THE (p, t, a)-INERTIAL GROUPS AS FINITE MONODROMY GROUPS
VERSION DE TRAVAIL DU January 8, 2025

SÉVERIN PHILIP

Abstract. The goal of this text is to provide a positive answer to a question of Silverberg and Zarhin on (p, t, a)-inertial groups in the case of p-groups.

1. Introduction

1.1.1. The finite monodromy groups of abelian varieties have been introduced by Grothendieck in [SGA7.1] exposé IX. They represent the local obstruction to semi-stable reduction. Silverberg and Zarhin studied these groups in [SZ98; SZ05], and the author in order to give an effective version of Grothendieck's semi-stable reduction theorem in [Phi22a; Phi22b; Phi24].

For a fixed natural integer $g \geq 1$, the list of finite groups which can be realized as finite monodromy groups of some abelian variety of dimension g is not known. An attempt to provide this list is made by the notion of (p, t, a)-inertial groups introduced in [SZ05].

Definition 1.1 ([SZ98])**.** Let p be a prime number or $p = 0$ and t, a positive integers. A finite group G is said to be (p, t, a)-inertial if it satisfies the two following conditions :

- (i) If $p = 0$ then G is cyclic otherwise G is a semi-product $\Gamma_p \rtimes \mathbf{Z}/n\mathbf{Z}$ with Γ_p a p-group and n an integer prime to p.
- (ii) For all primes $\ell \neq p$ there is an injection

$$G \hookrightarrow \mathrm{GL}_t(\mathbf{Z}) \times \mathrm{Sp}_{2a}(\mathbf{Q}_\ell)$$

such that the projection map onto the first factor is independent of ℓ and the characteristic polynomial of the projection of any element onto the second factor has integer coefficients independent of ℓ.

It follows from the definition that the character χ_ℓ of the $\mathbf{Q}_\ell$ representation given by the projection on the second factor has integer values and is independent of ℓ.

A precise statement is given by question 1.13 of *loc. cit.*. Specifically, the question is to realize a (p, t, a)-inertial group G as the finite monodromy group of an abelian variety A of dimension $g = t + a$ at a place of residue characteristic p. In the same paper they show that the set of (p, t, a)-inertial groups with $t + a = 2$ are realized as finite monodromy groups of abelian surfaces over local fields of equal characteristic p. Further Chrétien and Matignon in [CM13] have shown that this list is also realized with abelian surfaces over number fields by an ad hoc construction for the last unrealized $(2, 0, 2)$-inertial group.

1.1.2. In [Phi24], the first criterion for realizability of finite groups as finite monodromy groups of abelian varieties over number fields in arbitrary fixed dimension is given. This criterion provides a path to answer the question of Silverberg and Zarhin. From Théorème 1.1 of [Phi24], it is enough to show that a (p,t,a)-inertial group G can be embedded as a subgroup of the automorphism group of a semi-abelian variety A_0 of toric rank t and abelian rank a over a finite field of characteristic p which is stable by some polarization. The result for abelian surfaces can thus be recovered directly from this theorem and [SZ05].

A complete answer for p-groups follows.

Theorem 1.2. *Let G be a finite p-group. Then G is a (p,t,a)-inertial group if and only if it is the finite monodromy group of an abelian variety A of dimension $g = t + a$ over a p-adic field K such that for an extension L/K with A_L semi-stable the toric rank of the reduction of A_L is t and its abelian rank is a.*

This is obtained as a corollary of the following result.

Theorem 1.3. *Let G be a finite p-group. which is a (p,t,a)-inertial group. Then there is a polarized semi-abelian variety A_0 over a finite field of characteristic p and an embedding $\iota\colon G \hookrightarrow \operatorname{Aut}(A_0, \lambda_0)$ such that ι_ℓ is isomorphic to $T_\ell \circ \iota$ for all $\ell \neq p$ where T_ℓ is the injection $\operatorname{Aut} A_0 \hookrightarrow \operatorname{Aut} T_\ell A$.*

The proof of Theorem 1.2 is then given by applying Théorème 3.10 of [Phi24] to the embedding given by ι. The fact that p-groups are ramification groups is lemma 3.5 of [Phi22b]. The converse is given, for instance, by section 5 of [SZ98].

