\beta+e_j$ and β. If $\lambda(\beta)=0$ then by Lemma 3.4 we have $\lambda(\beta+e_i+e_j)=0$. Since also $\tilde{\lambda}(\beta)=0$ and since the components of $\tilde{\lambda}$ are univariate multiplier sequences we have by Lemma 3.1 that $\tilde{\lambda}(\beta+e_i+e_j)=0$, as was to be shown. If $\lambda(\beta)\neq 0$ we have by Lemma 3.3 and the induction hypothesis that

$$\lambda(\beta+e_i+e_j)=\frac{\lambda(\beta+e_i)\lambda(\beta+e_j)}{\lambda(\beta)}=\frac{\tilde{\lambda}(\beta+e_i)\tilde{\lambda}(\beta+e_j)}{\tilde{\lambda}(\beta)}=\tilde{\lambda}(\beta+e_i+e_j)$$

and then by iteration we get $\lambda(\gamma)=\tilde{\lambda}(\gamma)$ for all $\gamma\geq\alpha$. □

If $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ and $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{N}^n$ we define two new vectors $\alpha\vee\beta,\alpha\wedge\beta\in\mathbb{N}^n$ by setting $\alpha\vee\beta=(\max(\alpha_1,\beta_1),\dots,\max(\alpha_n,\beta_n))$ and $\alpha\wedge\beta=(\min(\alpha_1,\beta_1),\dots,\min(\alpha_n,\beta_n))$.

Lemma 3.6. *Let $\lambda:\mathbb{N}^n\to\mathbb{R}$ be a multivariate multiplier sequence and suppose that $\lambda(\alpha)\lambda(\beta)\neq 0$. Then $\lambda(\alpha\vee\beta)\neq 0$ if and only if $\lambda(\alpha\wedge\beta)\neq 0$.*

Proof. If $\lambda(\alpha)\lambda(\beta)\neq 0$ and $\lambda(\alpha\wedge\beta)\neq 0$ then Lemma 3.5 and Lemma 3.1 imply that $\lambda(\alpha\vee\beta)\neq 0$. Suppose that $\lambda(\alpha)\lambda(\beta)\neq 0$ and $\lambda(\alpha\vee\beta)\neq 0$. We prove that $\lambda(\alpha\wedge\beta)\neq 0$ by induction on $|\alpha-\beta|$. If $|\alpha-\beta|=0$ there is nothing to prove. Also, if α and β are comparable there is nothing to prove, so we may assume that there are indices i and j such that $\alpha_i<\beta_i$ and $\beta_j<\alpha_j$. Since $\alpha<\alpha+e_i\leq\alpha\vee\beta$ and $\beta<\beta+e_j\leq\alpha\vee\beta$ we have by Lemma 3.4 that $\lambda(\alpha+e_i)\lambda(\beta+e_j)\neq 0$. Consider the pairs $(\alpha+e_i,\beta+e_j)$, $(\alpha+e_i,\beta)$ and $(\alpha,\beta+e_j)$. The distance between each of them is smaller than $|\alpha-\beta|$, they all have to join $\alpha\vee\beta$, and the meets are

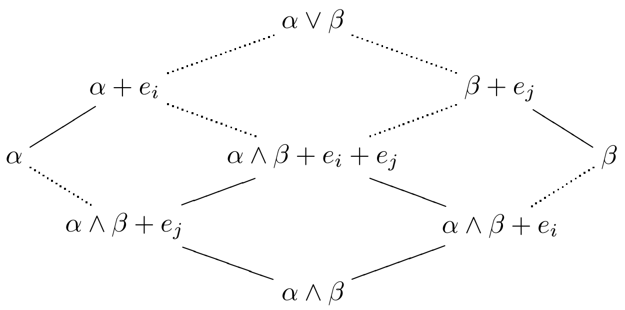

FIGURE 1. Illustration of the induction step in Lemma 3.6.

$\alpha \wedge \beta + e_i + e_j$, $\alpha \wedge \beta + e_j$ and $\alpha \wedge \beta + e_i$ respectively, see Fig. 1. By induction we have that $\lambda(\alpha\wedge\beta+e_i)\lambda(\alpha\wedge\beta+e_j)\lambda(\alpha\wedge\beta+e_i+e_j) \neq 0$. By Lemma 3.3 this gives

$$\lambda(\alpha \wedge \beta) = \frac{\lambda(\alpha \wedge \beta + e_i)\lambda(\alpha \wedge \beta + e_j)}{\lambda(\alpha \wedge \beta + e_i + e_j)} \neq 0,$$

which is the desired conclusion. $\square$

Recall that the *support* of a map $\lambda : \mathbb{N}^n \to \mathbb{R}$ is the set $\{\alpha \in \mathbb{N}^n : \lambda(\alpha) \neq 0\}$.

Lemma 3.7. *Let $\lambda : \mathbb{N}^n \to \mathbb{R}$ be a multivariate multiplier sequence. Then there exist univariate multiplier sequences $\lambda_i : \mathbb{N} \to \mathbb{R}$, $1 \leq i \leq n$, such that*

$$\lambda(\alpha) = \lambda_1(\alpha_1)\lambda_2(\alpha_2)\cdots\lambda_n(\alpha_n), \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n.$$

Proof. By Lemma 3.5 and Remark 3.1 it suffices to prove that the support of λ denoted by S has a unique minimal element. So far, by Lemma 3.4 and Lemma 3.6 we know that S is a disjoint union $S = \cup_{i=1}^m B_i$ of boxes $B_i = I_1^i \times \cdots \times I_n^i$, where I_n^i is an interval (possibly infinite) of nonnegative integers. Also, points in different boxes are incomparable.

Suppose that S does not have a unique minimal element. We claim that there exists an interval $[\alpha, \beta]$ such that $[\alpha, \beta] \cap S = \{\delta, \gamma\}$, where δ and γ are in different boxes. We postpone the proof of this statement for a while and show first how it leads to a contradiction. Let T be the (diagonal) linear operator associated to λ. We then have

$$T(z^\alpha(1+z)^{\beta-\alpha}) = \lambda(\delta)z^\delta + \lambda(\gamma)z^\gamma.$$

Now $|\delta - \gamma| \geq 3$ since otherwise δ and γ would be comparable or we would have $\gamma = \delta \wedge \gamma + e_i$ and $\delta = \delta \wedge \gamma + e_j$ for some i and j. This is impossible by Lemma 3.3 since γ and δ would then be in the same box. By assumption we have that

$$\lambda(\delta)z^{\delta-\delta\wedge\gamma} + \lambda(\gamma)z^{\gamma-\delta\wedge\gamma} \in \mathcal{H}_n(\mathbb{R})$$

so by setting all the variables in $z^{\delta-\delta\wedge\gamma}$ equal to t and setting all the variables in $z^{\gamma-\delta\wedge\gamma}$ equal to $-t^{-1}$ (which we may since $z^{\delta-\delta\wedge\gamma}$ and $z^{\gamma-\delta\wedge\gamma}$ contain no common variables) we obtain that

$$\lambda(\delta)t^{|\delta-\gamma|} \pm \lambda(\gamma) \in \mathcal{H}_1(\mathbb{R}).$$

This is a contradiction since $|\delta - \gamma| \geq 3$ and $\delta, \gamma \in S$ so $\lambda(\delta)\lambda(\gamma) \neq 0$.

It remains to prove the claim. Let d be the minimal distance between different boxes and suppose that δ and γ are two points that realize the minimal distance.

If $\kappa \in [\delta \wedge \gamma, \delta \vee \gamma]$ then $|\kappa - \delta| \leq d$ with equality only if $\kappa = \gamma$ and $|\kappa - \gamma| \leq d$ with equality only if $\kappa = \delta$. It follows that $[\delta \wedge \gamma, \delta \vee \gamma] \cap S = \{\delta, \gamma\}$. $\square$

3.2. **Affine differential contractions and multivariate compositions.** For the proof of Theorem 1.8 we need to establish first Theorem 3.11 below, which is the main purpose of this section.

The proof of Theorem 3.11 relies on some known results of Lieb and Sokal [31]. Let us introduce the following notation. Given $a, b \in \mathbb{C}$, $1 \leq i < j \leq n$ and

$$F(z_1, \ldots, z_n) = \sum a_{\alpha_1,\ldots,\alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in \mathbb{C}[z_1, \ldots, z_n]$$

let

$$F\left(z_1, \ldots, z_{i-1}, az_i + b\frac{\partial}{\partial z_j}, z_{i+1}, \ldots, z_j, \ldots, z_n\right) \tag{3.1}$$

denote the polynomial

$$\sum a_{\alpha_1,\ldots,\alpha_n} z_1^{\alpha_1} \cdots z_{i-1}^{\alpha_{i-1}} \left(az_i + b\frac{\partial}{\partial z_j}\right)^{\alpha_i} z_{i+1}^{\alpha_{i+1}} \cdots z_j^{\alpha_j} \cdots z_n^{\alpha_n}.$$

The next lemma follows from [31, Lemma 2.3] by a rotation of the variables.

Lemma 3.8. *If* $P_0(v), P_1(v) \in \mathbb{C}[v]$ *with* $P_0(v) + xP_1(v) \neq 0$ *for* $\mathfrak{Im}(v) \geq c$ *and* $\mathfrak{Im}(x) \geq d$ *then*

$$P_0(v) + \left(x - \frac{\partial}{\partial v}\right) P_1(v) \neq 0$$

for $\mathfrak{Im}(v) \geq c$ *and* $\mathfrak{Im}(x) \geq d$.

Using Lemma 3.8 and the Grace-Walsh-Szegö Coincidence Theorem [44] one can argue as in the proof of [31, Proposition 2.2] to show:

Proposition 3.9 (Lieb-Sokal). *Let* $(c_1, \ldots, c_n) \in \mathbb{R}^n$ *and* $F \in \mathbb{C}[z_1, \ldots, z_n]$ *be such that*

$$F(z_1, \ldots, z_n) \neq 0$$

if $\mathfrak{Im}(z_k) \geq c_k$, $1 \leq k \leq n$. *Then for any* $1 \leq i < j \leq n$ *one has*

$$F\left(z_1, \ldots, z_{i-1}, z_i - \frac{\partial}{\partial z_j}, z_{i+1}, \ldots, z_j, \ldots, z_n\right) \neq 0$$

whenever $\mathfrak{Im}(z_k) \geq c_k$, $1 \leq k \leq n$.

From Proposition 3.9 we immediately get the following.

Corollary 3.10 (Lieb-Sokal). *Suppose that* $1 \leq i < j \leq n$ *and* $F(z_1, \ldots, z_n) \in \mathcal{H}_n(\mathbb{C})$. *Then* $F(z_1, \ldots, z_{i-1}, -\partial_j, z_{i+1}, \ldots, z_j, \ldots, z_n) \in \mathcal{H}_{n-1}(\mathbb{C}) \cup \{0\}$.

We can now prove the following extension of a famous composition theorem of Schur [45] and related results of Maló-Szegö [14, 44] to the multivariate case. Further consequences of Theorem 3.11 will be given in §4.5 and §4.6.

