The Grothendieck construction and models for dependent types

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We show here that presheaves and the Grothendieck construction gives a natural model for dependent types, in the form of a contextual category. This is a variant of Hofmann's presheaf models. The difference is briefly that in Hofmann's models the category of contexts is the presheaf category $\mathrm{PSh}(\mathbb{C})$, whereas in the model here presented, the context objects are iterated Grothendieck constructions $\int(\cdots\int(\int(\mathbb{C},P_1),P_2),\ldots,P_n)$ and the context morphisms are functors with some restriction. [1]

1 Iterated Grothendieck constructions

Let $\mathbb{C}$ be a small category. Let P be a presheaf on $\mathbb{C}$. The *category of elements of* P, denoted

$$\Sigma(\mathbb{C}, P),$$

consists of objects (a, x) where a is an object in $\mathbb{C}$ and x is an element of $P(a)$. A morphism $\alpha : (a, x) \longrightarrow (b, y)$ is a $\mathbb{C}$-morphism $\alpha : a \longrightarrow b$, such that $P(\alpha)(y) = x$. $\Sigma(\mathbb{C}, P)$ is the well-known *Grothendieck construction* [3] and is usually denoted

$$\int_{\mathbb{C}} P \qquad \text{or perhaps} \qquad \int(\mathbb{C}, P).$$

This category is again small. There is a projection functor $\pi_P = \pi : \Sigma(\mathbb{C}, P) \longrightarrow \mathbb{C}$ defined by $\pi(a, x) = a$ and $\pi(\alpha) = \alpha$, for $\alpha : (a, x) \longrightarrow (b, y)$.

It is well-known that the Grothendieck construction gives the following equivalence

$$\mathrm{PSh}(\mathbb{C})/P \simeq \mathrm{PSh}(\Sigma(\mathbb{C}, P)).$$

[1] Henrik Forssell observed after seeing the first version of these notes (dated February 28, 2013) that these restrictions amount to imposing a fibration condition, so that new iterated presheaf category is actually equivalent to the standard presheaf category. See Section 3.

Thus objects Q over P can be regarded as presheaves over $\Sigma(\mathbb{C}, P)$. This suggests a relation to semantics of dependent types [2]. One can iterate the Grothendieck construction as follows. Write

$$\Sigma(\mathbb{C}) = \mathbb{C}$$

$$\Sigma(\mathbb{C}, P_1, P_2, \ldots, P_n) = \Sigma(\Sigma(\mathbb{C}, P_1, P_2, \ldots, P_{n-1}), P_n).$$

Here $P_{k+1} \in \mathrm{PSh}(\Sigma(\mathbb{C}, P_1, \ldots, P_k))$ for each $k = 0, \ldots, n-1$. Note that

$$\pi_{P_{k+1}} : \Sigma(\Sigma(\mathbb{C}, P_1, P_2, \ldots, P_k), P_{k+1}) \longrightarrow \Sigma(\mathbb{C}, P_1, P_2, \ldots, P_k).$$

Using these projections, define the iterated first projection functor

$$\pi^*_{P_1,P_2,\ldots,P_n} =_{\mathrm{def}} \pi_{P_1} \circ \pi_{P_2} \circ \cdots \circ \pi_{P_n} : \Sigma(\mathbb{C}, P_1, P_2, \ldots, P_n) \longrightarrow \mathbb{C}.$$

We explicate these constructions to see the connection to contexts of type theory. The objects of the category $\Sigma(\mathbb{C}, P_1, P_2, \ldots, P_n)$ have the form $(\cdots((a, x_1), x_2), \ldots, x_n)$ but we shall write them as $(a, x_1, \ldots, x_n)$. Thus $a \in \mathbb{C}$, $x_1 \in P_1(a)$, $x_2 \in P_2(a, x_1)$, $x_3 \in P_3(a, x_1, x_2)$, $\ldots$, $x_n \in P_n(a, x_1, \ldots, x_{n-1})$.

Proposition 1.1. *A morphism*

$$\alpha : (a, x_1, \ldots, x_n) \longrightarrow (b, y_1, \ldots, y_n)$$

in $\Sigma(\mathbb{C}, P_1, P_2, \ldots, P_n)$ *is given by a morphism* $\alpha : a \longrightarrow b$ *in* $\mathbb{C}$ *such that*

$$P_1(\alpha)(y_1) = x_1, P_2(\alpha)(y_2) = x_2, \ldots, P_n(\alpha)(y_n) = x_n.$$

Proof. Induction on n. For $n = 0$, this is trivial, since the condition is void. Suppose it holds for n. A morphism $\alpha : (a, x_1, \ldots, x_{n+1}) \longrightarrow (b, y_1, \ldots, y_{n+1})$ is by definition a morphism $\alpha : (a, x_1, \ldots, x_n) \longrightarrow (b, y_1, \ldots, y_n)$ such that $P_{n+1}(\alpha)(y_{n+1}) = x_{n+1}$. By the induction hypothesis α is a morphism $a \longrightarrow b$ such that

$$P_1(\alpha)(y_1) = x_1, P_2(\alpha)(y_2) = x_2, \ldots, P_n(\alpha)(y_n) = x_n.$$

Hence also

$$P_1(\alpha)(y_1) = x_1, P_2(\alpha)(y_2) = x_2, \ldots, P_n(\alpha)(y_n) = x_n, P_{n+1}(y_{n+1}) = x_{n+1}$$

as required. □

2 A category with attributes

Let $\mathbb{C}$ be a small category. We first define the category of contexts. Define a category $\mathrm{MPSh}(\mathbb{C}) = M$ to have as objects finite sequences $\overline{P} = [P_1, \dots, P_n]$, $n \geq 0$, such that $P_{k+1} \in \mathrm{PSh}(\Sigma(\mathbb{C}, P_1, \dots, P_k))$ for each $k = 0, \dots, n-1$. (In commutative diagrams we omit the rectangular brackets for typographical reasons.) We write $\Sigma(\mathbb{C}, \overline{P})$, or $\Sigma(\overline{P})$ when $\mathbb{C}$ is clear from the situation, for $\Sigma(\mathbb{C}, P_1, \dots, P_n)$. Define the set of morphisms $\mathrm{Hom}_M(\overline{P}, \overline{Q})$ as a subset of the functors from $\Sigma(\overline{P})$ to $\Sigma(\overline{Q})$, as follows:

$$\mathrm{Hom}_M(\overline{P}, \overline{Q}) = \{f \in \Sigma(\overline{Q})^{\Sigma(\overline{P})} : \pi^*_{\overline{Q}} \circ f = \pi^*_{\overline{P}}\}. \tag{1}$$

Notice that since $\pi^*_{\overline{Q}}(\beta) = \beta$ and $\pi^*_{\overline{P}}(\alpha) = \alpha$ for all arrows β in $\Sigma(\overline{Q})$, and α in $\Sigma(\overline{P})$, it holds for $f \in \mathrm{Hom}_M(\overline{P}, \overline{Q})$,

$$f(\alpha) = \alpha.$$

To see that M is category we need only to check that it has all identity morphisms and is closed under composition. If $f \in \mathrm{Hom}_M(\overline{P}, \overline{Q})$ and $g \in \mathrm{Hom}_M(\overline{Q}, \overline{R})$, then

$$\pi^*_{\overline{R}} \circ g \circ f = \pi^*_{\overline{Q}} \circ f = \pi^*_{\overline{P}}.$$

Thus closure under composition is clear. Further,

$$\pi^*_{\overline{P}} \circ \mathrm{id}_{\Sigma(\overline{P})} = \pi^*_{\overline{P}}$$

so $\mathrm{id}_{\Sigma(\overline{P})} \in \mathrm{Hom}_M(\overline{P}, \overline{P})$. Note that $\mathrm{Hom}_M(\overline{P}, [])$ consists only of the functor $\pi^*_{\overline{P}}$ since if $f \in \mathrm{Hom}_M(\overline{P}, [])$, then $\pi^*_{[]} \circ f = \pi^*_{\overline{P}}$. But $\pi^*_{[]}$ is the identity functor on $\mathbb{C}$, so $f = \pi^*_{\overline{P}}$. It seems reasonable to call $\mathrm{MPSh}(\mathbb{C})$ the *multivariable presheaves over* $\mathbb{C}$. We conclude from this

Theorem 2.1. $\mathrm{MPSh}(\mathbb{C})$ *is a category with terminal object* $[]$*, whenever* $\mathbb{C}$ *is a small category.* □

The objects form a tree structure via the immediate extension relation: $\overline{P} \lhd \overline{Q}$ if and only if $\overline{Q} = [\overline{P}, S]$ for some $S \in \mathrm{PSh}(\overline{P})$.

The following is immediate in view of the definition (1):

Lemma 2.2. *Let* $M = \mathrm{MPSh}(\mathbb{C})$*. The hom-set* $\mathrm{Hom}_M(\overline{P}, [Q_1, \dots, Q_{m+1}])$ *consists of those functors* $f : \Sigma(\overline{P}) \longrightarrow \Sigma(Q_1, \dots, Q_{m+1})$ *such that* $\pi_{Q_{m+1}} \circ f \in \mathrm{Hom}_M(\overline{P}, [Q_1, \dots, Q_m])$*.* □

The restriction in the hom-sets of $\mathrm{MPSh}(\mathbb{C})$ yields the following characterization.