1.1.3. The first part of the paper is devoted to the representations of algebras. We show the unicity part of the result. The second part gives the proof of our main result Theorem 1.3. The first step is a reduction to the case of $(p,0,a)$-inertial groups, in other words, the toric part plays no significant role. We then provide abelian varieties over finite fields with embeddings into their automorphism groups from the finite p-groups we consider using the results of Roquette [Roq58] and Honda-Tate theory.

2. On the rational representation of finite groups and algebras

2.1. Unicity of rational representations of algebras

2.1.1. Let E be a simple, finite $\mathbf{Q}$-algebra and $F \subset E$ its center. We shall deal with rational representations of E in the following sense.

Definition 2.1. A representation $\rho\colon E \to \mathrm{M}_{2g}(\mathbf{Q}_\ell)$ is said to be rational if the characteristic polynomials of the elements of E have rational coefficients.

The aim of this section is to show some unicity results for such rational representation and in the general case, where E is not assumed simple, when they come from ℓ-adic Tate module maps of abelian varieties over finite fields.

Proposition 2.2. *Let $g \in \mathbf{N} \setminus \{0\}$. There is at most one rational $\mathbf{Q}_\ell$-representation of E of dimension $2g$ up to isomorphism.*

Proof. Let $\rho\colon E \to \mathrm{M}_{2g}(\mathbf{Q}_\ell)$ be a rational representation of E. There is a factorization

$$E \to E \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \to \mathrm{M}_{2g}(\mathbf{Q}_\ell)$$

and E itself acts on $\mathbf{Q}_\ell^{2g}$. The algebra $E \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$ decomposes into a product $\prod_{i=1}^{r} E_i$ where E_i is central simple over F_i with $F \otimes_{\mathbf{Q}} \mathbf{Q}_\ell = \prod_{i=1}^{r} F_i$.

In particular there is a unique class of non trivial simple E_i-module up to isomorphism which we denote by S_i. By the structure theorem of Wedderbun we have $E_i = \mathrm{M}_{r_i}(D_i)$ with D_i a division algebra central over F_i. Considering the different dimensions we note $s_i = \dim_{F_i} S_i$, $a_i^2 = \dim_{F_i} E_i$, $d_i^2 = \dim_{F_i} D_i$. It follows that $a_i = d_i r_i$, $S_i = D_i^{r_i}$, $s_i = d_i^2 r_i$ and that $\dim_{\mathbf{Q}_\ell} S_i = d_i^2 r_i [F_i : \mathbf{Q}_\ell]$ depends only on E.

Now we also have

$$\mathbf{Q}_\ell^{2g} = \bigoplus_{i=1}^{r} S_i^{n_i}.$$

We will show that n_i depend only on E and g. By Lemma 2.1 of [Tam95] the $F \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$-module $\mathbf{Q}_\ell^{2g}$ is free, that is there is some integer m such that

$$\mathbf{Q}_\ell^{2g} \simeq (\prod_{i=1}^{r} F_i)^m.$$

A simple computation gives

$$m = \frac{2g}{[F : \mathbf{Q}]}.$$

It also follows that $S_i^{n_i} = F_i^m$ as F_i-vector space and thus $s_i n_i = m$ which gives that n_i does not depend on ρ but only on g and E. □

The following corollary thus follows directly.

Corollary 2.3. *Let k be a field, g a positive integer and E a simple finite $\mathbf{Q}$-algebra. Let $\mathcal{A}$ be the set of abelian varieties A over $\overline{k}$ such that* $\operatorname{End} A \otimes \mathbf{Q} \simeq E$ *taien up to isogeny. Then, the image of the map*

$$\mathcal{A} \longrightarrow \{\rho\colon E \to \mathrm{M}_{2g}(\mathbf{Q}_\ell) \mid \text{ rational and faithful}\}/\sim$$

induced by taking the representation given by the ℓ-adic Tate module has at most one element.