Theorem 3.11. *Assume that all the zeros of* $f(z) = \sum_{i=0}^{r} a_i z^i \in \mathcal{H}_1(\mathbb{R})$ *are non-positive and that* $F(z_1, \ldots, z_n) = \sum_{j=0}^{s} Q_j(z_2, \ldots, z_n) z_1^j \in \mathcal{H}_n(\mathbb{C})$ *and set* $m = \min(r, s)$. *Then*

$$\sum_{k=0}^{m} k! a_k Q_k(z_2, \ldots, z_n) z_1^k \in \mathcal{H}_n(\mathbb{C}) \cup \{0\}.$$

Proof. Suppose that f has all nonpositive zeros. Then $f(-z_0w_0) \in \mathcal{H}_2(\mathbb{R})$ so

$$G(w_0, z_0, \dots, z_n) := f(-z_0w_0)F(z_1, \dots, z_n) \in \mathcal{H}_{n+2}(\mathbb{C}).$$

By Corollary 3.10 we have that

$$H(z_0, z_1, \dots, z_n) := G\Big(-\frac{\partial}{\partial z_1}, z_0, \dots, z_n\Big) \in \mathcal{H}_{n+1}(\mathbb{C}) \cup \{0\}.$$

This means that

$$H(z_1, 0, z_2, \dots, z_n) = \sum_{k=0}^{m} k!a_kQ_k(z_2, \dots, z_n)z_1^m \in \mathcal{H}_n(\mathbb{C}) \cup \{0\},$$

as required. □

3.3. **Proofs of Theorems 1.8–1.9.** We can now settle the classification of multivariate multiplier sequences stated in Theorem 1.8.

Proof of Theorem 1.8. For the "only if" direction what remains to be proven is the statement about the signs. If it were false for some (multivariate) multiplier sequence λ then since λ is a product of univariate multiplier sequences whose entries either all have the same sign or alternate in sign there would exist $\alpha \in \mathbb{N}^n$ such that $\lambda(\alpha) \neq 0$ and $\lambda(\alpha + e_i)\lambda(\alpha + e_j) < 0$. Let T be the corresponding (diagonal) operator and apply it to $z^\alpha(1 - z_iz_j) \in \mathcal{H}_n(\mathbb{R})$. By Lemma 3.3 we get

$$T(z^\alpha(1 - z_iz_j)) = \lambda(\alpha)z^\alpha \left(1 - \frac{\lambda(\alpha + e_i)\lambda(\alpha + e_j)}{\lambda(\alpha)^2} z_iz_j\right).$$

Since $1 + az_1z_2 \in \mathcal{H}_2(\mathbb{R})$ if and only if $a \leq 0$ this is a contradiction.

Now $\alpha \mapsto \lambda(\alpha)$ is a multiplier sequence if and only if $\alpha \mapsto (-1)^{|\alpha|}\lambda(\alpha)$ is a multiplier sequence so we may assume that $\lambda(\alpha) \geq 0$ for all $\alpha \in \mathbb{N}^n$. By applying the λ_i's one at a time we may further assume that $\lambda_i \equiv 1$ for $2 \leq i \leq n$. Hence we have to show that if $f(z_1, \dots, z_n) := \sum_{i=0}^{M} Q_i(z_2, \dots, z_n)z_1^i \in \mathcal{H}_n(\mathbb{R})$ and $\lambda : \mathbb{N} \to \mathbb{R}$ is a nonnegative univariate multiplier sequence then

$$\sum_{i=0}^{M} \lambda(i)Q_i(z_2, \dots, z_n)z_1^i \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}.$$

By Theorem 1.7 there are polynomials

$$p_k(z) = \sum_{i=0}^{N_k} \frac{\lambda_{i,k}}{i!} z^i, \quad k \in \mathbb{N},$$

such that $p_k(z) \in \mathcal{H}_1(\mathbb{R})$ have all nonpositive zeros and

$$\lim_{k\to\infty} p_k(z) = \sum_{i=0}^{\infty} \frac{\lambda(i)}{i!} z^i.$$

Furthermore, by Theorem 3.11 we know that

$$\sum_{i=0}^{M} \lambda_{i,k}Q_i(z_2, \dots, z_n)z_1^i \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$$

for each k. Since $\lim_{k\to\infty} \lambda_{i,k} = \lambda(i)$ we get $\sum_{i=0}^{M} \lambda(i)Q_i(z_2, \dots, z_n)z_1^i \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ by Lemma 2.3, which settles the theorem. □

Let us finally prove the characterization of finite order multiplier sequences.

Proof of Theorem 1.9. The "if" direction is an immediate consequence of Theorem 1.3 whose proof is given in §4 below. To prove the converse statement note first that Theorem 1.8 implies that the symbol of T is the product of the symbols of the corresponding univariate operators. Hence it suffices to settle the case $n = 1$. Let $F(z, w) = \sum_{k=0}^{N} a_k z^k w^k$ be the symbol of T. By Theorem 1.7 we know that all zeros of

$$g_m(z) = T[(1 + z)^m] = \sum_{k=0}^{M} a_k (m)_k z^k (1 + z)^{m-k}$$

are real and have the same sign. Note that these zeros are actually nonpositive since $z = -1$ is a zero of $g_m(z)$ for all large m. Now

$$g_m(z/m) = \Big(1 + \frac{z}{m}\Big)^m \sum_{k=0}^{M} a_k \frac{(m)_k}{m^k} z^k (1 + z/m)^{-k}$$

and since $\lim_{m\to\infty} \frac{(m)_k}{m^k} = 1$ we have

$$\lim_{m\to\infty} g_m(z/m) = e^z \sum_{k=0}^{M} a_k z^k.$$

Hence by Lemma 2.3 the polynomial $\sum_{k=0}^{M} a_k z^k$ has all nonpositive zeros and the theorem follows. $\square$

4. Algebraic and geometric properties of stability preservers

4.1. **Sufficiency in Theorems 1.2–1.3.** Since $\mathcal{H}_n(\mathbb{R}) \subset \mathcal{H}_n(\mathbb{C})$ it is enough to prove only the sufficiency in Theorem 1.2. Recall the "affine differential contraction" of a polynomial $F \in \mathbb{C}[z_1, \dots, z_n]$ defined in (3.1) and note that the following consequence of Corollary 3.10 actually settles the sufficiency part in Theorem 1.2.

Corollary 4.1. *Let $T \in \mathcal{A}_n[\mathbb{C}]$ and suppose that $F_T(z, -w) \in \mathcal{H}_{2n}(\mathbb{C})$. Then $T \in \mathcal{A}_n(\mathbb{C})$.*

Proof. If $F_T(z, -w) \in \mathcal{H}_{2n}(\mathbb{C})$ and $f(v) \in \mathcal{H}_n(\mathbb{C})$ then $F_T(z, -w)f(v) \in \mathcal{H}_{3n}(\mathbb{C})$. By Corollary 3.10 if we exchange the variables w_i's for $-\frac{\partial}{\partial v_i}$'s the resulting polynomial will be in $\mathcal{H}_{2n}(\mathbb{C}) \cup \{0\}$. If we then replace each variable v_i with z_i, $1 \le i \le n$, we get a polynomial in $\mathcal{H}_n(\mathbb{C}) \cup \{0\}$. This polynomial is indeed $T(f)$. $\square$

4.2. **Necessity in Theorems 1.2–1.3.** Let $T = \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha \partial^\beta \in \mathcal{A}_n[\mathbb{R}]$. We may write T as a finite sum $T = \sum_\gamma z^\gamma T_\gamma$, where $T_\gamma = \sum_\beta a_{\gamma+\beta,\beta} z^\beta \partial^\beta$. It follows that T_γ acts on monomials as $T_\gamma(z^\alpha) = \lambda_\gamma(\alpha) z^\alpha$ for some function $\lambda_\gamma : \mathbb{N}^n \to \mathbb{R}$. The following lemma gives a sufficient condition for λ_γ to be a multiplier sequence.

Lemma 4.2. *Let $T = \sum_\gamma z^\gamma T_\gamma \in \mathcal{A}_n(\mathbb{R})$ and denote by $CH(T)$ the convex hull of the set $\{\gamma : T_\gamma \neq 0\}$. If $\kappa \in \mathbb{Z}^n$ is a vertex (face of dimension 0) of $CH(T)$ then T_κ is a multiplier sequence.*

Proof. Let $v \in \mathbb{R}_+^n$. If $f(z) \in \mathcal{H}_n(\mathbb{R})$ then $f(vz) = f(v_1 z_1, \dots, v_n z_n) \in \mathcal{H}_n(\mathbb{R})$ and

$$T[f(vz)] = \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha v^\beta (\partial^\beta f)(vz).$$

Hence

$$T^v := \sum_{\alpha,\beta} a_{\alpha\beta} v^{\alpha-\beta} z^\alpha \partial^\beta = \sum_\gamma v^\gamma z^\gamma T_\gamma \in \mathcal{A}_n(\mathbb{R}).$$

Let $\langle z, \mu\rangle = a$ be a supporting hyperplane of the vertex κ. Hence, up to replacing μ with $-\mu$, if necessary, we have $\langle \gamma - \kappa, \mu\rangle < 0$ for all $\gamma \in CH(T) \setminus \{\kappa\}$. Now let $v_i = v_i(t) = e^{\mu_i t}$. Then

$$v^{-\kappa} T^v = z^\kappa T_\kappa + \sum_{\gamma \neq \kappa} e^{t\langle \gamma-\kappa,\mu\rangle} z^\gamma T_\gamma.$$

By letting $t \to \infty$ we have that $z^\kappa T_\kappa \in \mathcal{A}_n(\mathbb{R})$ and the lemma follows. □

Let $f = \sum_\alpha a(\alpha) z^\alpha \in \mathbb{R}[z_1, \ldots, z_n]$. Define the *support* $\operatorname{supp}(f)$ of f to be the set $\{\alpha \in \mathbb{N}^n : a(\alpha) \neq 0\}$ and let $d = \max\{|\alpha| : \alpha \in \operatorname{supp}(f)\}$. We further define the *leading part* of f to be $a(\alpha)z^\alpha$, where α is the maximal element with respect to the lexicographical order on $\mathbb{Z}^n$ of the set $\{\alpha \in \operatorname{supp}(f) : |\alpha| = d\}$. Similarly, if $T = \sum_\gamma z^\gamma T_\gamma \in \mathcal{A}_n[\mathbb{R}]$ let $k = \max\{|\alpha| : T_\alpha \neq 0\}$ and let κ_0 be the maximal element of the set $\{\alpha : |\alpha| = k, T_\alpha \neq 0\}$ with respect to the lexicographical order. Since κ_0 is a vertex of $CH(T)$ we know that λ_{κ_0} is a multiplier sequence with a finite symbol whenever $T \in \mathcal{A}_n(\mathbb{R})$. We say that T_{κ_0} is the *dominating part* of T. Note that the dominating part of fg is the product of the dominating parts of f and g. Moreover, if $\lambda_{\kappa_0}(\alpha) \neq 0$ then the dominating part of $T(f)$ is $\lambda_{\kappa_0}(\alpha)a(\alpha)z^{\alpha+\kappa_0}$, where $a(\alpha)z^\alpha$ is the dominating part of f and T_{κ_0} is the dominating part of T.