Lemma 2.3. *For* $P, Q \in \mathrm{PSh}(\mathbb{C})$ *there is a bijection*

$$\mathrm{Hom}_{\mathrm{MPSh}(\mathbb{C})}([P], [Q]) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathbb{C})}(P, Q).$$

Proof. For $f \in \mathrm{Hom}_{\mathrm{MPSh}(\mathbb{C})}([P],[Q])$ we have by the restriction $\pi^*_{[Q]} \circ f = \pi^*_{(P)}$ that $f(a,x) = (a, \hat{f}_a(x))$ and $f(\alpha) = \alpha$. Thus if $x \in P(a)$, then $(a,x) \in \Sigma(\mathbb{C},P)$, so $\hat{f}_a(x) \in Q(a)$. This gives a family of maps $\hat{f}_a : P(a) \longrightarrow Q(a)$, $a \in \mathbb{C}$. We check that they form a natural transformation $\tau : P \longrightarrow Q$. Suppose $y \in P(b)$ and $\alpha : a \longrightarrow b$. Then $(a, P(\alpha)(y)) \in \Sigma(\mathbb{C},P)$ and $\alpha : (a, P(\alpha)(y)) \longrightarrow (b,y)$, so $f(\alpha) : f(a, P(\alpha)(y)) \longrightarrow f(b,y)$. This means

$$\alpha : (a, \hat{f}_a(P(\alpha)(y))) \longrightarrow (b, \hat{f}_b(y)).$$

Hence $Q(\alpha)(\hat{f}_b(y)) = \hat{f}_a(P(\alpha)(y))$, which verifies the naturally condition. Write $\hat{f}$ for the natural transformation constructed from f.

Conversely, suppose that $\tau : P \longrightarrow Q$ is a natural transformation. Define a functor $f : \Sigma(\mathbb{C},P) \longrightarrow \Sigma(\mathbb{C},Q)$ by

$$f(a,x) = (a, \tau_a(x)) \qquad f(\alpha) = \alpha.$$

Note that if $x \in P(a)$, then $\tau_a(x) \in Q(a)$, so it is well-defined on objects. If $\alpha : (a,x) \longrightarrow (b,y)$ then $P(\alpha)(y) = x$. Now we need to check that $f(\alpha) = \alpha : (a, \tau_a(x)) \longrightarrow (b, \tau_b(y))$, i.e. that $Q(\alpha)(\tau_b(y)) = \tau_a(x)$. Inserting $x = P(\alpha)(y)$, this is

$$Q(\alpha)(\tau_b(y)) = \tau_a(P(\alpha)(y))$$

which is exactly the naturality of τ. So f is well-defined on arrows as well. The functoriality of f is clear. Write $[\tau] = f$ for the morphism so constructed from τ.

Now for $g \in \mathrm{Hom}_{\mathrm{MPSh}(\mathbb{C})}([P],[Q])$,

$$[\hat{g}](a,x) = (a, \hat{g}_a(x)) = g(a,x)$$

and $[\hat{g}](\alpha) = \alpha = g(\alpha)$. Thus $[\hat{g}] = g$. Further for $\sigma \in \mathrm{Hom}_{\mathrm{PSh}(\mathbb{C})}(P,Q)$, we wish to prove $\widehat{[\sigma]} = \sigma$. For $(a,x) \in \Sigma(P)$, we have by definition

$$[\sigma](a,x) = (a, \sigma_a(x)),$$

and further by definition

$$\widehat{[\sigma]}_a(x) = \sigma_a(x).$$

Thus

$$\widehat{[\sigma]} = \sigma$$

and this shows that the operations are mutual inverses. □

The following may be considered as a secondary Yoneda embedding.

Theorem 2.4. $[\cdot] : \mathrm{PSh}(\mathbb{C}) \longrightarrow \mathrm{MPSh}(\mathbb{C})$ *is a full and faithful functor.*

Proof. In view of Lemma 2.3 we need only to check that the operation $[\cdot]$ is functorial. Consider identity natural transformation $\iota : P \longrightarrow P$ given by $\iota_a = \mathrm{id}_{P(a)}$. We have

$$[\iota](a,x) = (a, \iota_a(x)) = (a,x) \qquad [\iota](\alpha) = \alpha,$$

so clearly (ι) is the identity $[P] \longrightarrow [P]$. Suppose that $\sigma : P \longrightarrow Q$ and $\tau : Q \longrightarrow R$ are natural transformations. We have for objects (a,x) in $\Sigma(P)$:

$$[\tau \cdot \sigma](a,x) = (a, (\tau \cdot \sigma)_a(x)) = (a, \tau_a(\sigma_a(x)))$$

and on the other hand we get the same result evaluating

$$([\tau] \circ [\sigma])(a,x) = [\tau]([\sigma](a,x)) = [\tau](a, \sigma_a(x)) = (a, \tau_a(\sigma_a(x))).$$

For a morphism $\alpha : (a,x) \longrightarrow (b,y)$ in $\Sigma(P)$, we have by definition

$$[\tau \cdot \sigma](\alpha) = \alpha = [\tau](\alpha) = [\tau]((([\sigma])(\alpha)) = ([\tau] \circ [\sigma])(\alpha).$$

This means that $[\cdot]$ is functorial. $\square$

Composing the Yoneda embedding with the secondary embedding we get:

Corollary 2.5. $[\cdot] \circ \mathbf{y} : \mathbb{C} \longrightarrow \mathrm{MPSh}(\mathbb{C})$ *is a full and faithful functor.* $\square$

Theorem 2.6. *Let* $\overline{P} = [P_1, \ldots, P_n]$ *and* $\overline{Q} = [Q_1, \ldots, Q_m]$ *be objects of* $\mathrm{MPSh}(\mathbb{C})$*. An* $\mathrm{MPSh}(\mathbb{C})$*-morphism* $f : \overline{P} \longrightarrow \overline{Q}$ *is given by* m *components* $f_1, \ldots, f_m$*, which are such that for objects* $(a, \overline{x})$ *of* $\Sigma(\overline{P})$*:*

$$f(a,\overline{x}) = (a, f_1(a,\overline{x}), \ldots, f_m(a,\overline{x})) \tag{2}$$

and

$$f_1(a,\overline{x}) \in Q_1(a), f_2(a,\overline{x}) \in Q_2(a, f_1(a,\overline{x})), \ldots, f_m(a,\overline{x}) \in Q_m(a, f_1(a,\overline{x}), \ldots, f_{m-1}(a,\overline{x})) \tag{3}$$

Moreover for each morphism $\alpha : (a,\overline{x}) \longrightarrow (b,\overline{y})$ *in* $\Sigma(\overline{P})$*, the following* naturality equations *hold*

$$\begin{array}{rcl} Q_1(\alpha)(f_1(b,\overline{y})) & = & f_1(a, P_1(\alpha)(y_1), \ldots, P_n(\alpha)(y_n)) \\ & \vdots & \\ Q_m(\alpha)(f_m(b,\overline{y})) & = & f_m(a, P_1(\alpha)(y_1), \ldots, P_n(\alpha)(y_n)) \end{array}$$

Proof. Induction on m. For $m = 0$, we have $f = \pi^*_{\overline{P}}$ since $[]$ is the terminal object. Now $\pi^*_{\overline{P}}(a,\overline{x}) = a$ and $\pi^*_{\overline{P}}(\alpha) = \alpha$. Since for $m = 0$ there are no side conditions or naturality equations, we are done. Suppose that the characterization holds for m. Let $f : \overline{P} \longrightarrow [Q_1, \ldots, Q_{m+1}]$ be a MPSh($\mathbb{C}$)-morphism. Write

$$f(a, \overline{x}) = (a, f_1(a, \overline{x}), \ldots, f_{m+1}(a, \overline{x})).$$

By the definition of the domain, (3) is satisfied for $m + 1$. According to Lemma 2.2 we have that $\pi_{Q_{m+1}} \circ f : \overline{P} \longrightarrow [Q_1, \ldots, Q_m]$ so applying the inductive hypothesis to this we get the naturality equations for $Q_1, \ldots, Q_m$. It remains to prove the naturally equation for Q_{m+1}. We have by definition

$$\begin{aligned} f(a, \overline{x}) &= ((\pi_{Q_{m+1}} \circ f)(a, \overline{x}), f_{m+1}(a, \overline{x})) \\ f(b, \overline{y}) &= ((\pi_{Q_{m+1}} \circ f)(b, \overline{y}), f_{m+1}(b, \overline{y})) \end{aligned}$$

and for a morphism $\alpha : (a, \overline{x}) \longrightarrow (b, \overline{y})$, we have in $\Sigma(\Sigma(\mathbb{C}, Q_1, \ldots, Q_m), Q_{m+1})$ the morphism

$$f(\alpha) = \alpha : ((\pi_{Q_{m+1}} \circ f)(a, \overline{x}), f_{m+1}(a, \overline{x})) \longrightarrow ((\pi_{Q_{m+1}} \circ f)(b, \overline{y}), f_{m+1}(b, \overline{y})).$$

This implies $Q_{m+1}(\alpha)(f_{m+1}(b, \overline{y})) = f_{m+1}(a, \overline{x})$. Since

$$\overline{x} = P_1(\alpha)(y_1), \ldots, P_n(\alpha)(y_n),$$

we are done.