We will now specify our treatment to the case of finite fields but generalize to any finite algebra over $\mathbf{Q}$.

First let us consider again the case where $E = \mathrm{M}_n(D)$ is simple. In that case, we know from Tate's theorem that if $E = \operatorname{End} A \otimes \mathbf{Q}$ for some abelian variety A over a finite field then $2 \dim A = ndf$ where $f = [F : \mathbf{Q}]$ with F the center of D and $d^2 = \dim_F D$. In particular, the dimension g of the variety is completely determined by E.

Second let us remark that since we are working up to isogeny, we can read the isogeny decomposition of A on $\operatorname{End} A \otimes \mathbf{Q}$. So let us consider E a finite algebra over $\mathbf{Q}$ such that $E = \prod_{i=1}^{r} E_i$ with E_i simple over $\mathbf{Q}$ such that $E_i = \mathrm{M}_{n_i}(D_i)$ for some skew field D_i and denote by f_i, d_i the corresponding invariants. We thus have that A is isogenous to a product of abelian varieties $A_1, \dots, A_r$ such that $\operatorname{End} A_i \otimes \mathbf{Q} \simeq E_i$ and A_i is of dimension $g_i = 1/2 \cdot n_i d_i f_i$.

Both remarks leads to the following theorem, completing the previous corollary in the case of finite fields.

Theorem 2.4. *Let k be a finite field, g a positive integer and E a finite $\mathbf{Q}$-algebra. Let $\mathcal{A}$ be the set of abelian varieties A over $\overline{k}$ such that* $\operatorname{End} A \otimes \mathbf{Q} \simeq E$ *taken up to isogeny. Then, the image of the map*

$$\mathcal{A} \longrightarrow \{\rho\colon A \to \mathrm{M}_{2g}(\mathbf{Q}_\ell) \mid \ \textit{rational and faithful}\}/\sim$$

factors through the set

$$\{\rho\colon A \to \mathrm{M}_{2g}(\mathbf{Q}_\ell) \mid \rho = \prod_{i=1}^{r} \rho_i,\ \rho_i\colon E_i \to \mathrm{M}_{2g_i}(\mathbf{Q}_\ell),\ g_i = n_i d_i f_i \ \textit{and} \ \sum_{i=1}^{r} g_i = g\}.$$

In particular, the image is always empty except possibly for $g = \sum_{i=1}^{r} g_i$ where it has at most one element.

3. The realization of (p, t, a)-inertial groups as finite monodromy groups

3.1. Some context and notations

3.1.1. The notion of polarized semi-abelian variety is the one from [Phi24], based on the chapter 2 of [FC90]. We recall it here.

Definition 3.1. Let A_0 and A_0^t be semi-abelian varieties over a finite field. A polarization $\lambda_0\colon A_0 \to A_0^t$ is a map of semi abelian varieties

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_0 & \longrightarrow & A_0 & \xrightarrow{p} & B_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda_{T_0}} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_{B_0}} & & \\ 0 & \longrightarrow & T_0^t & \longrightarrow & A_0^t & \xrightarrow{p} & B_0^\vee & \longrightarrow & 0 \end{array}$$

such that the induced maps λ_{T_0} is an isogeny and λ_{B_0} is a polarization. A semi-abelian variety A_0 equipped with a polarization λ_0 is said to be polarized.

In the next sections of the text, we will conserve the notations T_0 and B_0 for the maximal torus and abelian quotient of a semi-abelian variety A_0.

3.2. A first reduction step

3.2.1. We show that, in order to realize a (p, t, a)-inertial group G as a finite monodromy group, it is enough to realize its projection as a subgroup of the automorphism group of an abelian variety over a finite field. A (p, t, a)-inertial group G is given with embeddings

$$\iota_\ell\colon G \longrightarrow \mathrm{GL}_t(\mathbf{Z}) \times \mathrm{Sp}_{2a}(\mathbf{Q}_\ell)$$

for all $\ell \neq p$. We denote by p_ℓ the composition of ι_ℓ with the projection on the factor $\mathrm{Sp}_{2a}(\mathbf{Q}_\ell)$.