We are now ready to prove that a real stability preserver also preserves proper position. Equivalently, Theorem 4.3 below asserts that $\mathcal{A}_n(\mathbb{R}) \subset \mathcal{A}_n(\mathbb{C})$.

Theorem 4.3. *Suppose that $T \in \mathcal{A}_n(\mathbb{R})$ and that $f, g \in \mathcal{H}_n(\mathbb{R})$ are such that $f \ll g$. Then $T(f) \ll T(g)$ or $T(f) = T(g) \equiv 0$.*

Proof. Let T_{κ_0} be the dominating part of T. We first assume that $f, g \in \mathcal{H}_n(\mathbb{R})$ are such that $f \ll g$, $0 \leq \deg(f) < \deg(g)$ and $T_{\kappa_0}(f)T_{\kappa_0}(g) \neq 0$. Let the leading part of f and g be $a(\alpha)z^\alpha$ and $b(\beta)z^\beta$, respectively. By considering $f(vt), g(vt) \in \mathcal{H}_1(\mathbb{R})$ as $v \in \mathbb{R}^n_+$ tends to infinity according to the lexicographical order we will have $\deg g(vt) = \deg f(vt) + 1$ for large v and the signs of the leading coefficients of $g(vt)$ and $f(vt)$ will be the same as the signs of $b(\beta)$ and $a(\alpha)$, respectively, for such large v. Since also $f(vt) \ll g(vt)$ we infer that $a(\alpha)b(\beta) > 0$.

Now since $T_{\kappa_0}(f)T_{\kappa_0}(g) \neq 0$ it follows that the leading parts of $T(f)$ and $T(g)$ are $\lambda_{\kappa_0}(\alpha)a(\alpha)z^{\kappa_0+\alpha}$ and $\lambda_{\kappa_0}(\beta)b(\beta)z^{\kappa_0+\beta}$, respectively. By Theorem 1.6 (the multivariate Obreschkoff theorem) we know that either $T(f) \ll T(g)$ or $T(g) \ll T(f)$. As pointed out in the paragraph preceding Theorem 4.3 dominating parts are necessarily multivariate multiplier sequences and so by Theorem 1.9 we have that $\lambda_{\kappa_0}(\alpha)\lambda_{\kappa_0}(\beta) > 0$. From the above discussion it follows that for some large $v \in \mathbb{R}^n_+$ we must have $T(f)(vt) \ll T(g)(vt)$ with $\deg(T(f)(vt)) < \deg(T(g)(vt))$, so that $T(f) \ll T(g)$.

If $\deg(f) > \deg(g)$ we may simply repeat the arguments using $-f$ and g and in the case when $\deg(f) = \deg(g)$ we consider f and $g + \epsilon z_1 f$ with $\epsilon > 0$. Indeed, $\deg(f) < \deg(g + \epsilon z_1 f)$ and $f \ll g + \epsilon z_1 f$ by Lemma 2.6 and we may let $\epsilon \to 0$.

Suppose now that $T_{\kappa_0}(f)T_{\kappa_0}(g) = 0$. There is nothing to prove if $fg \equiv 0$. Let $h_\epsilon(z_1, \ldots, z_n) = (1 + \epsilon z_1)^{\xi_1} \cdots (1 + \epsilon z_n)^{\xi_n}$ with $\xi_i \in \mathbb{N}$, $1 \leq i \leq n$, and let $f_\epsilon = h_\epsilon f$ and $g_\epsilon = h_\epsilon g$. If $\xi = (\xi_1, \ldots, \xi_n)$ is large enough then $T_{\kappa_0}(f_\epsilon)T_{\kappa_0}(g_\epsilon) \neq 0$. The theorem follows from Lemma 2.3 by letting $\epsilon \to 0$. □

4.3. **Homotopy transformations for symbols of stability preservers.** To complete the proof of the necessity part in Theorems 1.2–1.3 we need to establish first a key property for symbols of (real) stability preservers.

Lemma 4.4. *Suppose that* $F(z,w) \in \mathbb{R}[z_1,\dots,z_n,w_1,\dots,w_n]$ *is the symbol of an operator in* $\mathcal{A}_n(\mathbb{R})$ *and let* $\lambda \in (0,1)^n$. *Then* $F(z,\lambda w)$ *is also the symbol of an operator in* $\mathcal{A}_n(\mathbb{R})$.

Proof. Suppose that $T \in \mathcal{A}_n(\mathbb{R})$ has symbol $F(z_1,\dots,z_n,w_1,\dots,w_n)$. We claim that if $\delta \geq 0$ then the linear operator $\mathcal{E}_1^\delta T$ defined by

$$\mathcal{E}_1^\delta T(f) = \sum_{m=0}^{\infty} \frac{\delta^m z_1^m T(\partial_1^m f)}{m!}$$

is an operator $\mathcal{E}_1^\delta T : \mathcal{H}_n(\mathbb{R}) \to \mathcal{H}_n(\mathbb{R}) \cup \{0\}$. If T_δ is the linear operator with symbol $F(z_1,\dots,z_n,w_1/(1+\delta),\dots,w_n)$ then a simple calculation shows that

$$T_\delta(f) = \mathcal{E}_1^\delta T(f(z_1(1+\delta),\dots,z_n)).$$

Hence the claim would prove the lemma.

In order to prove the remaining claim let $\delta \geq 0$ and define a linear operator $R_\delta T : \mathbb{R}[z_1,\dots,z_n] \to \mathbb{R}[z_1,\dots,z_n]$ by

$$R_\delta T(f) = T(f) + \delta z_1 T(\partial_1 f).$$

Suppose that $f \in \mathcal{H}_n(\mathbb{R})$ and that $T(\partial_1 f) \neq 0$. Since $1 - iw_1 \in \mathcal{H}_n(\mathbb{C})$ we know by Corollary 4.1 that $1 + i\partial_1 \in \mathcal{A}_n(\mathbb{C})$, so $f + i\partial_1 f \in \mathcal{H}_n(\mathbb{C})$, i.e., $\partial_1 f \ll f$. By Theorem 4.3 we know that $T(\partial_1 f) \ll T(f)$ and $T(\partial_1 f) \ll z_1 T(\partial_1 f)$, which by Lemma 2.6 gives $R_\delta T(f) \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$. If $T(\partial_1 f) = 0$ then $R_\delta T(f) = T(f) \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$, so $R_\delta T \in \mathcal{A}_n(\mathbb{R})$.

An elementary computation shows that when we apply R_δ to T m times we get

$$R_\delta^m T(f) = \sum_{k=0}^{m} \binom{m}{k} \delta^k z_1^k T(\partial_1^k f).$$

Hence $R_{\delta/m}^m : \mathcal{H}_n(\mathbb{R}) \to \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ for all $m \in \mathbb{N}$. Now

$$\begin{aligned} R_{\delta/m}^m T(f) &= \sum_{k=0}^{m} \binom{m}{k} m^{-k} \delta^k z_1^k T(\partial_1^k f) \\ &= \sum_{k=0}^{m} (1-1/m)(1-2/m)\cdots(1-(k-1)/m) \frac{\delta^k z_1^k T(\partial_1^k f)}{k!}. \end{aligned}$$

It follows that $R_{\delta/m}^m T(f)$ tends uniformly to $\mathcal{E}^\delta T(f)$ on any compact subset of $\mathbb{C}^n$ as $m \to \infty$. Thus $\mathcal{E}^\delta T(f) : \mathcal{H}_n(\mathbb{R}) \to \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ by Lemma 2.3. ☐

From Lemma 4.4 one can easily see that symbols of (real) stability preservers actually satisfy the following homotopical property:

Theorem 4.5. *If* $F(z,w) \in \mathbb{R}[z_1,\dots,z_n,w_1,\dots,w_n]$ *is the symbol of an operator in* $\mathcal{A}_n(\mathbb{R})$ *then* $F(\mu z,\lambda w)$ *is also the symbol of an operator in* $\mathcal{A}_n(\mathbb{R})$ *for any* $(\mu,\lambda) \in [0,1]^n \times [0,1]^n$. *Moreover, the corresponding statement holds for symbols of operators in* $\mathcal{A}_n(\mathbb{C})$.

4.4. **Necessity in Theorems 1.2–1.3, continued.** We now have all the tools to prove the necessity in Theorem 1.3.

Proof of Theorem 1.3. The final step in the proof is to show that $F(z,\mu z^{-1})\neq 0$ whenever F is the symbol of an operator $T\in\mathcal{A}_n(\mathbb{R})$, $\mu\in\mathbb{R}_+^n$ and $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ is such that $\mathfrak{Im}(z_i)>0$, $1\leq i\leq n$. Indeed, since $F(z,w)$ is the symbol of a real stability preserver if and only if $F(z+\alpha,w)$ is the symbol of a real stability preserver for all $\alpha\in\mathbb{R}^n$ the claim implies that $F(z+\alpha,\mu z^{-1})\neq 0$ whenever F is the symbol of an operator $T\in\mathcal{A}_n(\mathbb{R})$, $\alpha\in\mathbb{R}^n$, $\mu\in\mathbb{R}_+^n$ and $z\in\mathbb{C}^n$ is such that $\mathfrak{Im}(z_i)>0$ for $1\leq i\leq n$. But it is straightforward to see that any pair $Z,W\in\mathbb{C}^n$ such that $\mathfrak{Im}(Z_i)>0$ and $\mathfrak{Im}(W_i)<0$ can be written as $Z_i=\alpha_i+z_i$ and $W_i=\mu_i z_i^{-1}$, where $\mathfrak{Im}(z_i)>0$, $\alpha_i\in\mathbb{R}$ and $\mu_i>0$, $1\leq i\leq n$, and the theorem would follow.

Let $T=\sum_{\alpha,\beta}a_{\alpha\beta}z^\alpha\partial^\beta\in\mathcal{A}_n(\mathbb{R})$ and let F be its symbol. By multiplying with a large monomial we may assume that $a_{\alpha\beta}=0$ if $\alpha\not\geq\beta$. Let $v\in\mathbb{R}_+^n$ and denote by ${}^vT$ the operator with symbol $F(z,vw)$. By Lemma 4.4 we have that

$$\begin{aligned}{}^vT(z^\gamma)z^{-\gamma}&=\sum_{\alpha,\beta}a_{\alpha\beta}v^\beta z^{\alpha-\beta}(\gamma)_\beta\\&=\sum_{\alpha,\beta}a_{\alpha\beta}(v\gamma)^\beta z^{\alpha-\beta}(\gamma)_\beta\gamma^{-\beta}\in\mathcal{H}_n(\mathbb{R})\cup\{0\}\end{aligned}\tag{4.1}$$

for all $v\in(0,1)^n$. Fix now $\mu\in\mathbb{R}_+^n$ and let v in (4.1) be of the form $\mu\gamma^{-1}$ with $\gamma=(\gamma_1,\ldots,\gamma_n)\in\mathbb{N}^n$, where $\gamma^{-1}=(\gamma_1^{-1},\ldots,\gamma_n^{-1})$. Then $v\in(0,1)^n$ for large γ. Letting γ tend to infinity and observing that $(\gamma)_\beta\gamma^{-\beta}\to 1$ we find by Lemma 2.3 that

$$\sum_{\alpha,\beta}a_{\alpha\beta}\mu^\beta z^{\alpha-\beta}=F(z,\mu z^{-1})\in\mathcal{H}_n(\mathbb{R})\cup\{0\}.$$

We have to prove that $F(z,\mu z^{-1})$ is not identically zero. To do this observe that

$$F(z,\mu z^{-1})=\sum_\kappa z^\kappa\sum_\beta a_{\beta+\kappa,\beta}\mu^\beta.$$

By Lemma 4.2 the dominating part, $T_{\kappa_0}=\sum_\beta a_{\beta+\kappa_0,\beta}z^\beta\partial^\beta$, of T is an operator associated to a multiplier sequence with finite symbol. Hence the nonzero coefficients $a_{\beta+\kappa_0,\beta}$ are all of the same sign by Theorem 1.9. This means that the coefficient of z^{κ_0} in $F(z,\mu z^{-1})$ is nonzero and proves the theorem. ☐

The proof of the necessity part in Theorem 1.2 now follows easily.