Conversely, suppose that $f_1, \ldots, f_{m+1}$ are satisfying (3) and the naturally equations for $m + 1$. Thus these conditions are also satisfied for $f_1, \ldots, f_m$. Define

$$g(a, \overline{x}) = (a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x})).$$

By the inductive hypothesis $g : \overline{P} \longrightarrow [Q_1, \ldots, Q_m]$ is a morphism. Let

$$f(a, \overline{x}) = (g(a, \overline{x}), f_{m+1}(a, \overline{x})) \qquad f(\alpha) = \alpha.$$

Note that $\pi_{Q_{m+1}} \circ f = g$, so we need only to check that f is a functor

$$\Sigma(\mathbb{C}, P_1, \ldots, P_n) \longrightarrow \Sigma(\Sigma(\mathbb{C}, Q_1, \ldots, Q_m), Q_{m+1})$$

We have $f(a, \overline{x}) = (g(a, \overline{x}), f_{m+1}(a, \overline{x}))$ and $g(a, \overline{x}) \in \Sigma(\mathbb{C}, Q_1, \ldots, Q_m))$ so objects are sent to objects. Now since $f(\alpha) = \alpha$ the functoriality is automatic, and we need only to check that a morphism $\alpha : (a, \overline{x}) \longrightarrow (b, \overline{y})$ also forms a morphism

$$\alpha : (g(a, \overline{x}), f_{m+1}(a, \overline{x})) \longrightarrow (g(b, \overline{y}), f_{m+1}(b, \overline{y})).$$

Assume $\alpha : (a, \overline{x}) \longrightarrow (b, \overline{y})$. We have $\alpha : a \longrightarrow b$ and

$$\overline{x} = (P_1(\alpha)(y_1), \ldots, P_n(\alpha)(y_n)). \tag{4}$$

By the induction hypothesis $\alpha : g(a, \overline{x}) \longrightarrow g(b, \overline{y})$ is a morphism. Thus it is enough to show $Q_{m+1}(f_{m+1}(b, \overline{y})) = f_{m+1}(a, \overline{x}))$. But by the assumption and using (4) we get

$$\begin{aligned} Q_{m+1}(\alpha)(f_{m+1}(b, \overline{y})) &= f_{m+1}(a, P_1(\alpha)(y_1), \dots, P_n(\alpha)(y_n)) \\ &= f_{m+1}(a, \overline{x}). \end{aligned}$$

This concludes the proof. □

Remark 2.7. Note that for $m = n = 1$ the naturality equations become just the usual condition that f_1 is a natural transformation. It may be reasonable to call the conditions in the general case *multinaturality.*

Example 2.8. For $R \in \mathrm{PSh}(\Sigma(\mathbb{C}, \overline{P}))$, the projection functor π_R is morphism $[\overline{P}, R] \longrightarrow \overline{P}$ in $\mathrm{MPSh}(\mathbb{C})$.

Example 2.9. (Sections.) Let $Q \in \mathrm{PSh}(\Sigma(\mathbb{C}, \overline{P}))$, where $\overline{P} = [P_1, \dots, P_n]$. Consider a $\mathrm{MPSh}(\mathbb{C})$-morphism $s : \overline{P} \longrightarrow [\overline{P}, Q]$ which is a *section* of π_Q, that is, it satisfies $\pi_Q \circ s = \mathrm{id}_{\overline{P}}$. By Theorem 2.6 it follows that s is specified by s' such that

$$s(a, \overline{x}) = (a, \overline{x}, s'(a, \overline{x})),$$

where $s'(a, \overline{x}) \in Q(a, \overline{x})$ and $(a, \overline{x}) \in \Sigma(\mathbb{C}, \overline{P})$, and for $\alpha : (a, \overline{x}) \longrightarrow (b, \overline{y})$,

$$Q(\alpha)(s'(b, \overline{y})) = s'(a, P_1(\alpha)(y_1), \dots, P_n(\alpha)(y_n)).$$

For $n = 0$, this is

$$Q(\alpha)(s'(b)) = s'(a).$$

For any object $\overline{P}$ of $\mathrm{MPSh}(\mathbb{C})$ define the presheaf $\Sigma^*(\overline{P})$ on $\mathbb{C}$ by letting

$$\Sigma^*(\overline{P})(a) = \{(x_1, \dots, x_n) : x_1 \in P_1(a), \dots, x_n \in P_n(a, x_1, \dots, x_n)\},$$

and for $\alpha : b \longrightarrow a$, assigning

$$\Sigma^*(\overline{P})(\alpha)((x_1, \dots, x_n)) = (P_1(\alpha)(x_1), \dots, P_n(\alpha)(x_n)).$$

The following will give the *types* in a context $\overline{P}$. Define for each $\overline{P} \in \mathrm{MPSh}(\mathbb{C})$,

$$\mathrm{T}(\overline{P}) = \mathrm{PSh}(\Sigma(\mathbb{C}, \overline{P})).$$

For a morphism $f : \overline{Q} \longrightarrow \overline{P}$, and $S \in \mathrm{T}(\overline{P})$, let

$$\mathrm{T}(f)(S) = S \circ f \in \mathrm{T}(\overline{Q}).$$

Thus T is a contravariant functor. We write $S\{f\}$ for $S \circ f$.

For $Q \in \mathrm{T}$ define its set of *elements* as the sections of π_Q

$$\mathrm{E}(\overline{P}, Q) = \{s : \overline{P} \longrightarrow [\overline{P}, Q] : \pi_Q \circ s = \mathrm{id}_{\overline{P}}\}$$

These data give rise to a category with attributes.

Theorem 2.10. *Let $M = \mathrm{MPSh}(\mathbb{C})$ for a small category $\mathbb{C}$. For $S \in \mathrm{T}(\overline{P})$ and $f : \overline{Q} \longrightarrow \overline{P}$, the functor $q_{S,f} = q : [\overline{Q}, S \circ f] \longrightarrow [\overline{P}, S]$ defined by*

$$q(a, \overline{x}, u) = (f(a, \overline{x}), u) \qquad \textit{and} \qquad q(\alpha) = f(\alpha) \quad (\alpha : (a, \overline{x}, u) \longrightarrow (b, \overline{y}, v))$$

makes the following into a pullback square in M:

$$\begin{array}{ccc} \overline{Q}, S \circ f & \xrightarrow{q_{S,f}} & \overline{P}, S \\ {\scriptstyle \pi_{S \circ f}} \downarrow & & \downarrow {\scriptstyle \pi_S} \\ \overline{Q} & \xrightarrow[f]{} & \overline{P} \end{array} \tag{5}$$

Further, if $f = \mathrm{id}_{\Sigma(\overline{P})} : \overline{P} \longrightarrow \overline{P}$, then

$$q_{S, \mathrm{id}_{\Sigma(\overline{P})}} = \mathrm{id}_{\Sigma(\overline{P}, S)}. \tag{6}$$

Suppose $g : \overline{A} \longrightarrow \overline{Q}$, where

$$\begin{array}{ccc} \overline{A}, S \circ f \circ g & \xrightarrow{q_{S \circ f, g}} & \overline{Q}, S \circ f \\ {\scriptstyle \pi_{S \circ f \circ g}} \downarrow & & \downarrow {\scriptstyle \pi_{S \circ f}} \\ \overline{A} & \xrightarrow[g]{} & \overline{Q} \end{array} \tag{7}$$

is a pullback. Then

$$q_{S \circ f, g} \circ q_{S,f} = q_{S, f \circ g} \tag{8}$$

where the associated pullback to $q_{S, f \circ g}$ is

$$\begin{array}{ccc} \overline{A}, S \circ f \circ g & \xrightarrow{q_{S, f \circ g}} & \overline{P}, S \\ {\scriptstyle \pi_{S \circ f \circ g}} \downarrow & & \downarrow {\scriptstyle \pi_S} \\ \overline{A} & \xrightarrow[f \circ g]{} & \overline{P} \end{array} \tag{9}$$

Proof. For $\alpha : (a, \overline{x}, u) \longrightarrow (b, \overline{y}, v)$ in $[\overline{Q}, S \circ f]$ we have $\alpha : (a, \overline{x}) \longrightarrow (b, \overline{y})$, and since f is a morphism, this gives

$$f(\alpha) = \alpha : f(a, \overline{x}) \longrightarrow f(b, \overline{y}).$$

Moreover $(S \circ f)(\alpha)(v) = u$. Hence

$$q(\alpha) = \alpha : (f(a, \overline{x}), v) \longrightarrow (f(b, \overline{y}), u).$$

Since $q(\alpha) = \alpha$, q is clearly a functor. It remains to verify the final condition for q being a morphism, this amounts to checking

$$\pi^*_{[\overline{P},S]} \circ q = \pi^*_{[\overline{Q},S\circ f]},$$

i.e. $\pi^*_{\overline{P}} \circ \pi_S \circ q = \pi^*_{\overline{Q}} \circ \pi_{S\circ f}$. We have

$$(\pi^*_{\overline{P}} \circ \pi_S \circ q)(a,\overline{x}) = \pi^*_{\overline{P}}(\pi_S(q(a,\overline{x}))) = \pi^*_{\overline{P}}(f(a,\overline{x})) = \pi^*_{\overline{Q}}(a,\overline{x}),$$

where the last step uses that f is a morphism. Moreover

$$(\pi^*_{\overline{P}} \circ \pi_S \circ q)(\alpha) = \pi^*_{\overline{P}}(\pi_S(\alpha)) = \alpha = \pi^*_{\overline{Q}}(\alpha).$$