Lemma 3.2. *Let G be a (p, t, a)-inertial group. Then the groups $p_\ell(G)$ are independent of ℓ. In particular, $p_\ell(G)$ is $(p, 0, a)$-inertial.*

Proof. Let $\ell \neq p$ be fixed. The character of the representation of G on $\mathbf{Q}_\ell^{2a}$ is independent of ℓ. In particular, its kernel $H_\ell \subset G$ is also independent of ℓ. It follows that $p_\ell(G) \simeq G/H_\ell$ is independent of ℓ.

We thus have induced representations of $p_\ell(G)$ on $\mathbf{Q}_{\ell'}^{2a}$ for all $\ell' \neq p$, which satisfies the conditions of definition 1.1 with $t = 0$. ☐

Proposition 3.3. *Let G be a (p,t,a)-inertial group. Then there is a polarized semi-abelian variety (A_0,λ_0) with toric rank t and abelian rank a and an embedding $\iota\colon G \hookrightarrow \mathrm{Aut}(A_0,\lambda_0)$ such that ι_ℓ is isomorphic to $T_\ell \circ \iota$ if and only if the same statement holds for $p_\ell(G)$ as $(p,0,a)$-inertial group.*

Proof. Assume first there is a polarized semi-abelian variety (A_0,λ_0) and an embedding $\iota\colon G \hookrightarrow \mathrm{Aut}(A_0,\lambda_0)$ with the desired property. We first remark that this leads to an embedding

$$G \hookrightarrow \mathrm{Aut}\, T_0 \times \mathrm{Aut}(B_0,\lambda_{B_0})$$

with the notations of section 3.1. It is clear that the map T_ℓ respects this decomposition so that the the second projection induces an embedding $p_\ell(G) \hookrightarrow \mathrm{Aut}(B_0,\lambda_{B_0})$ which concludes.

For the converse, we have an embedding $\iota'\colon p_\ell(G) \hookrightarrow \mathrm{Aut}(B_0,\lambda_{B_0}$ for some polarized abelian variety B_0 of dimension a over a finite field which satisfies that $T_\ell \circ \iota'$ is isomorphic to $p_\ell \circ \iota_\ell$ for all $\ell \neq p$. Consider now the semi-abelian variety $\mathbf{G}_m^t \times B_0$ with the product polarization $(\mathrm{id},\lambda_{B_0})$. Its automorphism group is given by

$$\mathrm{GL}_t(\mathbf{Z}) \times \mathrm{Aut}(B_0,\lambda_{B_0})$$

And since G embeds as a subset of $\mathrm{GL}_t \times p_\ell(G)$ it has an embedding in $\mathrm{Aut}(A_0,\lambda_0)$ which satisfies the conditions. □

From this result we should now only be concerned about constructing embeddings into the automorphism groups of polarized abelian varieties over finite fields.

3.3. The case of p-groups

3.3.1. Let G be a finite p-group. Let us first recall some basic results on group algebras in order to introduce notations and context. I t follows from the results of Roquette in [Roq58] that the rational group algebra $\mathbf{Q}[G]$ is a product of matrix algebras over fields with the only possible skew fields appearing being of the form $F \otimes \mathbf{H}$ where F is a finite field extension of $\mathbf{Q}$ and $\mathbf{H}$ the standard quaternion algebra over $\mathbf{Q}$. This last case can only happen when p is even. More precisely, the fields appearing in such a decomposition are of the form $\mathbf{Q}(\mu_{p^n})$ for some $n \geq 0$.

3.3.2. We can now prove our main result.

Proof. (of Theorem 1.3)

By Proposition 3.3 it is enough to consider G a p-group which is $(p,0,a)$-inertial. Now remark that it is enough to show that G embeds in a polarized abelian variety of dimension $g \leq a$ over some finite field. Indeed, it is clear that any group that has such an embedding also has one into a polarized abelian variety of dimension a.