Proof of Theorem 1.2. Let $T\in\mathcal{A}_n(\mathbb{C})$ and write the symbol of T as $F(z,w)=F_R(z,w)+iF_I(z,w)$, where $F_R(z,w)$ and $F_I(z,w)$ have real coefficients. Let further T_R and T_I be the corresponding operators. Now $T:\mathcal{H}_n(\mathbb{R})\to\mathcal{H}_n(\mathbb{C})\cup\{0\}$ so by Lemma 2.2 we have that $T_R+z_{n+1}T_I:\mathcal{H}_n(\mathbb{R})\to\mathcal{H}_{n+1}(\mathbb{R})\cup\{0\}$. Hence by Lemma 2.1 we know that $T_R+(\lambda z_1+\alpha)T_I\in\mathcal{A}_n(\mathbb{R})\cup\{0\}$ for every $\lambda\in\mathbb{R}_+$ and $\alpha\in\mathbb{R}$. Suppose that $T_R+(\lambda z_1+\alpha)T_I=0$. Then $T=(i-\alpha-\lambda z_1)T_I$, so $T_I=0$ since $i-\alpha-\lambda z_1\notin\mathcal{H}_1(\mathbb{C})$. We thus have $T_R+(\lambda z_1+\alpha)T_I\in\mathcal{A}_n(\mathbb{R})$ for every $\lambda\in\mathbb{R}_+$ and $\alpha\in\mathbb{R}$, which by Theorem 1.3 gives $F_R+(\lambda z_1+\alpha)F_I\in\mathcal{H}_n(\mathbb{R})$ for every $\lambda\in\mathbb{R}_+$ and $\alpha\in\mathbb{R}$. By Lemma 2.1 and Lemma 2.2 this implies that $F=F_R+iF_I\in\mathcal{H}_n(\mathbb{C})$, as was to be shown. ☐

4.5. **The Weyl product and Schur-Maló-Szegö type theorems.** The results of §3.2 provide a unifying framework for most of the classical composition theorems for univariate hyperbolic polynomials [14, 33, 44, 45]. Moreover, they lead to natural multivariate extensions of these composition theorems. Let us for instance consider two operators $S, T \in \mathcal{A}_n[\mathbb{C}]$ with symbols $F_S(z,w)$ and $F_T(z,w)$, respectively. The well-known product formula in the Weyl algebra asserts that the symbol of the composite operator ST is given by

$$F_{ST}(z,w) = \sum_{\kappa \in \mathbb{N}^n} \frac{1}{\kappa!} \partial_w^\kappa F_S(z,w) \partial_z^\kappa F_T(z,w). \tag{4.2}$$

This suggests the following definition.

Definition 4.1. Let $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$. The Weyl product of two polynomials $f(z,w), g(z,w) \in \mathbb{C}[z,w]$ is given by

$$(f \star g)(z,w) = \sum_{\kappa \in \mathbb{N}^n} \frac{(-1)^\kappa}{\kappa!} \partial_z^\kappa f(z,w) \partial_w^\kappa g(z,w).$$

Theorems 1.2–1.3 and (4.2) imply that the Weyl product of polynomials defined above preserves (real) stability:

Theorem 4.6. *If $f(z,w)$ and $g(z,w)$ are (real) stable polynomials in the variables $z_1, \dots, z_n$ and $w_1, \dots, w_n$ then their Weyl product $(f \star g)(z,w)$ is also (real) stable.*

Example 4.1 (Schur-Maló-Szegö theorem)**.** Suppose that $S, T \in \mathcal{A}_1[\mathbb{R}]$ are such that $F_S(z,w) = f(\lambda^{-1}zw)$ and $F_T(z,w) = g(\lambda z)$, where $f \in \mathcal{H}_1(\mathbb{R})$ has all non-positive zeros, $g \in \mathcal{H}_1(\mathbb{R})$ and $\lambda > 0$. Then $S, T \in \mathcal{A}_1(\mathbb{R})$ by Theorem 1.3 hence $ST \in \mathcal{A}_1(\mathbb{R})$ and therefore

$$F_{ST}(z,-w) = \sum_{k \geq 0} \frac{1}{k!} z^k f^{(k)}(-\lambda^{-1}zw) g^{(k)}(\lambda z) \in \mathcal{H}_2(\mathbb{R})$$

by (4.2). Letting $w = 0$ and $\lambda \to 0$ it follows that

$$\sum_{k \geq 0} k! \frac{f^{(k)}(0)}{k!} \frac{g^{(k)}(0)}{k!} z^k \in \mathcal{H}_1(\mathbb{R}),$$

which is a well-known result of Schur [45], Maló and Szegö [14, 44].

Finally, we note that by using Theorems 1.2–1.3 we can derive yet another property of (real) stability preservers:

Proposition 4.7. *Let $T \in \mathcal{A}_n[\mathbb{C}]$ with $F_T(z,w) = \sum_{\alpha \in \mathbb{N}^n} Q_\alpha(z) w^\alpha$, where as before $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$. If $T \in \mathcal{A}_n(\mathbb{C})$ (respectively, $\mathcal{A}_n(\mathbb{R})$) then $Q_\alpha(z) \in \mathcal{H}_n(\mathbb{C}) \cup \{0\}$ (respectively, $\mathcal{H}_n(\mathbb{R}) \cup \{0\}$) for all $\alpha \in \mathbb{N}^n$.*

Proof. If $T \in \mathcal{A}_n(\mathbb{C})$ then $F_T(z_1, \dots, z_n, -w_1, \dots, -w_n) \in \mathcal{H}_{2n}(\mathbb{C})$ by Theorem 1.2. It follows that for any polynomial $P(v_1, \dots, v_n) \in \mathcal{H}_n(\mathbb{C})$ one has

$$P(v_1, \dots, v_n) F_T(z_1, \dots, z_n, -w_1, \dots, -w_n) \in \mathcal{H}_{3n}(\mathbb{C})$$

hence

$$P\left(-\frac{\partial}{\partial w_1}, \dots, -\frac{\partial}{\partial w_n}\right) F_T(z_1, \dots, z_n, -w_1, \dots, -w_n) \in \mathcal{H}_{2n}(\mathbb{C}) \cup \{0\}$$

by Corollary 3.10. Now the polynomial $P_\alpha(v_1,\dots,v_n) := v_1^{\alpha_1}\cdots v_n^{\alpha_n}$ clearly belongs to $\mathcal{H}_n(\mathbb{C})$ for any $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, so that by the above one has

$$\alpha! Q_\alpha(z) =$$
$$P_\alpha\Big(-\frac{\partial}{\partial w_1},\dots,-\frac{\partial}{\partial w_n}\Big)F_T(z_1,\dots,z_n,-w_1,\dots,-w_n)\Big|_{w_1=\cdots=w_n=0} \in \mathcal{H}_n(\mathbb{C})\cup\{0\},$$

as required. The case when $T \in \mathcal{A}_n(\mathbb{R})$ is treated similarly. □

4.6. **Duality, Pólya's curve theorem and generalizations.** Let us first establish the duality property stated in Theorem 1.11.

Proof of Theorem 1.11. By Theorem 1.2 we have that $T \in \mathcal{A}_n(\mathbb{C})$ if and only if

$$G(z,w) := F_T(z,-w) \in \mathcal{H}_n(\mathbb{C}).$$

But $F_{T^*}(z,-w) = \overline{G(-\overline{w},-\overline{z})} \in \mathcal{H}_n(\mathbb{C})$ so the desired conclusion follows from Theorem 1.2. The same arguments combined with Theorem 1.3 prove the analogous statement for $\mathcal{A}_n(\mathbb{R})$. □

In view of Theorem 1.11 one can both recover known results and deduce new ones by a simple dualization procedure, as illustrated in the following examples.

Example 4.2 (Hermite-Poulain-Jensen theorem)**.** Let $p(z) = \sum_{k=0}^n a_k z^k \in \mathbb{R}[z] \setminus \{0\}$, $T = p\big(\frac{d}{dz}\big) = \sum_{k=0}^n a_k \frac{d^k}{dz^k} \in \mathcal{A}_1[\mathbb{R}]$, and let T_p be the linear operator on $\mathbb{R}[z]$ defined by $T_p(f)(z) = p(z)f(z)$. Then $T^* = T_p$ so by Theorem 1.11 one has $T \in \mathcal{A}_1(\mathbb{R})$ if and only if $T_p \in \mathcal{A}_1(\mathbb{R})$, which clearly holds if and only if $p \in \mathcal{H}_1(\mathbb{R})$.

Example 4.3. The main result of [1] (Theorem 1.4 in *op. cit.*) shows that any operator in $\mathcal{A}_1(\mathbb{R})$ that commutes with the "inverted plane differentiation" operator $D_\sharp = z^2 D$, where $D = \frac{d}{dz}$, is of the form $\alpha D_\sharp^k$ for some $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Therefore, by Theorem 1.11 we conclude that any operator in $\mathcal{A}_1(\mathbb{R})$ that commutes with the operator $zD^2 = D_\sharp^*$ is of the form $\alpha(zD^2)^k$ for some $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$.

As we will now explain, both Theorem 1.3 and Theorem 1.11 admit natural geometric interpretations that lead to further interesting consequences. For simplicity's sake, we will only focus on the case $n = 1$.