It is clear that (5) commutes. Suppose that $h : \overline{R} \longrightarrow \overline{Q}$ and $k : \overline{R} \longrightarrow [\overline{P}, S]$ are morphisms such that $f \circ h = \pi_S \circ k$. Define $t : \overline{R} \longrightarrow [\overline{Q}, S \circ f]$ by on objects $(a,\overline{x})$ letting

$$t(a,\overline{x}) = (h(a,\overline{x}), k_2(a,\overline{x})),$$

where $k(a,\overline{x}) = (k_1(a,\overline{x}), k_2(a,\overline{x}))$. We have $k_2(a,\overline{x}) \in S(k_1(a,\overline{x}))$. Now $f(h(a,\overline{x})) = \pi_S(k(a,\overline{x})) = k_1(a,\overline{x})$, so $k_2(a,\overline{x}) \in (S \circ f)(h(a,\overline{x}))$. Thus $t(a,\overline{x})$ is well-defined on objects. For an arrow $\alpha : (a,\overline{x}) \longrightarrow (b,\overline{y})$, we define (as usual)

$$t(\alpha) = \alpha.$$

Need to check that $\alpha : t(a,\overline{x}) \longrightarrow t(b,\overline{y})$, i.e. that

$$h(\alpha) = \alpha : h(a,\overline{x}) \longrightarrow h(b,\overline{y}) \text{ and } ((S \circ f)(h(\alpha)))(k_2(b,\overline{y})) = k_2(a,\overline{x}). \qquad (10)$$

The first statement of (10) follows since h is a functor. As k is a morphism we have $S(k_1(\alpha))(k_2(b,\overline{y})) = k_2(a,\overline{x})$, but

$$\begin{aligned} ((S \circ f)(h(\alpha)))(k_2(b,\overline{y})) &= S(f(h(\alpha)))(k_2(b,\overline{y})) \\ &= S(\pi_S(k(\alpha)))(k_2(b,\overline{y})) \\ &= S(k_1(\alpha))(k_2(b,\overline{y})) \\ &= k_2(a,\overline{x}). \end{aligned}$$

That t is functorial is trivial since $t(\alpha) = \alpha$. Next we check that t is a morphism $\overline{R} \longrightarrow [\overline{Q}, S \circ f]$, and for this it remains to verify that $\pi^*_{\overline{Q},S\circ f} \circ t = \pi^*_{\overline{R}}$. This amounts to checking $\pi^*_{\overline{Q}} \circ \pi_{S\circ f} \circ t = \pi^*_{\overline{R}}$. Now

$$(\pi^*_{\overline{Q}} \circ \pi_{S\circ f} \circ t)(a,\overline{x}) = \pi^*_{\overline{Q}}(h(a,\overline{x})) = \pi^*_{\overline{R}}(a,\overline{x})$$

where using in the last step, the fact that h is a morphism. Moreover, for $\mathbb{C}$-morphisms α

$$\begin{aligned}(\pi^*_{\overline{Q}} \circ \pi_{S \circ f} \circ t)(\alpha) &= \pi^*_{\overline{Q}}(\pi_{S \circ f}(\alpha)) \\ &= \pi^*_{\overline{Q}}(\pi_{S \circ f}(h(\alpha))) \\ &= \pi^*_{\overline{Q}}(h(\alpha)) \\ &= \pi^*_{\overline{R}}(\alpha)\end{aligned}$$

The last step used that h is a morphism. Further

$$q_{S,f}(t(a,\overline{x})) = (f(h(a,\overline{x})), k_2(a,\overline{x})) = (k_1(a,\overline{x}), k_2(a,\overline{x})) = k(a,\overline{x}) \qquad \pi_{S \circ f}(t(a,\overline{x})) = h(a,\overline{x}).$$

and

$$q_{S,f}(t(\alpha)) = f(t(\alpha)) = f(h(\alpha)) = k_1(\alpha) = k(\alpha) \qquad \pi_{S \circ f}(t(\alpha)) = \pi_{S \circ f}(h(\alpha)) = h(\alpha).$$

Thus t is a mediating morphism for the diagram. We check that it is unique: suppose that $t' : \overline{R} \longrightarrow [\overline{Q}, S \circ f]$ is such that

$$q_{S,f}(t'(a,\overline{x})) = k(a,\overline{x}) \qquad \pi_{S \circ f}(t'(a,\overline{x})) = h(a,\overline{x})$$

and

$$q_{S,f}(t'(\alpha)) = k(\alpha) \qquad \pi_{S \circ f}(t'(\alpha)) = h(\alpha). \tag{11}$$

Writing $t'(a,\overline{x}) = (t'_1(a,\overline{x}), t'_2(a,\overline{x}))$ we see that $q_{S,f}(t'(a,\overline{x})) = (f(t'_1(a,\overline{x})), t'_2(a,\overline{x})) = k(a,\overline{x}) = (k_1(a,\overline{x}), k_2(a,\overline{x}))$, and $\pi_{S \circ f}(t'(a,\overline{x})) = t'_1(a,\overline{x}) = h(a,\overline{x})$. Hence $t'(a,\overline{x}) = t(a,\overline{x})$. From (11) we get $g(t'(\alpha)) = k(\alpha)$ and $t'(\alpha) = h(\alpha)$. Thus also $t'(\alpha) = t(\alpha)$. □

Suppose that a pullback square as in (5) is given. For an element $t \in \mathrm{E}(\overline{P}, S)$ we have $\pi_S \circ t \circ f = f \circ \mathrm{id}_{\overline{Q}}$. Let $t\{f\} : \overline{Q} \longrightarrow [\overline{Q}, S \circ f]$ be the unique map such that

$$\pi_{S \circ f} \circ t\{f\} = \mathrm{id}_{\overline{Q}} \text{ and } q_{S,f} \circ t\{f\} = t \circ f.$$

Then $t\{f\} \in \mathrm{E}(\overline{Q}, S\{f\})$, which is the element obtained from t by carrying out the substitution f. What does this look like in its components? Suppose $\overline{Q} = [Q_1, \ldots, Q_n]$ and $\overline{P} = [P_1, \ldots, P_m]$. Write

$$f(a,\overline{x}) = (a, f_1(a,\overline{x}), \ldots, f_m(a,\overline{x})).$$

Moreover write

$$t(a,\overline{y}) = (a, \overline{y}, t'(a,\overline{y})).$$

By Theorem 2.10 above

$$t\{f\}(a,\overline{x}) = (a, \overline{x}, t'(a, f_1(a,\overline{x}), \ldots, f_m(a,\overline{x}))).$$

Question: What are the categorical closure conditions of MPSh($\mathbb{C}$) in analogy to the closure conditions of PSh($\mathbb{C}$) (which is a topos)?

3 Equivalence with standard presheaves

It was noted by Henrik Forssell that the functor $[] : \mathrm{PSh}(\mathbb{C}) \longrightarrow \mathrm{MPSh}(\mathbb{C})$ is actually an equivalence of categories. Its inverse can be constructed explicitly.

For a morphism $f : \overline{P} \longrightarrow \overline{Q}$ in $\mathrm{MPSh}(\mathbb{C})$ define a natural transformation

$$\Sigma^*(f) : \Sigma^*(\overline{P}) \longrightarrow \Sigma^*(\overline{Q})$$

by letting

$$\Sigma^*(f)_a((x_1, \ldots, x_n)) = (f_1(a, x_1, \ldots, x_n), \ldots, f_m(a, x_1, \ldots, x_n)).$$

Here $f_1, \ldots, f_m$ are as in Theorem 2.6.

Lemma 3.1. $\Sigma^* : \mathrm{MPSh}(\mathbb{C}) \longrightarrow \mathrm{PSh}(\mathbb{C})$ *is a functor.*

Proof. It is clear that Σ^* sends objects to objects. We check that it is also well-defined on arrows by verifying that $\Sigma^*(f)$ is a natural transformation for $f : \overline{P} \longrightarrow \overline{Q}$. Let $\alpha : b \longrightarrow a$ and $\overline{x} \in \Sigma^*(\overline{P})(a)$. Then by definition and since $\alpha : (b, \overline{P}(\alpha)(\overline{x})) \longrightarrow (a, \overline{x})$ we get by Theorem 2.6

$$\begin{aligned}
\Sigma^*(\overline{Q})(\alpha)(\Sigma^*(f)_a(\overline{x})) &= (Q_1(\alpha)(f_1(a, \overline{x})), \ldots, Q_m(\alpha)(f_m(a, \overline{x}))) \\
&= (f_1(b, \overline{P}(\alpha)\overline{x}), \ldots, f_m(b, \overline{P}(\overline{x}))) \\
&= \Sigma^*(f)_b(\overline{P}(\alpha)(\overline{x})) \\
&= \Sigma^*(f)_b(\Sigma^*(\overline{P})(\alpha)(\overline{x})
\end{aligned}$$

as required.

If f is the identity, then $f_k(a, \overline{x}) = x_k$ and hence $\Sigma^*(f)_a(\overline{x}) = \overline{x}$.