For some $\ell \neq p$ consider the finite $\mathbf{Q}$-algebra E generated by the image of G in $\mathrm{M}_{2a}(\mathbf{Q}_\ell)$. We have a decomposition

$$E \simeq \prod_{i=1}^{r} \mathrm{M}_{n_i}(D_i)$$

where D_i is a central simple algebra over F_i, a finite extension of $\mathbf{Q}$. Moreover, since E is a quotient of $\mathbf{Q}[G]$ we know from the previous description that either $D_i = F_i = \mathbf{Q}(\mu_{p^n})$ or D_i is a base change of the standard quaternion algebra and F_i is of the previous form. In any case, the fields F_i are CM fields.

The existence of Weil q-integers in the CM fields F_i with the desired properties follow for instance from Proposition A.4.8.4 of [CCO14] which by Honda-Tate theory provides abelian varieties A'_i over a finite field of characteristic p such that ${A'_i}^{n_i}$ has endomorphism algebra $\mathrm{M}_{n_i}(D_i)$. Since G embeds in the order generated by $\mathbf{Z} \cdot G$ of E, by Theorem 3.13 of [Wat69] there is an abelian variety A such that $\operatorname{End} A \otimes \mathbf{Q} \simeq E$ and $G \hookrightarrow \operatorname{Aut} A$.

Moreover, the symplectic structure provides E with a positive involution compatible with the embedding of G. This gives us the desired polarization.

The unicity part of the statement follow directly from Theorem 2.4. It also provides that $\dim A \leq a$. □

References

[CCO14] C.-L. Chai, B. Conrad, and F. Oort, *Complex multiplication and lifting problems* (Math. Surv. Monogr.). Providence, RI: American Mathematical Society (AMS), 2014, vol. 195 (cit. on p. 6).

[CM13] P. Chrétien and M. Matignon, "Maximal wild monodromy in unequal characteristic," *J. Number Theory*, vol. 133, no. 4, pp. 1389–1408, 2013 (cit. on p. 1).

[FC90] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties* (Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]). Springer-Verlag, Berlin, 1990, vol. 22, pp. xii+316, With an appendix by David Mumford (cit. on p. 4).

[SGA7.1] A. Grothendieck, *Modèles de Néron et monodromie.* Sémin. Géom. Algébrique, Bois-Marie 1967–1969, SGA 7 I, Exp. No. 9, Lect. Notes Math. 288, 313-523. Avec un appendice par M. Raynaud, 1972 (cit. on p. 1).

[Phi22a] S. Philip, "On the semi-stability degree for abelian varieties," *Bull. Lond. Math. Soc.*, vol. 54, no. 6, pp. 2174–2187, 2022 (cit. on p. 1).

[Phi22b] ——, "Variétés abéliennes CM et grosse monodromie finie sauvage," *J. Number Theory*, vol. 240, pp. 163–195, 2022 (cit. on pp. 1, 2).

[Phi24] ——, "Groupes de monodromie finie des variétés abéliennes," *(submitted) arxiv:2412.11900*, p. 33, 2024 (cit. on pp. 1, 2, 4).

[Roq58] P. Roquette, "Realisierung von Darstellungen endlicher nilpotenter Gruppen.," *Arch. Math.*, vol. 9, pp. 241–250, 1958 (cit. on pp. 2, 5).

[SZ98] A. Silverberg and Y. G. Zarhin, "Subgroups of inertia groups arising from abelian varieties," *J. Algebra*, vol. 209, no. 1, pp. 94–107, 1998 (cit. on pp. 1, 2).

[SZ05] ——, "Inertia groups and abelian surfaces," *J. Number Theory*, vol. 110, no. 1, pp. 178–198, 2005 (cit. on pp. 1, 2).

[Tam95] A. Tamagawa, "The Eisenstein quotient of the Jacobian variety of a Drin'feld modular curve.," *Publ. Res. Inst. Math. Sci.*, vol. 31, no. 2, pp. 203–246, 1995 (cit. on p. 3).

[Wat69] W. C. Waterhouse, "Abelian varieties over finite fields," *Ann. Sci. École Norm. Sup. (4)*, vol. 2, pp. 521–560, 1969 (cit. on p. 6).