Definition 4.2. Let $f(z,w) \in \mathbb{R}[z,w]$ be a nonzero polynomial in two variables of (total) degree d and define the real algebraic curve Γ_f (of degree d) by

$$\Gamma_f = \{(z,w) \in \mathbb{R}^2 \,:\, f(z,w) = 0\}.$$

We say that f, or equivalently Γ_f, has the *intersection property* $(\mathcal{I}_+)$ if Γ_f has d real intersection points (counted with multiplicities) with any line in $\mathbb{R}^2$ of the form

$$w = \alpha z + \beta, \ \text{ where } \alpha > 0,\ \beta \in \mathbb{R}.$$

Similarly, we say that f (or Γ_f) has the *intersection property* $(\mathcal{I}_-)$ if Γ_f has d real intersection points (counted with multiplicities) with any line in $\mathbb{R}^2$ of the form

$$w = \alpha z + \beta, \ \text{ where } \alpha < 0,\ \beta \in \mathbb{R}.$$

The *symbol curve* of an operator $T \in \mathcal{A}_1[\mathbb{R}]$ with symbol $F_T(z,w) \in \mathbb{R}[z,w]$ of degree d is the real algebraic curve (of degree d) given by

$$\Gamma_T = \{(z,w) \in \mathbb{R}^2 \,:\, F_T(z,w) = 0\}.$$

From Lemma 2.1 and Definition 4.2 we get:

Corollary 4.8. *Let f be a nonzero real polynomial in two variables. Then $f \in \mathcal{H}_2(\mathbb{R})$ if and only if Γ_f has the intersection property $(\mathcal{I}_+)$.*

Therefore, in the univariate case Theorem 1.3 may be restated as follows.

Corollary 4.9. *Let $T \in \mathcal{A}_1[\mathbb{R}]$. Then $T \in \mathcal{A}_1(\mathbb{R})$ if and only if its symbol curve Γ_T has the intersection property $(\mathcal{I}_-)$.*

As depicted in Figure 2 below, Corollary 4.9 essentially allows one to visualize whether an operator $T \in \mathcal{A}_1[\mathbb{R}]$ preserves hyperbolicity by checking if any line in $\mathbb{R}^2$ with negative slope has the required number of interesection points with Γ_T.

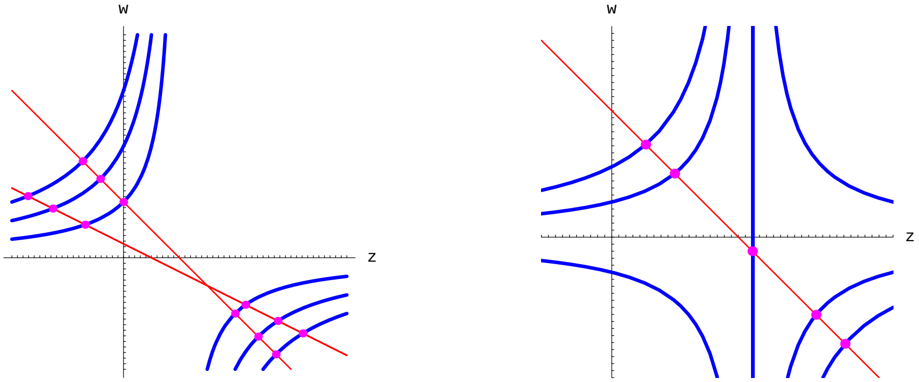

Figure 2. Left picture: the symbol curve of degree $d = 6$ of an operator in $\mathcal{A}_1(\mathbb{R})$. Right picture: the symbol curve of degree $d = 7$ of an operator in $\mathcal{A}_1[\mathbb{R}]$ for which property $(\mathcal{I}_-)$ fails.

In the same spirit, a simple geometric interpretation and proof of Theorem 1.11 for $n = 1$ is as follows: if $T \in \mathcal{A}_1[\mathbb{R}]$ then $F_{T^*}(z, w) = F_T(w, z)$ so Γ_{T^*} is just the reflection of Γ_T in the main diagonal in the zw-plane (i.e., the line $w = z$). Since the intersection property $(\mathcal{I}_-)$ is clearly invariant under this reflection we conclude that Γ_T and Γ_{T^*} have the aforementioned property simultaneously.

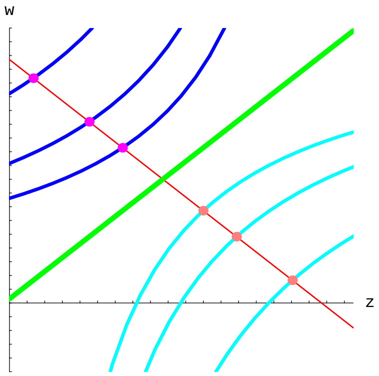

Figure 3. The symbol curve Γ_T of degree $d = 3$ of an operator $T \in \mathcal{A}_1(\mathbb{R})$ and its dual curve Γ_{T^*}

These geometric reformulations of Theorem 1.3 and Theorem 1.11 provide a unifying framework for "curve type theorems" that include and considerably strengthen Pólya's original result [41] and its various known generalizations [16, 40]. We illustrate this with a few examples where we derive some known as well as new results as direction applications of Theorem 1.3 (note that in fact we do not even need to make full use of Theorem 1.3 but just to argue as in the proof of Theorem 3.11).

Example 4.4. In [41] Pólya proved the following result that he considered as the most general theorem on the reality of roots of algebraic equations known at that moment (1916). He writes "Dieser Satz gehört wohl zu den allgemeinsten bekannten Sätzen über Wurzelrealität". This situation apparently remained more or less unchanged untill the early 1980's (cf. [16]).

Theorem 4.10 (Pólya's curve theorem)**.** *Let $f(x)$ be a (nonzero) hyperbolic polynomial of degree n, and let $b_0+b_1x+\ldots+b_{n+m}x^{n+m}$, where $m \geq 0$, be a hyperbolic polynomial with $b_i > 0$ for all $0 \leq i \leq n$. Set*

$$G_1(x,y) = b_0 f(y) + b_1 x f'(y) + b_2 x^2 f''(y) + \ldots + b_n f^{(n)}(y) \in \mathbb{R}[x,y].$$

Then G has the intersection property $(\mathcal{I}_+)$.

Proof. By assumption the polynomial $q(x) = \sum_{k=0}^{n+m} b_k x^k$ has all real and nonpositive zeros hence $q(-xz) \in \mathcal{H}_2(\mathbb{R})$ and thus the polynomial in variables x, y, z given by $q(-xz)f(y)$ belongs to $\mathcal{H}_3(\mathbb{R})$. From Corollary 3.10 we then get $q(xD_y)f(y) \in \mathcal{H}_2(\mathbb{R})$, where $D_y = \frac{d}{dy}$, and the result follows by Corollary 4.8. □

Example 4.5. Theorem 4.10 was generalized in [40, Theorem 147] as follows.

Theorem 4.11. *Fix $\alpha \geq 0$ and suppose that $q(x) = \sum_{k=0}^{n} b_k x^k$ is a (nonzero) hyperbolic polynomial with nonnegative coefficients. Let further $f(x)$ be a (nonzero) hyperbolic polynomial and set*

$$G(x,y) = \sum_{k=0}^{n} b_k (x - \alpha D_y)^k f^{(k)}(y) \in \mathbb{R}[x,y],$$

where $D_y = \frac{d}{dy}$. Then G has the intersection property $(\mathcal{I}_+)$.

Proof. Note that $q(-z(x+\alpha z)) \in \mathcal{H}_2(\mathbb{R})$ and thus $q(-z(x+\alpha z))f(y) \in \mathcal{H}_3(\mathbb{R})$. By Corollary 3.10 we get $q((x-\alpha D_y)D_y)f(y) \in \mathcal{H}_2(\mathbb{R})$. □

Remark 4.1. One can obtain further extensions of Pólya's Theorem 4.10 in the same spirit as Theorem 4.11, e.g. by using the fact that in the notation of the latter theorem one has $q(\alpha - xz) \in \mathcal{H}_2(\mathbb{R})$ whenever $\alpha \geq 0$.

5. Strict stability and strict real stability preservers

A natural question in the present context is to characterize all finite order linear differential operators that preserve *strict stability* and *strict real stability*, respectively. These notions are defined as follows: note first that the set of real stable univariate polynomials coincides with the set of hyperbolic univariate polynomials. Denote by $\mathcal{H}_1^s(\mathbb{R})$ the set of all *strictly hyperbolic* univariate polynomials, i.e., polynomials in $\mathcal{H}_1(\mathbb{R})$ with all simple zeros.

Definition 5.1. A polynomial $f \in \mathcal{H}_n(\mathbb{R})$ is called *strictly real stable* if $f(\alpha+vt) \in \mathcal{H}_1^s(\mathbb{R})$ for any $\alpha \in \mathbb{R}^n$ and $v \in \mathbb{R}_+^n$. One calls a polynomial $g \in \mathcal{H}_n(\mathbb{C})$ *strictly stable* if $g(z_1,\ldots,z_n) \neq 0$ for all n-tuples $(z_1,\ldots,z_n) \in \mathbb{C}^n$ with $\mathfrak{Im}(z_j) \geq 0$.

Let $\mathcal{H}_n^s(\mathbb{R})$ denote the set of all strictly real stable polynomials in n variables and let $\mathcal{H}_n^s(\mathbb{C})$ be the set of all strictly stable polynomials in n variables. Denote by $\mathcal{A}_n^s(\mathbb{C})$ and $\mathcal{A}_n^s(\mathbb{R})$ the submonoids of $\mathcal{A}_n[\mathbb{C}]$ and $\mathcal{A}_n[\mathbb{R}]$ consisting of all strict stability and strict real stability preservers, respectively, i.e., $\mathcal{A}_n^s(\mathbb{C}) = \{T \in \mathcal{A}_n[\mathbb{C}] : T(\mathcal{H}_n^s(\mathbb{C})) \subseteq \mathcal{H}_n^s(\mathbb{C}) \cup \{0\}\}$ and $\mathcal{A}_n^s(\mathbb{R}) = \{T \in \mathcal{A}_n[\mathbb{R}] : T(\mathcal{H}_n^s(\mathbb{R})) \subseteq \mathcal{H}_n^s(\mathbb{R}) \cup \{0\}\}$. In this section we give necessary and sufficient conditions in order for a linear operator to belong to either of these submonoids.

Theorem 5.1. *Let* $T \in \mathcal{A}_n[\mathbb{C}]$*. If* $T \in \mathcal{A}_n^s(\mathbb{C})$ *then* $F_T(z_1, \ldots, z_n, -w_1, \ldots, -w_n) \neq 0$ *whenever* $\mathfrak{Im}(z_j) \geq 0$ *and* $\mathfrak{Im}(w_k) > 0$*,* $1 \leq j, k \leq n$*.*

Theorem 5.2. *Let* $T \in \mathcal{A}_n[\mathbb{R}]$*. If* $T \in \mathcal{A}_n^s(\mathbb{R})$ *then* $F_T(z_1, \ldots, z_n, -w_1, \ldots, -w_n) \neq 0$ *whenever* $\mathfrak{Im}(z_j) \geq 0$ *and* $\mathfrak{Im}(w_k) > 0$*,* $1 \leq j, k \leq n$*.*

To prove Theorems 5.1–5.2 we need to establish a multivariate extension of the following classical result.