Suppose that $f : \overline{P} \longrightarrow \overline{Q}$ and $g : \overline{Q} \longrightarrow \overline{R}$ are morphisms and write

$$f(a, \overline{x}) = (a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x}))$$

and

$$g(a, \overline{y}) = (a, g_1(a, \overline{y}), \ldots, f_k(a, \overline{x}))$$

Then

$$\begin{aligned}
\Sigma^*(g)_a(\Sigma^*(f)_a(\overline{x})) &= \Sigma^*(g)_a(f_1(a, \overline{x}), \ldots, f_m(a, \overline{x})) \\
&= (g_1(a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x})), \ldots, g_k(a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x})))
\end{aligned}$$

But we have

$$\begin{aligned}
(g \circ f)(a, \overline{x}) &= g(f(a, \overline{x})) \\
&= g(a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x})) \\
&= (a, g_1(a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x})), \ldots, f_k(a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x})))
\end{aligned}$$

Hence

$$\Sigma^*(g)_a(\Sigma^*(f)_a(\overline{x})) = \Sigma^*(g \circ f)_a(\overline{x}))$$

as was to be proved. □

Theorem 3.2. *The functors* $[] : \mathrm{PSh}(\mathbb{C}) \longrightarrow \mathrm{MPSh}(\mathbb{C})$ *and* $\Sigma^* : \mathrm{MPSh}(\mathbb{C}) \longrightarrow \mathrm{PSh}(\mathbb{C})$ *form an equivalence of categories witnessed by the natural isomorphisms*

$$\varepsilon : \Sigma^*([-]) \longrightarrow \mathrm{Id}_{\mathrm{PSh}(\mathbb{C})} \text{ and } \eta : [\Sigma^*(-)] \longrightarrow \mathrm{Id}_{\mathrm{MPSh}(\mathbb{C})}$$

where

$$(\varepsilon_P)_a((x)) = x$$

and

$$\eta_{\overline{P}} : \Sigma(\mathbb{C}, \Sigma^*(P_1, \dots, P_n)) \longrightarrow \Sigma(\mathbb{C}, P_1, \dots, P_n)$$

is given by $\eta_{\overline{P}}(a, (x_1, \dots, x_n)) = (a, x_1, \dots, x_n)$.

Proof. Clearly $(\varepsilon_P)_a : \Sigma^*([P])(a) \longrightarrow P(a)$ is a bijection. For $\alpha : b \longrightarrow a$,

$$P(\alpha)((\varepsilon_P)_a((x))) = P(\alpha)(x) = (\varepsilon_P)_b((P(\alpha)(x))) = (\varepsilon_P)_b(\Sigma^*([P])(\alpha)((x))).$$

Thus $\varepsilon_P : \Sigma^*([P]) \longrightarrow P$ is a natural isomorphism, so an iso in $\mathrm{PSh}(\mathbb{C})$. We check that ε is natural in P. Let $\tau : P \longrightarrow Q$ be a natural transformation. We need to verify

$$\tau \cdot \varepsilon_P = \varepsilon_Q \cdot \Sigma^*([\tau]),$$

i.e. $\tau_a((\varepsilon_P)_a((x))) = (\varepsilon_Q)_a((\Sigma^*([\tau]))_a((x)))$. Now

$$\tau_a((\varepsilon_P)_a((x))) = \tau_a(x).$$

On the other hand

$$(\varepsilon_Q)_a((\Sigma^*([\tau]))_a((x))) = (\varepsilon_Q)_a((\tau_a(x))) = \tau_a(x).$$

Hence ϵ is a natural transformation.

We verify that η is an natural isomorphism. First check that $\eta_{\overline{P}}$ is a morphism $[\Sigma^*(P_1, \dots, P_n)] \longrightarrow [P_1, \dots, P_n]$ in $\mathrm{MPSh}(\mathbb{C})$ by verifying the multinaturality of Theorem 2.6: Let $\alpha : (b, (\overline{y})) \longrightarrow (a, (\overline{x}))$. We should have

$$\begin{aligned} P_1(\alpha)(f_1(b, (\overline{x}))) &= f_1(a, \Sigma^*(P_1, \dots, P_n)(\alpha)((\overline{x}))) \\ &\vdots \\ P_n(\alpha)(f_m(b, (\overline{x}))) &= f_n(a, \Sigma^*(P_1, \dots, P_n)(\alpha)((\overline{x}))) \end{aligned}$$

where $f_k(a, (\overline{x})) = x_k$. But $\Sigma^*(P_1, \dots, P_n)(\alpha)((\overline{x})) = (P_1(\alpha)(x_1), \dots, P_n(\alpha)(x_n))$ so this is clear. We claim that $g(a, x_1, \dots, x_n) = (a, (x_1, \dots, x_n))$ defines an inverse morphism to $\eta_{\overline{P}}$. It is clearly an inverse, so it remains to verify it is a morphism. Let $\alpha : (b, \overline{y}) \longrightarrow (a, \overline{x})$. We need to verify

$$\Sigma^*(P_1, \ldots, P_n)(\alpha)(f(a, \overline{x})) = f(b, P_1(\alpha)(x_1), \ldots, P_n(\alpha)(x_n)) \tag{12}$$

where $f(\overline{x}) = (\overline{x})$. But (12) is

$$\Sigma^*(P_1, \ldots, P_n)(\alpha)((\overline{x})) = (P_1(\alpha)(x_1), \ldots, P_n(\alpha)(x_n)) \tag{13}$$

which follows by definition. Thus each $\eta_{\overline{P}}$ is an isomorphism. We check that $\eta_{\overline{P}}$ is natural in $\overline{P}$. Suppose that $f : \overline{P} \longrightarrow \overline{Q}$ is a morphism. We need to verify that

$$f \circ \eta_{\overline{P}} = \eta_{\overline{Q}} \circ [\Sigma^*(f)].$$

Write

$$f(a, \overline{x}) = (a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x}))$$

We have

$$f(\eta_{\overline{P}}(a, (\overline{x}))) = f(a, \overline{x}) = (a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x}))$$

and on the other hand

$$\eta_{\overline{Q}}([\Sigma^*(f)](a, (\overline{x}))) = \eta_{\overline{Q}}(a, \Sigma^*(f)_a((\overline{x})))) = \eta_{\overline{Q}}(a, (f_1(a, \overline{x}), \ldots, f_m(a, \overline{x}))) = (a, f_1(a, \overline{x}), \ldots, f_m(a, \overline{x})).$$

Thus we are done. □

4 Π-construction

Let $\mathbb{C}$ be any small category and let $\overline{R} = [R_1, \ldots, R_n] \in \mathrm{MPSh}(\mathbb{C})$. Let $P \in \mathrm{PSh}(\Sigma(\overline{R}))$ and $Q \in \mathrm{PSh}(\Sigma(\overline{R}, P))$. We define a presheaf $\Pi(P, Q)$ over $\Sigma(\overline{R})$ as follows. For $(a, \overline{x}) \in \Sigma(\overline{R})$, let

$$\begin{aligned} \Pi(P, Q)(a, \overline{x}) \;=\; \Big\{ & h \in (\Pi b \in \mathbb{C})(\Pi f : b \to a)(\Pi v \in P(b, \overline{R}(f)(\overline{x}))Q(b, \overline{R}(f)(\overline{x}), v) \mid \\ & \forall b \in \mathbb{C}, \forall f : b \to a, \forall v \in P(b, \overline{R}(f)(\overline{x})), \\ & \forall c \in \mathbb{C}, \forall \beta : c \to b, \\ & \quad Q(\beta)(h(b, f, v)) = h(c, f \circ \beta, P(\beta)(v)) \Big\} \end{aligned}$$

We have written $\overline{R}(f)(\overline{x})$ for $R_1(f)(x_1), \ldots, R_n(f)(x_n)$. For $\alpha : (a', \overline{x}') \to (a, \overline{x})$ and $h \in \Pi(P, Q)(a, \overline{x})$ define $\Pi(P, Q)(\alpha)(h) = h'$ by

$$h'(b, f, v) = h(b, \alpha \circ f, v), \tag{14}$$

for $b \in \mathbb{C}, f : b \to a', v \in P(b, \overline{R}(f)(\overline{x}'))$. It is straightforward to verify that $\Pi(P, Q)$ is a presheaf over $\mathbb{C}$.

Let $s \in \mathrm{E}([\overline{R}, P], Q)$. Thus there is s' such that for all $(b, \overline{u}, v) \in \Sigma(\overline{R}, P)$,

$$s(b, \overline{u}, v) = (b, \overline{u}, v, s'(b, \overline{u}, v))$$

where $s'(b, \overline{u}, v) \in Q(b, \overline{u}, v)$ and further for all $\alpha : (a, \overline{x}, y) \longrightarrow (b, \overline{u}, v)$

$$Q(\alpha)(s'(b, \overline{u}, v)) = s'(a, \overline{R}(\alpha)(\overline{u}), P(\alpha)(v)). \tag{15}$$

Define

$$\hat{s}(a, \overline{x}) = \lambda b \in \mathbb{C}.\lambda f : b \to a.\lambda v \in P(b, \overline{R}(f)(\overline{x})).s'(b, \overline{R}(f)(\overline{x}), v).$$

We check that $\hat{s}(a, \overline{x}) \in \Pi(P, Q)(a, \overline{x})$: For $b \in \mathbb{C}$, $f : b \to a$, $v \in P(b, \overline{R}(f)(\overline{x}))$ we need to verify that for any $\beta : c \longrightarrow b$,

$$Q(\beta)(\hat{s}(a, \overline{x})(b, f, v)) = \hat{s}(a, \overline{x})(c, f \circ \beta, P(\beta)(v)).$$

Indeed, using (18), the following calculation proves this.

$$\begin{aligned} Q(\beta)(\hat{s}(a, \overline{x})(b, f, v)) &= Q(\beta)(s'(b, \overline{R}(f)(\overline{x}), v)) \\ &= s'(c, \overline{R}(\beta)(\overline{R}(f)(\overline{x})), P(\beta)(v)) \\ &= s'(c, \overline{R}(f \circ \beta)(\overline{x}), P(\beta)(v)) \\ &= \hat{s}(a, \overline{x})(c, f \circ \beta, P(\beta)(v)). \end{aligned}$$

Next, define

$$\boldsymbol{\lambda}_{P,Q}(s)(a, \overline{x}) = (a, \overline{x}, \hat{s}(a, \overline{x})).$$

We wish to verify that $\boldsymbol{\lambda}_{P,Q}(s) \in \mathrm{E}(\overline{R}, \Pi(P, Q))$. For this it suffices to check that for $\alpha : (a', \overline{x}') \longrightarrow (a, \overline{x})$,

$$\Pi(P, Q)(\alpha)(\hat{s}(a, \overline{x})) = \hat{s}(a', \overline{R}(\alpha)(\overline{x})). \tag{16}$$

Evaluate the left hand side at $b \in \mathbb{C}, f : b \to a', v \in P(b, \overline{R}(f)(\overline{x}'))$,

$$\begin{aligned} \Pi(P, Q)(\alpha)(\hat{s}(a, \overline{x}))(b, f, v) &= \hat{s}(a, \overline{x})(b, \alpha \circ f, v) \\ &= s'(b, \overline{R}(\alpha \circ f)(\overline{x}), v) \\ &= s'(b, \overline{R}(f)(\overline{R}(\alpha)(\overline{x})), v) \\ &= \hat{s}(a, \overline{R}(\alpha)(\overline{x}))(b, f, v) \end{aligned}$$

This verifies (16).

For $f \in \mathrm{E}(\overline{R}, \Pi(P, Q))$ and $t \in \mathrm{E}(\overline{R}, P)$ we write

$$f(a, \overline{x}) = (a, \overline{x}, f'(a, \overline{x})) \qquad t(a, \overline{x}) = (a, \overline{x}, t'(a, \overline{x})).$$

Thus $f'(a,\overline{x}) \in \Pi(P,Q)(a,\overline{x})$ and $t'(a,\overline{x}) \in P(a,\overline{x})$. It holds that

$$f'(a,\overline{x})(a,1_a,t'(a,\overline{x})) \in Q(a,\overline{x},t'(a,\overline{x})) = (Q \circ t)(a,\overline{x}).$$

Define

$$\mathrm{App}_{P,Q}(f,t)(a,\overline{x}) = (a,\overline{x},f'(a,\overline{x})(a,1_a,t'(a,\overline{x}))).$$

We wish to prove that

$$\mathrm{App}_{P,Q}(f,t) \in \mathrm{E}(\overline{R},Q \circ t).$$

By the form of the definition it suffices to check the naturality condition: for $\alpha : (a,\overline{x}) \longrightarrow (b,\overline{y})$,

$$(Q \circ t)(\alpha)(f'(b,\overline{y})(b,1_b,t'(b,\overline{y}))) = f'(a,\overline{R}(\alpha)(\overline{y}))(a,1_a,t'(a,\overline{R}(\alpha)(\overline{y}))).$$

We use the naturality conditions for f, t and naturality of elements in $\Pi(P,Q)(b,\overline{y})$ to verify this:

$$\begin{array}{rcl}
(Q \circ t)(\alpha)(f'(b,\overline{y})(b,1_b,t'(b,\overline{y}))) & = & Q(t(\alpha))(f'(b,\overline{y})(b,1_b,t'(b,\overline{y}))) \\
& = & f'(b,\overline{y})(a,1_b \circ \alpha, P(\alpha)(t'(b,\overline{y}))) \\
& = & f'(b,\overline{y})(a,1_b \circ \alpha, t'(a,\overline{R}(\alpha)(\overline{y}))) \\
& = & f'(b,\overline{y})(a,\alpha \circ 1_a, t'(a,\overline{R}(\alpha)(\overline{y}))) \\
& = & (\Pi(P,Q)(\alpha)(f'(b,\overline{y})))(a,1_a,t'(a,\overline{R}(\alpha)(\overline{y}))) \\
& = & f'(a,\overline{R}(\alpha)(\overline{y}))(a,1_a,t'(a,\overline{R}(\alpha)(\overline{y}))).
\end{array}$$

The λ-computation rule is verified as follows

$$\begin{array}{rcl}
\mathrm{App}_{P,Q}(\boldsymbol{\lambda}_{P,Q}(s),t)(a,\overline{x}) & = & (a,\overline{x},\hat{s}(a,\overline{x})(a,1_a,t'(a,\overline{x}))) \\
& = & (a,\overline{x},s'(a,\overline{x},t'(a,\overline{x}))) \\
& = & s\{t\}(a,\overline{x}).
\end{array}$$

Thus $\mathrm{App}_{P,Q}(\boldsymbol{\lambda}_{P,Q}(s),t) = s\{t\}$.

It remains to check that all constructs commute with substitutions. Fix a morphism $f : \overline{S} \longrightarrow \overline{R}$, where $\overline{S} = [S_1,\ldots,S_k]$ and $\overline{R} = [R_1,\ldots,R_n]$. Then write

$$f(d,\overline{w}) = (d,f_1(d,\overline{w}),\ldots,,f_n(d,\overline{w})).$$

The components satisfy the naturality conditions: for each morphism $\alpha : (e,\overline{z}) \longrightarrow (d,\overline{w})$ in $\Sigma(\overline{S})$, the following equations hold

$$\begin{array}{rcl}
R_1(\alpha)(f_1(d,\overline{w})) & = & f_1(e,\overline{S}(\alpha)(\overline{w})) \\
& \vdots & \\
R_n(\alpha)(f_n(d,\overline{w})) & = & f_n(e,\overline{S}(\alpha)(\overline{w}))
\end{array}$$

Π-substitution: $P \in \mathrm{PSh}(\Sigma(\overline{R}))$ and $Q \in \mathrm{PSh}(\Sigma(\overline{R}, P))$. We need to check that $\Pi(P,Q)\{f\} = \Pi(P\{f\}, Q\{q_{P,f}\})$ as presheaves. Let $(d, \overline{w}) \in \Sigma(\overline{S})$. We have

$$\begin{aligned}
\Pi(P,Q)\{f\}(d,\overline{w}) \;=\; \Big\{ & h \in (\Pi b \in \mathbb{C})(\Pi g : b \to d)(\Pi v \in P(b, \overline{R}(g)(f_1(d,\overline{w}), \ldots, , f_n(d,\overline{w})))\\
& \quad Q(b, \overline{R}(g)(f_1(d,\overline{w}), \ldots, , f_n(d,\overline{w})), v) \mid \\
& \forall b \in \mathbb{C}, \forall g : b \to d, \forall v \in P(b, \overline{R}(g)(f_1(d,\overline{w}), \ldots, , f_n(d,\overline{w}))),\\
& \forall c \in \mathbb{C}, \forall \beta : c \to b,\\
& \quad Q(\beta)(h(b,g,v)) = h(c, g \circ \beta, P(\beta)(v)) \Big\}
\end{aligned}$$

By the naturality condition

$$\overline{R}(g)(f_1(d,\overline{w}), \ldots, f_n(d,\overline{w})) = (f_1(b, \overline{S}(g)(\overline{w})), \ldots, f_n(b, \overline{S}(g)(\overline{w}))). \tag{17}$$

Thus

$$\begin{aligned}
P(b, \overline{R}(g)(f_1(d,\overline{w}), \ldots, , f_n(d,\overline{w})) &= P(b, (f_1(b, \overline{S}(g)(\overline{w})), \ldots, , f_n(b, \overline{S}(g)(\overline{w}))))\\
&= P(f(b, \overline{S}(g)(\overline{w})))\\
&= (P\{f\})(b, \overline{S}(g)(\overline{w}))
\end{aligned}$$

and moreover

$$\begin{aligned}
Q(b, \overline{R}(g)(f_1(d,\overline{w}), \ldots, f_n(d,\overline{w})), v) &= Q(b, f_1(b, \overline{S}(g)(\overline{w})), \ldots, f_n(b, \overline{S}(g)(\overline{w})), v)\\
&= Q(f(b, \overline{S}(g)(\overline{w})), v)\\
&= (Q\{q_{P,f}\})(\overline{S}(g)(\overline{w}), v)
\end{aligned}$$

We have thereby

$$\begin{aligned}
\Pi(P,Q)\{f\}(d,\overline{w}) \;=\; \Big\{ & h \in (\Pi b \in \mathbb{C})(\Pi g : b \to d)(\Pi v \in (P\{f\})(b, \overline{S}(g)(\overline{w})))\\
& \quad (Q\{q_{P,f}\})(\overline{S}(g)(\overline{w}), v) \mid \\
& \forall b \in \mathbb{C}, \forall g : b \to d, \forall v \in (P\{f\})(b, \overline{S}(g)(\overline{w})),\\
& \forall c \in \mathbb{C}, \forall \beta : c \to b,\\
& \quad Q(\beta)(h(b,g,v)) = h(c, g \circ \beta, P(\beta)(v)) \Big\}
\end{aligned}$$

But $Q(\beta) = Q(q_{P,f}(\beta))$ and $P(\beta) = P(f(\beta))$, so

$$\Pi(P,Q)\{f\}(d,\overline{w}) = \Pi(P\{f\}, Q\{q_{P,f}\})(d,\overline{w}).$$

Suppose $\alpha : (e, \overline{z}) \longrightarrow (d, \overline{w})$ in $\Sigma(\overline{S})$,

$$\Pi(P,Q)\{f\}(\alpha)(h)(b,f,v) = \Pi(P,Q)(f(\alpha))(h)(b,f,v) = \Pi(P,Q)(\alpha)(h)(b,f,v) = (b, \alpha \circ f, v)$$

and on the other hand

$$\Pi(P\{f\}, Q\{q_{P,f}\})(\alpha)(h)(b, f, v) = (b, \alpha \circ f, v).$$

Hence $\Pi(P, Q)\{f\} = \Pi(P\{f\}, Q\{q_{P,f}\})$.