Theorem 5.3 (Strict Obreschkoff theorem)**.** *Let* $f, g \in \mathbb{R}[z]$*. Then*

$$\{\alpha f + \beta g : \alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 > 0\} \subset \mathcal{H}_1^s(\mathbb{R})$$

if and only if $f + ig \in \mathcal{H}_1^s(\mathbb{C})$ *or* $g + if \in \mathcal{H}_1^s(\mathbb{C})$*.*

Theorem 5.4. *Let* $f, g \in \mathbb{R}[z_1, \ldots, z_n]$*. Then*

$$\{\alpha f + \beta g : \alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 > 0\} \subset \mathcal{H}_n^s(\mathbb{R})$$

if and only if $f + ig \in \mathcal{H}_n^s(\mathbb{C})$ *or* $g + if \in \mathcal{H}_n^s(\mathbb{C})$*.*

Proof. This is an immediate consequence of Definition 5.1 and Theorem 5.3. □

Proof of Theorem 5.1. Suppose that $T \in \mathcal{A}_n^s(\mathbb{C})$ and let $f \in \mathcal{H}_n(\mathbb{C})$. Then

$$f_\epsilon(z_1, \ldots, z_n) := f(z_1 + i\epsilon, \ldots, z_n + i\epsilon) \in \mathcal{H}_n^s(\mathbb{C})$$

for all $\epsilon > 0$, so $T(f_\epsilon) \in \mathcal{H}_n^s(\mathbb{C})$. Letting $\epsilon \to 0$ it follows from Hurwitz' theorem that $T(f) \in \mathcal{H}_n(\mathbb{C}) \cup \{0\}$ and thus $T \in \mathcal{A}_n(\mathbb{C})$. Now Theorem 1.2 implies that

$$F_T(z_1, \ldots, z_n, -w_1, \ldots, -w_n) := \sum_{\alpha \in \mathbb{N}^n} Q_\alpha(z_1, \ldots, z_n)(-w)^\alpha \in \mathcal{H}_{2n}(\mathbb{C}),$$

where $Q_\alpha \in \mathbb{C}[z_1, \ldots, z_n]$ are not identically zero only for a finite number of multi-indices $\alpha \in \mathbb{N}^n$. Fix $(z_1^0, \ldots, z_n^0) \in \mathbb{C}^n$ with $\mathfrak{Im}(z_i^0) \geq 0$, $1 \leq i \leq n$. Then for any $\epsilon > 0$ one has $\mathfrak{Im}(z_k^0 + i\epsilon) > 0$, $1 \leq k \leq n$, hence

$$\sum_{\alpha \in \mathbb{N}^n} Q_\alpha(z_1^0 + i\epsilon, \ldots, z_n^0 + i\epsilon)(-w)^\alpha \in \mathcal{H}_n(\mathbb{C})$$

and by letting $\epsilon \to 0$ we deduce that

$$\sum_{\alpha \in \mathbb{N}^n} Q_\alpha(z_1^0, \ldots, z_n^0)(-w)^\alpha \in \mathcal{H}_n(\mathbb{C}) \cup \{0\}.$$

If $\sum_{\alpha \in \mathbb{N}^n} Q_\alpha(z_1^0, \ldots, z_n^0)(-w)^\alpha \equiv 0$ then $Q_\alpha(z_1^0, \ldots, z_n^0) = 0$ for all $\alpha \in \mathbb{N}^n$ and consequently $T(f)(z_1^0, \ldots, z_n^0) = 0$ for all polynomials f, which contradicts the assumption that $T \in \mathcal{A}_n^s(\mathbb{C})$ and $\mathfrak{Im}(z_i^0) \geq 0$, $1 \leq i \leq n$. □

Proof of Theorem 5.2. Suppose that $T \in \mathcal{A}_n^s(\mathbb{R})$. If the symbol is not as in the statement of the theorem then by arguing as in the proof of the necessity in Theorem 5.1 we see that there exists $Z^0 = (z_1^0, \dots, z_n^0) \in \mathbb{C}^n$ with $\mathfrak{Im}(z_i^0) \geq 0$, $1 \leq i \leq n$, and

$$F(Z^0, -w) = 0 \tag{5.1}$$

as a polynomial in $w = (w_1, \dots, w_n)$. Choose a polynomial $f \in \mathcal{H}_n^s(\mathbb{R})$ of sufficiently high degree so that $T(\alpha f + \beta \partial_1 f) \neq 0$ whenever $\alpha^2 + \beta^2 > 0$. By Theorem 5.4 we have $T(f) + iT(\partial_1 f) \in \mathcal{H}_n^s(\mathbb{C})$. This is however a contradiction since by (5.1) we have $T(f)(Z^0) + iT(\partial_1 f)(Z^0) = 0$. □

The next two theorems give sufficient conditions for operators in the Weyl algebra to be strict stability or strict real stability preserving, respectively.

Theorem 5.5. *Let $T \in \mathcal{A}_n[\mathbb{C}]$. If $F_T(z_1, \dots, z_n, -w_1, \dots, -w_n) \in \mathcal{H}_{2n}^s(\mathbb{C})$ then $T \in \mathcal{A}_n^s(\mathbb{C})$.*

Theorem 5.6. *Let $T \in \mathcal{A}_n[\mathbb{R}]$. If $F_T(z_1, \dots, z_n, -w_1, \dots, -w_n) \in \mathcal{H}_{2n}^s(\mathbb{R})$ then $T \in \mathcal{A}_n^s(\mathbb{R})$.*

Proof of Theorem 5.5. Let $T \in \mathcal{A}_n[\mathbb{C}]$ and suppose that

$$F_T(z_1, \dots, z_n, -w_1, \dots, -w_n) \neq 0$$

whenever $\mathfrak{Im}(z_i) \geq 0$ and $\mathfrak{Im}(w_j) \geq 0$, $1 \leq i, j \leq n$. If $f(v_1, \dots, v_n) \in \mathcal{H}_n^s(\mathbb{C})$ then

$$F_T(z_1, \dots, z_n, -w_1, \dots, -w_n) f(v_1, \dots, v_n) \neq 0$$

provided that $\mathfrak{Im}(z_i) \geq 0$, $\mathfrak{Im}(w_j) \geq 0$ and $\mathfrak{Im}(v_k) \geq 0$ for $1 \leq i, j, k \leq n$. By Proposition 3.9 we may replace each variable w_j with $w_j - \frac{\partial}{\partial v_j}$, $1 \leq j \leq n$, to get

$$F_T\left(z_1, \dots, z_n, \frac{\partial}{\partial v_1} - w_1, \dots, \frac{\partial}{\partial v_n} - w_n\right) f(v_1, \dots, v_n) \neq 0$$

whenever $\mathfrak{Im}(z_i) \geq 0$, $\mathfrak{Im}(v_i) \geq 0$, $\mathfrak{Im}(w_i) \geq 0$ for $1 \leq i \leq n$. If we now exchange each variable v_j by z_j, $1 \leq j \leq n$, and let $w_i = 0$, $1 \leq i \leq n$, we see that

$$T(f)(z_1, \dots, z_n) = F_T\left(z_1, \dots, z_n, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\right) f(z_1, \dots, z_n) \neq 0$$

whenever $\mathfrak{Im}(z_i) \geq 0$, $1 \leq i \leq n$, hence $T(f) \in \mathcal{H}_n^s(\mathbb{C})$. □

Proof of Theorem 5.6. If F_T is as in the statement of the theorem then by Theorem 5.5 we have that $T \in \mathcal{A}_n^s(\mathbb{C})$. Consider $f \in \mathcal{H}_n^s(\mathbb{R})$. The case when f is a nonzero constant, say $f(z) \equiv c \in \mathbb{R} \setminus \{0\}$, is immediate since $T(f)(z_1, \dots, z_n) = cF_T(z_1, \dots, z_n, 0, \dots, 0) \neq 0$ whenever $\mathfrak{Im}(z_i) \geq 0$, $1 \leq i \leq n$, hence $T(f) \in \mathcal{H}_n^s(\mathbb{R})$. If f is not a constant polynomial we may assume that $\partial_1 f \not\equiv 0$. Then $f + i\partial_1 f \in \mathcal{H}_n^s(\mathbb{C})$, so $T(f) + iT(\partial_1 f) \in \mathcal{H}_n^s(\mathbb{C}) \cup \{0\}$. By Theorem 5.4 we have that $T(f) \in \mathcal{H}_n^s(\mathbb{R}) \cup \{0\}$, as required. □

To close this section we note that in general the necessary conditions stated in Theorems 5.1–5.2 are not sufficient while the sufficient conditions given in Theorems 5.5–5.6 are not necessary. This may be seen already in the univariate case from the following simple examples. The operator $T = \frac{d}{dz}$ is clearly strict (real) stability preserving but its symbol $F_T(z, w) = w$ does not satisfy the sufficient conditions stated in Theorems 5.5–5.6. Consider now the operator $S = 2z + 1 + (z^2 + z)\frac{d}{dz}$. One can easily check that $F_S(z, -w) \neq 0$ whenever $\mathfrak{Im}(z) \geq 0$ and $\mathfrak{Im}(w) > 0$ and also that S preserves strictly real stable (i.e., strictly hyperbolic) polynomials. However, $S(1) = 2z + 1 \notin \mathcal{H}_1^s(\mathbb{C})$ so $S \notin \mathcal{A}_1^s(\mathbb{C})$. A characterization of strict (real) stability preservers would therefore require conditions that are "intermediate" between those of Theorems 5.1–5.2 and Theorems 5.5–5.6.

6. Multivariate matrix pencils and applications

In this section we consider several examples and applications of the above results. First we prove Proposition 1.12 claiming that the polynomial

$$f(z_1, \dots, z_n) := \det\left(\sum_{i=1}^{n} z_i A_i + B\right)$$

with A_i, $1 \le i \le n$, positive semidefinite matrices and B a Hermitian matrix of the same order is either real stable or identically zero.

Proof of Proposition 1.12. By a standard continuity argument using Hurwitz' theorem it suffices to prove the result only in the case when all matrices $A_1, A_2, \dots, A_n$ are positive definite. Set $z(t) = \alpha + \lambda t$ with $\alpha \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_+^n$ and $t \in \mathbb{R}$. Note that $P := \lambda_1 A_1 + \ldots + \lambda_n A_n$ is positive semidefinite and thus it has a square root. Then

$$f(z(t)) = \det(P)\det(tI + P^{-1/2} H P^{-1/2}),$$

where $H := B + \alpha_1 A_1 + \ldots + \alpha_n A_n$ is Hermitian. Hence $f(z(t))$ is a constant multiple of the characteristic polynomial of a Hermitian matrix and so it must have all real zeros. □

6.1. **A stable multivariate extension of the Cauchy-Poincaré theorem.**

Let A be any $n \times n$ complex matrix and define a polynomial $C(A, z) = \det(Z - A) \in \mathbb{C}[z_1, \dots, z_n]$, where $z = (z_1, \dots, z_n)$ and Z is the (diagonal) matrix with entries $Z_{ij} = z_i \delta_{ij}$. Given $1 \le i, j \le n$ let A^{ij} be the submatrix of A obtained by deleting row i and column j and set $C_{ij}(A, z) = \det\big((Z - A)^{ij}\big)$. For $z = (z_1, \dots, z_n)$ and $1 \le i \le n$ let $z \setminus z_i = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n)$, so that $C_{ii}(A, z) = C(A^{ii}, z \setminus z_i)$.

Lemma 6.1. *For $1 \le j \le n$ one has $T_j := 1 + i\frac{\partial}{\partial z_j} \in \mathcal{A}_n(\mathbb{C})$.*

Proof. The symbol $F_{T_j}(z, w)$ of T_j clearly satisfies $F_{T_j}(z, -w) = 1 - i w_j$ and the latter polynomial is stable since it is obviously nonvanishing if $\mathfrak{Im}(w_j) > 0$. The assertion now follows from Theorem 1.2. □

Theorem 6.2. *If A is a complex Hermitian $n \times n$ matrix then $C(A, z) \in \mathcal{H}_n(\mathbb{R})$ and*

$$C(A^{jj}, z \setminus z_j) \ll C(A, z)$$

for $1 \le j \le n$.

Proof. Note that since A is Hermitian $C(A, z)$ is real stable by Proposition 1.12. Now

$$C(A^{jj}, z \setminus z_j) = C_{jj}(A, z) = \frac{\partial}{\partial z_j} C(A, z) \ll C(A, z)$$

by Lemma 6.1 and Corollary 2.4. □

The above theorem generalizes the classical Cauchy-Poincaré theorem stating that the eigenvalues of a Hermitian matrix and those of any of its degeneracy one principal submatrices interlace.

An alternative proof of Theorem 6.2 may be obtained by using the following consequence of the Christoffel-Darboux identity [21]:

Lemma 6.3. *Let A be any $n \times n$ matrix with $n \geq 2$ and let $1 \leq i, j \leq n$. Then*

$$C(A,y)C_{ij}(A,x) - C(A,x)C_{ij}(A,y) = \sum_{k=1}^{n}(y_k - x_k)C_{ik}(A,x)C_{kj}(A,y). \tag{6.1}$$

Proof. Let $X = (x_i\delta_{ij})$ and $Y = (x_i\delta_{ij})$. The identity

$$(X-A)^{-1} - (Y-A)^{-1} = (X-A)^{-1}(Y-X)(Y-A)^{-1}$$

obtains by multiplying on the left with $(X - A)$ and on the right with $(Y - A)$. Taking the ij-th entry on both sides in the above identity and multiplying by $C(A,x)C(A,y)$ yields formula (6.1). $\square$

Let now A be a complex Hermitian $n \times n$ matrix with $n \geq 2$ and let $y = z$, $x = \bar{z}$ and $i = j$ in (6.1). Note that $C_{ij}(A,\bar{z}) = \overline{C_{ji}(A,z)}$ and since $C_{ii}(A,z) = C(A^{ii}, z \setminus z_i)$ we get

$$\mathfrak{Im}(C(A,z)C(A^{ii},\overline{z \setminus z_i})) = \sum_{k=0}^{n}\mathfrak{Im}(z_k)|C_{ik}(A,z)|^2. \tag{6.2}$$

Theorem 6.2 is obviously true for $n = 1$ and the general case follows by induction on n. Indeed, let $\mathfrak{Im}(z_j) > 0$ for $1 \leq j \leq n$, where $n \geq 2$. By the induction hypothesis we have $C(A^{ii}, z \setminus z_i) \in \mathcal{H}_{n-1}(\mathbb{C})$ and then from (6.2) we deduce that

$$\mathfrak{Im}\left(\frac{C(A,z)}{C(A^{ii}, z \setminus z_i)}\right) = \mathfrak{Im}\left(\frac{C(A,z)C(A^{ii},\overline{z \setminus z_i})}{|C(A^{ii}, z \setminus z_i)|^2}\right) \geq \mathfrak{Im}(z_i) > 0.$$

Hence $C(A,z) \neq 0$ and the desired conclusion follows from Lemma 2.2.

6.2. **Lax conjecture for real stable polynomials in two variables.** Here we will prove that all real stable polynomials in two variables x, y can be written as $\pm\det(xA + yB + C)$, where A and B are positive semidefinite (PSD) matrices and C is a symmetric matrix. The proof relies on the Lax conjecture that was recently settled in [30] by using in an essential way the results of [25].

Theorem 6.4 ([25, 30])**.** *A homogeneous polynomial $p \in \mathbb{R}[x,y,z]$ is hyperbolic of degree d with respect to the vector $e = (1,0,0)$ if and only if there exist two symmetric $d \times d$ matrices B, C such that*

$$p(x,y,z) = p(e)\det(xI + yB + zC).$$

We will also need Proposition 1.1 that we proceed to prove.

Proof of Proposition 1.1. If $f \in \mathbb{R}[z_1,\dots,z_n]$ is of degree d then its *homogenization* – i.e., the unique homogeneous polynomial $f_H \in \mathbb{R}[z_1,\dots,z_{n+1}]$ of degree d such that $f_H(z_1,\dots,z_n,1) = f(z_1,\dots,z_n)$ – is simply

$$f_H(z_1,\dots,z_{n+1}) = z_{n+1}^d f(z_1 z_{n+1}^{-1},\dots,z_n z_{n+1}^{-1}).$$

If f_H is hyperbolic with respect to every vector $v \in \mathbb{R}^{n+1}$ such that $v_{n+1} = 0$ and $v_i > 0$, $1 \leq i \leq n$, it follows in particular that for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and $(v_1, \dots, v_n) \in \mathbb{R}_+^n$ the univariate (real) polynomial in t given by

$$f_H((\alpha_1, \dots, \alpha_n, 1) + (v_1, \dots, v_n, 0)t) = f(\alpha + vt)$$

is not identically zero (since $\lim_{t\to\infty} t^{-d} f(\alpha + vt) = f_H(v_1, \dots, v_n, 0) \neq 0$) and has all real zeros hence it belongs to $\mathcal{H}_1(\mathbb{R})$. Thus $f \in \mathcal{H}_n(\mathbb{R})$ by Lemma 2.1.

Conversely, suppose that $f \in \mathcal{H}_n(\mathbb{R})$ has degree d and is given by

$$f(z) = \sum_{\kappa \in \mathbb{N}^n} a_\kappa z^\kappa, \quad z = (z_1, \dots, z_n).$$

Let $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in \mathbb{R}^{n+1}$ and $v = (v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+1}$ with $v_{n+1} = 0$, $v_i > 0$, $1 \leq i \leq n$. Since $a_\kappa \neq 0$ for some $\kappa \in \mathbb{N}^n$ with $|\kappa| = d$ Hurwitz' theorem yields

$$g(z) := \lim_{t\to\infty} t^{-d} f(tz) = \sum_{\kappa \in \mathbb{N}^n,\, |\kappa| = d} a_\kappa z^d \in \mathcal{H}_n(\mathbb{R}).$$

Moreover, g is a homogeneous polynomial so by the "same phase property" established in [13] all nonzero a_κ's with $|\kappa| = d$ have the same sign. Therefore

$$f_H(v) = g(v_1, \dots, v_n) = \sum_{\kappa \in \mathbb{N}^n,\, |\kappa| = d} a_\kappa v_1^{\kappa_1} \cdots v_n^{\kappa_n} \neq 0$$

since $v_i > 0$, $1 \leq i \leq n$. Now, if $\alpha_{n+1} = 0$ then the univariate polynomial

$$t \mapsto f_H(\alpha + vt) = g(\alpha_1 + v_1 t, \dots, \alpha_n + tv_n)$$

has all real zeros by Lemma 2.1, while if $\alpha_{n+1} > 0$ then again by Lemma 2.1 the univariate polynomial

$$t \mapsto f_H(\alpha + vt) = \alpha_{n+1}^d f(\alpha_1 \alpha_{n+1}^{-1} + v_1 \alpha_{n+1}^{-1} t, \dots, \alpha_n \alpha_{n+1}^{-1} + v_n \alpha_{n+1}^{-1} t)$$

has all real zeros. By the last part of Lemma 2.1, the same holds when $\alpha_{n+1} < 0$. Hence f_H is hyperbolic with respect to all vectors $v \in \mathbb{R}^{n+1}$ as above. $\square$

Lemma 6.5. *Suppose that $p \in \mathbb{R}[x, y, z]$ is a homogeneous polynomial of degree d which is hyperbolic with respect to any $(v_1, v_2, 0) \in \mathbb{R}^3$ with $v_1, v_2 \in \mathbb{R}_+$. Then*

$$p(x, y, 0) = x^d \sum_{i=0}^d a_i \left(\frac{y}{x}\right)^i,$$

where the a_i's are such that $\sum_{i=0}^d a_i t^i$ is a polynomial with all nonpositive zeros.

Proof. By letting $z \to 0$ it follows from Hurwitz' theorem that $p(x, y, 0)$ is hyperbolic with respect to all $v \in \mathbb{R}_+^2$ and must therefore be of the form

$$p(x, y, 0) = x^d \sum_{i=0}^d a_i \left(\frac{y}{x}\right)^i,$$

where $a_0 + a_1 t + \cdots + a_d t^d$ is a polynomial with all nonpositive zeros. $\square$

Theorem 6.6. *A homogeneous polynomial $p \in \mathbb{R}[x, y, z]$ of degree d is hyperbolic with respect to all vectors of the form $(v_1, v_2, 0)$ with $v_1, v_2 \in \mathbb{R}_+$ if and only if there exist two positive semidefinite $d \times d$ matrices A and B and a symmetric $d \times d$ matrix C such that*

$$p(x, y, z) = \alpha \det(xA + yB + zC),$$

where $\alpha \in \mathbb{R}$*. Moreover,* A *and* B *can be chosen so that* $A + B = I$*.*

Proof. Let p be hyperbolic of degree d with respect to all vectors of the form $(v_1, v_2, 0)$ with $v_1, v_2 \in \mathbb{R}_+$ and let $\alpha := p(1,1,0) \in \mathbb{R}\backslash\{0\}$. Consider the polynomial $f(x,y,z) = p(x, x+y, z)$. Then $f(x,y,z)$ is hyperbolic of degree d with respect to all vectors $(v_1, v_2, 0)$, where $v_1, v_2 \in \mathbb{R}_+$. Moreover, it is hyperbolic with respect to the vector $e = (1,0,0)$. Hence by Theorem 6.4 there exist two symmetric $d \times d$ matrices B, C such that

$$f(x,y,z) = f(e)\det(xI + yB + zC).$$

Since f is hyperbolic with respect to all vectors of the form $(v_1, v_2, 0)$ with $v_1, v_2 \in \mathbb{R}_+$ we know by Lemma 6.5 that all the eigenvalues of B are nonnegative. Hence B is a PSD matrix. Let $A = I - B$. Then

$$p(x,y,z) = \alpha\det(xA + y(I-A) + zC),$$

and by Lemma 6.5 all zeros of the polynomial

$$r(t) := \alpha^{-1}p(1,t,0) = (1-t)^d \det\left(A + \frac{t}{1-t}I\right) \in \mathbb{R}[t]$$

are nonpositive. Inverting this we have

$$\det(A + tI) = (1+t)^d r\left(\frac{t}{1+t}\right),$$

which implies that A has all nonnegative eigenvalues, so A is a PSD matrix. □

From Theorem 6.6 and Proposition 1.1 we deduce the following converse to Proposition 1.12 for real stable polynomials in two variables.