λ-substitution: Let $s \in \mathrm{E}([\overline{R}, P], Q)$. Thus there is s' such that for all $(b, \overline{u}, v) \in \Sigma(\overline{R}, P)$,

$$s(b, \overline{u}, v) = (b, \overline{u}, v, s'(b, \overline{u}, v))$$

where $s'(b, \overline{u}, v) \in Q(b, \overline{u}, v)$ and further for all $\alpha : (a, \overline{x}, y) \longrightarrow (b, \overline{u}, v)$

$$Q(\alpha)(s'(b, \overline{u}, v)) = s'(a, \overline{R}(\alpha)(\overline{u}), P(\alpha)(v)). \tag{18}$$

We have

$$\hat{s}(a, \overline{x}) = \lambda b \in \mathbb{C}.\lambda f : b \to a.\lambda v \in P(b, \overline{R}(f)(\overline{x})).s'(b, \overline{R}(f)(\overline{x}), v).$$

and

$$\boldsymbol{\lambda}_{P,Q}(s)(a, \overline{x}) = (a, \overline{x}, \hat{s}(a, \overline{x})).$$

Further,

$$\boldsymbol{\lambda}_{P,Q}(s)\{f\}(d, \overline{w}) = (d, \overline{w}, \hat{s}(d, f_1(a, \overline{w}), \ldots, f_m(d, \overline{w})))$$

Now $s\{q_{P,f}\} \in \mathrm{E}([\overline{S}, P\{f\}], Q\{q_{P,f}\})$, and so

$$\boldsymbol{\lambda}_{P\{f\}, Q\{q_{P,f}\}}(s\{q_{P,f}\}) \in \mathrm{E}(\overline{S}, \Pi(P\{f\}, Q\{q_{P,f}\})).$$

and

$$\boldsymbol{\lambda}_{P\{f\}, Q\{q_{P,f}\}}(s\{q_{P,f}\})(d, \overline{w}) = (d, \overline{w}, \widehat{s\{q_{P,f}\}}(d, \overline{w}))$$

We have

$$q_{P,f}(d, \overline{w}, v) = (f(d, \overline{w}), v) = (d, f_1(d, \overline{w}), \ldots, f_m(d, \overline{w}), v).$$

By construction of substitution on terms

$$s\{q_{P,f}\}(d, \overline{w}, v) = (d, \overline{w}, v, s'(d, f_1(d, \overline{w}), \ldots, f_m(d, \overline{w}), v))$$

Thus

$$\widehat{s\{q_{P,f}\}}(d, \overline{w}) = \lambda b \in \mathbb{C}.\lambda g : b \to d.\lambda v \in P\{f\}(b, \overline{S}(g)(\overline{w})).s'(b, f_1(b, \overline{S}(g)(\overline{w})), \ldots, f_m(b, \overline{S}(g)(\overline{w})), v)).$$

We compare

$$\widehat{s\{q_{P,f}\}}(d, \overline{w})(b, g, v) = s'(b, f_1(b, \overline{S}(g)(\overline{w})), \ldots, f_m(b, \overline{S}(g)(\overline{w})), v))$$

and

$$\hat{s}(d, f_1(a,\overline{w}),\ldots,f_m(d,\overline{w}))(b,g,v) \quad = \quad s'(b, \overline{R}(g)(f_1(a,\overline{w}),\ldots,f_m(d,\overline{w})),v)$$

By the condition (17) we see that the two expressions are equal.

App-substitution: For $g \in \mathrm{E}(\overline{R}, \Pi(P,Q))$ and $t \in \mathrm{E}(\overline{R}, P)$ we write

$$g(a,\overline{x}) = (a,\overline{x},g'(a,\overline{x})) \qquad t(a,\overline{x}) = (a,\overline{x},t'(a,\overline{x})).$$

Thus $g'(a,\overline{x}) \in \Pi(P,Q)(a,\overline{x})$ and $t'(a,\overline{x}) \in P(a,\overline{x})$. It holds that

$$g'(a,\overline{x})(a,1_a,t'(a,\overline{x})) \in Q(a,\overline{x},t'(a,\overline{x})) = (Q \circ t)(a,\overline{x}).$$

We have by definition

$$\mathrm{App}_{P,Q}(g,t)(a,\overline{x}) = (a,\overline{x},g'(a,\overline{x})(a,1_a,t'(a,\overline{x}))).$$

We shall prove

$$\mathrm{App}_{P,Q}(g,t)\{f\} = \mathrm{App}_{P\{f\},Q\{q_{P,f}\}}(g\{f\},t\{f\})$$

On the one hand

$$\begin{aligned} & \mathrm{App}_{P,Q}(g,t)\{f\}(d,\overline{w}) \\ = \quad & (d,\overline{w},g'(d,f_1(a,\overline{w}),\ldots,f_m(d,\overline{w}))(d,1_d,t'(d,f_1(a,\overline{w}),\ldots,f_m(d,\overline{w})))). \end{aligned}$$

We have further

$$g\{f\}(d,\overline{w}) = (d,\overline{w},g'(d,f_1(a,\overline{w}),\ldots,f_m(d,\overline{w})))$$

and

$$t\{f\}(d,\overline{w}) = (d,\overline{w},t'(d,f_1(a,\overline{w}),\ldots,f_m(d,\overline{w})))$$

Now on the other hand

$$\begin{aligned} & \mathrm{App}_{P\{f\},Q\{q_{P,f}\}}(g\{f\},t\{f\})(d,\overline{w}) \\ & = (d,\overline{w},g'(d,f_1(a,\overline{w}),\ldots,f_m(d,\overline{w}))(d,1_d,t'(d,f_1(a,\overline{w}),\ldots,f_m(d,\overline{w})))) \end{aligned}$$

which is indeed the same.

5 Σ-construction

Let $\mathbb{C}$ be any small category and let $\overline{R} = [R_1,\ldots,R_n] \in \mathrm{MPSh}(\mathbb{C})$. Let $P \in \mathrm{PSh}(\Sigma(\overline{R}))$ and $Q \in \mathrm{PSh}(\Sigma(\overline{R},P))$. We define a presheaf $\dot{\Sigma}(P,Q)$ over $\Sigma(\overline{R})$ as follows. For $(a,\overline{x}) \in \Sigma(\overline{R})$, let

$$\dot{\Sigma}(P,Q)(a,\overline{x}) = \{(u,v) : u \in P(a,\overline{x}), v \in Q(a,\overline{x},u)\}.$$

For $\alpha : (a',\overline{x}') \to (a,\overline{x})$ and $h \in \dot{\Sigma}(P,Q)(a,\overline{x})$ define

$$\dot{\Sigma}(P,Q)(\alpha)(u,v) = (P(\alpha)(u),Q(\alpha)(v)).$$

It is straightforward to verify that $\dot{\Sigma}(P,Q)$ is a presheaf over $\mathbb{C}$.

6 Explication of the constructions over some categories

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Suppose that $\mathbb{C}$ is the category $0 \to 2 \leftarrow 1$, where all other arrows are identities. Let $\overline{R} = [R_1, \ldots, R_n] \in \mathrm{MPSh}(\mathbb{C})$. Let $P \in \mathrm{PSh}(\Sigma(\overline{R}))$ and $Q \in \mathrm{PSh}(\Sigma(\overline{R}, P))$. Now the definition of $\Pi(P, Q)$ simples for $a = 0, 1$, since there are only identity arrows into a, the naturality condition becomes void, so we have:

$$\begin{aligned}
\Pi(P,Q)(a,\overline{x}) &= (\Pi b \in \mathbb{C})(\Pi f : b \to a)(\Pi v \in P(b, \overline{R}(f)(\overline{x}))Q(b, \overline{R}(f)(\overline{x}), v) \\
&\cong (\Pi v \in P(b, \overline{x})Q(b, \overline{x}, v)
\end{aligned}$$

For $a = 2$, the naturality condition has a few nontrivial cases:

$$\begin{aligned}
\Pi(P,Q)(2,\overline{x}) &= \Big\{ h \in (\Pi b \in \mathbb{C})(\Pi f : b \to 2)(\Pi v \in P(b, \overline{R}(f)(\overline{x}))Q(b, \overline{R}(f)(\overline{x}), v) \mid \\
&\qquad \forall b \in \mathbb{C}, \forall f : b \to 2, \forall v \in P(b, \overline{R}(f)(\overline{x})), \\
&\qquad \forall c \in \mathbb{C}, \forall \beta : c \to b, \\
&\qquad\quad Q(\beta)(h(b, f, v)) = h(c, f \circ \beta, P(\beta)(v)) \Big\}
\end{aligned}$$