Corollary 6.7. *Let* $f(x,y) \in \mathbb{R}[x,y]$ *be of degree* n*. Then* f *is real stable if and only if there exist two* $n \times n$ *PSD matrices* A, B *and a symmetric* $n \times n$ *matrix* C *such that*

$$f(x,y) = \pm\det(xA + yB + C).$$

6.3. **Hyperbolicity preservers via determinants and homogenized symbols.** Using Theorem 1.3 with $n = 1$ and Corollary 6.7 we immediately get the following determinantal description of finte order linear preservers of univariate real stable (i.e., hyperbolic) polynomials.

Theorem 6.8. *Let* $T \in \mathcal{A}_1[\mathbb{R}]$*. Then* $T \in \mathcal{A}_1(\mathbb{R})$ *if and only if there exist* $\alpha \in \mathbb{R}$*,* $d \in \mathbb{N}$*, two positive semidefinite* $d \times d$ *matrices* A *and* B *and a symmetric* $d \times d$ *matrix* C *such that*

$$T = \alpha\det(zA - wB + C)\Big|_{w=\frac{\partial}{\partial z}}.$$

From Theorem 6.8 and Proposition 1.1 we deduce yet another characterization of univariate hyperbolicity preservers involving real homogeneous (Gårding) hyperbolic polynomials in 3 variables:

Theorem 6.9. *Let* $T \in \mathcal{A}_1[\mathbb{R}]$ *with symbol* $F_T(z,w)$ *of degree* d *and let* $\tilde{F}_T(y,z,w)$ *be the (unique) homogeneous degree* d *polynomial such that* $\tilde{F}_T(1,z,w) = F_T(z,w)$*. Then* $T \in \mathcal{A}_1(\mathbb{R})$ *if and only if the following conditions hold:*

(i) $\tilde{F}_T(y,z,w)$ *is hyperbolic with respect to* $(0,1,1)$*,*
(ii) *all zeros of* $\tilde{F}_T(0,t,1)$ *lie in* $(-\infty, 0]$*.*

7. Further remarks and open problems

7.1. A characterization of real stable polynomials has recently been obtained in [10] by using the above multivariate extension of Obreschkoff's theorem (Theorem 1.6) and of the Hermite-Biehler theorem (Corollary 2.4) combined with the Grace-Walsh-Szegö Coincidence Theorem [44]. To formulate this result we need some notation. Given $f \in \mathbb{C}[z_1, \dots, z_n]$ and $1 \leq i, j \leq n$ define the following quadratic differential expression:

$$\Delta_{ij}(f) = \frac{\partial f}{\partial z_i} \cdot \frac{\partial f}{\partial z_j} - \frac{\partial^2 f}{\partial z_i \partial z_j} \cdot f = -f^2 \frac{\partial^2}{\partial z_i \partial z_j} \left[\log |f|\right].$$

(In the spirit of [7, 47], one may call $\Delta_{ij}(f)$ the ij-th Rayleigh form of f.) Assume further that $\deg_{z_i} f = d_i$, $1 \leq i \leq n$. The *polarization* $\mathcal{P}(f)$ of f is the unique polynomial in the variables $\{z_{ij} : 1 \leq i \leq n, 1 \leq j \leq d_i\}$ satisfying

1. $\mathcal{P}(f)$ is multi-affine (i.e., it has degree 1 in each variable),
2. $\mathcal{P}(f)$ is symmetric in the variables $z_{i1}, \dots, z_{id_i}$ for all $1 \leq i \leq n$,
3. if we set $z_{ij} = z_i$ for all i, j in $\mathcal{P}(f)$ we recover f.

Theorem 7.1 ([10])**.** *Let $f \in \mathbb{R}[z_1, \dots, z_n]$ be of degree $d = \sum_{i=1}^{n} d_i$, where $\deg_{z_i} f = d_i$, $1 \leq i \leq n$. The following assertions are equivalent:*

- (i) $f \in \mathcal{H}_n(\mathbb{R}) \cup \{0\}$,
- (ii) *for all $x \in \mathbb{R}^d$ and $1 \leq i, j \leq d$ one has $\Delta_{ij}(\mathcal{P}(f))(x) \geq 0$.*

Note that in the univariate case Theorem 7.1 gives a characterization of hyperbolic (i.e., real zero) polynomials by means of a single equation in $n - 2$ variables, n being the degree of the polynomial under consideration.

7.2. Two of the main challenges in the theory of univariate polynomials have been to describe all hyperbolicity (respectively, stability) preserving linear operators, that is, all linear operators T on $\mathbb{R}[z]$ (respectively, $\mathbb{C}[z]$) such that $T(\mathcal{H}_1(\mathbb{R})) \subseteq \mathcal{H}_1(\mathbb{R}) \cup \{0\}$ (respectively, $T(\mathcal{H}_1(\mathbb{C})) \subseteq \mathcal{H}_1(\mathbb{C}) \cup \{0\}$). These fundamental problems originate from Laguerre's work [27] and Pólya-Schur's characterization of multiplier sequences of the first kind [42], see also [14, 17]. Both these problems have recently been solved in [8]. However, the natural multivariate analogs of these questions are still open at the moment:

Problem 7.1. *Characterize all linear operators T on $\mathbb{R}[z_1, \dots, z_n]$ (respectively, $\mathbb{C}[z_1, \dots, z_n]$) that satisfy $T(\mathcal{H}_n(\mathbb{R})) \subseteq \mathcal{H}_n(\mathbb{R}) \cup \{0\}$ (respectively, $T(\mathcal{H}_n(\mathbb{C})) \subseteq \mathcal{H}_n(\mathbb{C}) \cup \{0\}$).*

Note that Theorems 1.2 and 1.3 solve Problem 7.1 for operators in the (n-th) Weyl algebra $\mathcal{A}_n[\mathbb{R}]$ (respectively, $\mathcal{A}_n[\mathbb{C}]$).

7.3. The finite degree analogs of the problems discussed in §7.2 are also important in view of their numerous applications, see, e.g., [14]. These questions deal with describing all linear operators acting on the finite-dimensional linear space of univariate real (respectively, complex) polynomials of degree at most d that preserve hyperbolicity (respectively, stability). Complete solutions to these problems have recently been obtained in [8]. Notwithstanding, the multivariate analogs of these questions remain open:

Problem 7.2. *Characterize all linear operators acting on the complex (respectively, real) space of polynomials in n variables of total degree at most d that preserve stability (respectively, real stability).*

7.4. It is well known that the Lax conjecture fails in four or more variables for rather obvious dimensional reasons. Indeed, the dimension of the space of matrix pencils is smaller than the dimension of the space of homogeneous polynomials of the appropriate degree. Helton and Vinnikov recently proposed in [25] a higher dimensional version of the Lax conjecture that may be stated as follows:

Conjecture 7.3. *Let $P(z_0, z_1, \dots, z_m)$ be a real homogeneous polynomial hyperbolic with respect to $c = (c_0, c_1, \dots, c_m) \in \mathbb{R}^{m+1}$ and L be a real linear form in $z_0, z_1, \dots, z_m$ with $L(c) \neq 0$. Then there exists an integer N such that*

$$L(z_0, z_1, \dots, z_m)^N P(z_0, z_1, \dots, z_m) = \det(z_0 A_0 + z_1 A_1 + \dots + z_m A_m)$$

for some real symmetric matrices $A_0, A_1, \dots, A_m$ with $c_0 A_0 + c_1 A_1 + \dots + c_m A_m > 0$.

A natural question in this context is whether any real stable (homogeneous) polynomial admits a Lax type determinantal representation. We recently proposed in [9] precise versions of this problem. These may be described as follows. Given a square matrix A of order n and $\mathcal{S} \subseteq \{1, \dots, n\}$ denote by $A[\mathcal{S}]$ the $|\mathcal{S}| \times |\mathcal{S}|$ principal submatrix of A whose rows and columns are indexed by $\mathcal{S}$. As in [9, Definition 2.5] we define the mixed determinant of an m-tuple $(A_1, \dots, A_m)$ of $n \times n$ matrices by

$$\eta(A_1, \dots, A_m) = \sum_{(\mathcal{S}_1, \dots, \mathcal{S}_m)} \det(A_1[\mathcal{S}_1]) \cdots \det(A_m[\mathcal{S}_m]),$$

where the summation is taken over all ordered partitions of $\{1, \dots, n\}$ into m parts i.e., all m-tuples $(\mathcal{S}_1, \dots, \mathcal{S}_m)$ of pairwise disjoint subsets of $\{1, \dots, n\}$ satisfying $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_m = \{1, \dots, n\}$. Let now $\ell, m, n \geq 1$ be integers. For $1 \leq j \leq m$ define matrix pencils

$$\mathcal{L}_j := \mathcal{L}_j(z_1, \dots, z_\ell) = \sum_{k=1}^{\ell} A_{jk} z_k + B_j, \tag{7.1}$$

where A_{jk}, $1 \leq k \leq \ell$, are positive semidefinite $n \times n$ matrices and B_j is a Hermitian $n \times n$ matrix. Then by [9, Theorem 2.6] we know that $\eta(\mathcal{L}_1, \dots, \mathcal{L}_m) \in \mathcal{H}_\ell(\mathbb{R}) \cup \{0\}$.

Problem 7.4. *Is the converse statement true, namely: if f is a real stable polynomial of degree n in ℓ variables then there exist a positive integer m and matrix pencils $\mathcal{L}_j$, $1 \leq j \leq m$, of the form (7.1) such that $f = \eta(\mathcal{L}_1, \dots, \mathcal{L}_m)$?*

Note that by Corollary 6.7 the answer to Problem 7.4 is affirmative (at least) in the case $\ell = 2$. The homogeneous version of Problem 7.4 is as follows.

Problem 7.5. *Let f be a real stable homogeneous polynomial of degree n in ℓ variables. Is it true that there exist a positive integer m and matrix pencils $\mathcal{L}_j$, $1 \leq j \leq m$, of the form (7.1) with $B_j = 0$, $1 \leq j \leq m$, such that $f = \eta(\mathcal{L}_1, \dots, \mathcal{L}_m)$?*

7.5. As we already mentioned in the introduction, principal symbols of hyperbolic partial differential equations are an important and rich source of examples of real stable polynomials. The present study naturally leads to arguably the most general linear preserver problem for such classes of polynomials whose applications would encompass PDE theory as well as many other areas. This fundamental question may be stated as follows.

Problem 7.6. *Let V be a cone in $\mathbb{R}^n$ and denote by $\mathcal{H}_n(V)$ the set of homogeneous polynomials in n variables that are hyperbolic with respect to any vector $v \in V$. Describe all linear operators T on $\mathbb{R}[z_1, \dots, z_n]$ such that $T(\mathcal{H}_n(V)) \subseteq \mathcal{H}_n(V) \cup \{0\}$.*

In view of Proposition 1.1 the description of finite order real stability preservers given in Theorem 1.3 is actually intimately related to Problem 7.6 for the cone $V = \{0\} \times \mathbb{R}_+^{n-1}$. In fact our methods seem appropriate for dealing with Problem 7.6 (at least) in the case of the Weyl algebra $\mathcal{A}_n[\mathbb{R}]$.

7.6. The problems discussed in §7.2–§7.5 as well as several other related questions made the object of the "Pólya-Schur-Lax Workshop" recently held at the American Institute of Mathematics, see [6], and are currently under investigation.

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