Writing out the cases explicitly we get

$$\begin{aligned}
\Pi(P,Q)(2,\overline{x}) &= \Big\{ h \in (\Pi b \in \mathbb{C})(\Pi f : b \to 2)(\Pi v \in P(b, \overline{R}(f)(\overline{x}))Q(b, \overline{R}(f)(\overline{x}), v) \mid \\
&\qquad \forall f : 0 \to 2, \forall v \in P(0, \overline{R}(f)(\overline{x})), \\
&\qquad \forall c \in \mathbb{C}, \forall \beta : c \to 0, \\
&\qquad\quad Q(\beta)(h(0, f, v)) = h(c, f \circ \beta, P(\beta)(v)), \\
&\qquad \forall f : 1 \to 2, \forall v \in P(b, \overline{R}(f)(\overline{x})), \\
&\qquad \forall c \in \mathbb{C}, \forall \beta : c \to 1, \\
&\qquad\quad Q(\beta)(h(1, f, v)) = h(c, f \circ \beta, P(\beta)(v)) \\
&\quad \forall f : 2 \to 2, \forall v \in P(2, \overline{R}(f)(\overline{x})), \\
&\qquad \forall c \in \mathbb{C}, \forall \beta : c \to 2, \\
&\qquad\quad Q(\beta)(h(2, f, v)) = h(c, f \circ \beta, P(\beta)(v)) \Big\}
\end{aligned}$$

Simplifying this the first two conditions become void.

$$\begin{aligned}
\Pi(P,Q)(2,\overline{x}) &= \Big\{ h \in (\Pi b \in \mathbb{C})(\Pi f : b \to 2)(\Pi v \in P(b, \overline{R}(f)(\overline{x}))Q(b, \overline{R}(f)(\overline{x}), v) \mid \\
&\quad \forall f : 2 \to 2, \forall v \in P(2, \overline{R}(f)(\overline{x})), \\
&\qquad \forall c \in \mathbb{C}, \forall \beta : c \to 2, \\
&\qquad\quad Q(\beta)(h(2, f, v)) = h(c, f \circ \beta, P(\beta)(v)) \Big\}
\end{aligned}$$

Further, simplifying the remaining condition

$$\begin{aligned}\Pi(P,Q)(2,\overline{x}) &= \Big\{h \in (\Pi b \in \mathbb{C})(\Pi f : b \to 2)(\Pi v \in P(b, \overline{R}(f)(\overline{x}))Q(b, \overline{R}(f)(\overline{x}), v) \mid \\ &\qquad \forall v \in P(2,\overline{x}), \forall c \in \mathbb{C}, \forall \beta : c \to 2, \\ &\qquad\quad Q(\beta)(h(2, 1_2, v)) = h(c, \beta, P(\beta)(v))\Big\}\end{aligned}$$

Finally, instantiating c to $0, 1, 2$ what remains after simplification ($c = 2$ gives an empty condition):

$$\begin{aligned}\Pi(P,Q)(2,\overline{x}) &= \Big\{h \in (\Pi b \in \mathbb{C})(\Pi f : b \to 2)(\Pi v \in P(b, \overline{R}(f)(\overline{x}))Q(b, \overline{R}(f)(\overline{x}), v) \mid \\ &\qquad \forall v \in P(2,\overline{x}), \\ &\qquad\quad Q(f_{02})(h(2, 1_2, v)) = h(0, f_{02}, P(f_{02})(v)), \\ &\qquad\quad Q(f_{12})(h(2, 1_2, v)) = h(1, f_{12}, P(f_{12})(v))\Big\}\end{aligned}$$

6.1 Simplicial sets

Let Δ be the category whose objects are the natural number $\mathbb{N} = \{0, 1, 2, \ldots\}$. Denote by $[n]$ the set $\{0, \ldots, n\}$ for $n \in \mathbb{N}$. A morphism $f : m \longrightarrow n$ in Δ is a monotone function $f : [m] \longrightarrow [n]$. The presheaves over Δ, is called the category of simplicial sets. The Yoneda embedding $y : \Delta \longrightarrow \mathrm{PSh}(\Delta)$ satisfies by the Yoneda lemma

$$\mathrm{Hom}_{\mathrm{PSh}(\mathbb{C})}(y(n), F) \cong F(n)$$

for any $n \in \mathbb{N}$ and any $F \in \mathrm{PSh}(\mathbb{C})$. The canonical n-simplex is $\Delta^n = y(n)$.

For $i = 0, \ldots, n+1$, let $\delta_i^n : [n] \longrightarrow [n+1]$ be the unique monotone function such that $\delta_i^n[\{0, \ldots, n\}] = \{0, \ldots, i-1, i+1, \ldots, n+1\}$. $F(\delta_i^n) : F(n+1) \longrightarrow F(n)$ is the *ith face map.*

The presheaf $F \in \mathrm{PSh}(\Delta)$ is a *Kan complex* if for any n and any $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1} \in F(n)$ such that $F(\delta_i)(x_j) = F(\delta_{j-1})(x_i)$ for all $i < j$, $i \neq k$, $j \neq k$, there exists $x \in F(n+1)$ such that $F(\delta_i)(x) = x_i$ for all $i \neq k$.

Spelling out:

$n = 1, k = 0$: for any x_1, x_2 with $F(\delta_1)(x_2) = F(\delta_1)(x_1)$, there is $x \in F(2)$ such that $F(\delta_1)(x) = x_1$, $F(\delta_2)(x) = x_2$.

$n = 1, k = 1$: for any x_0, x_2 with $F(\delta_0)(x_2) = F(\delta_1)(x_0)$, there is $x \in F(2)$ such that $F(\delta_0)(x) = x_0$, $F(\delta_2)(x) = x_2$.

$n = 1, k = 2$: for any x_0, x_1 with $F(\delta_0)(x_1) = F(\delta_0)(x_0)$, there is $x \in F(2)$ such that $F(\delta_0)(x) = x_0$, $F(\delta_1)(x) = x_1$.

$n = 2, k = 0$: for any x_1, x_2, x_3 with $F(\delta_1)(x_2) = F(\delta_1)(x_1)$, $F(\delta_1)(x_3) = F(\delta_2)(x_1)$, $F(\delta_2)(x_3) = F(\delta_2)(x_2)$, there is $x \in F(3)$ such that $F(\delta_1)(x) = x_1$, $F(\delta_2)(x) = x_2$, $F(\delta_3)(x) = x_3$.

References

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Appendix

Definition 6.1. A *category with attributes* (cwa) consists of the data

(a) A category $\mathcal{C}$ with a terminal object 1. This is the called the *category of contexts and substitutions.*

(b) A functor $T : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$. This functor is intended to assign to each context Γ a set $T(\Gamma)$ of types in the context and tells how substitutions act on these types. For $f : B \longrightarrow \Gamma$ and $\sigma \in T(\Gamma)$ we write

$$\sigma\{f\} \text{ for } T(f)(\sigma).$$

(c) For each $\sigma \in T(\Gamma)$, an object $\Gamma.\sigma$ in $\mathcal{C}$ and a morphism

$$\mathsf{p}(\sigma) = \mathsf{p}_\Gamma(\sigma) : \Gamma.\sigma \longrightarrow \Gamma \text{ in } \mathcal{C}.$$

This tells that each context can be extended by a type in the context, and that there is a projection from the extended context to the original one.

(d) The final datum tells how substitutions interact with context extensions: For each $f : B \longrightarrow \Gamma$ and $\sigma \in T(\Gamma)$, there is a morphism $\mathsf{q}(f, \sigma) = \mathsf{q}_\Gamma(f, \sigma) : B.(T(f)(\sigma)) \longrightarrow \Gamma.\sigma$ in $\mathcal{C}$ such that

$$\begin{array}{ccc} B.(\sigma\{f\}) & \xrightarrow{\mathsf{q}(f,\sigma)} & \Gamma.\sigma \\ {\scriptstyle \mathsf{p}(\sigma\{f\})}\downarrow & & \downarrow{\scriptstyle \mathsf{p}(\sigma)} \\ B & \xrightarrow[f]{} & \Gamma \end{array}$$

is a pullback, and furthermore

(d.1) $\mathsf{q}(1_\Gamma, \sigma) = 1_{\Gamma.\sigma}$

(d.2) $\mathsf{q}(f \circ g, \sigma) = \mathsf{q}(f, \sigma) \circ \mathsf{q}(g, \sigma\{f\})$ for $A \xrightarrow{g} B \xrightarrow{f} \Gamma$.

From [2] we take the following definition, but adapt it in the obvious way to cwas.

Definition 6.2. A cwa *supports* Π-*types* if for $\sigma \in T(\Gamma)$ and $\tau \in T(\Gamma.\sigma)$ there is a type

$$\Pi(\sigma, \tau) \in T(\Gamma),$$

and moreover for every $P \in E(\Gamma.\sigma, \tau)$ there is an element

$$\lambda_{\sigma,\tau}(P) \in E(\Gamma, \Pi(\sigma, \tau)),$$

and furthermore for any $M \in E(\Gamma, \Pi(\sigma, \tau))$ and any $N \in E(\Gamma, \sigma)$ there is an element

$$\mathsf{App}_{\sigma,\tau}(M, N) \in E(\Gamma, \tau\{N\}),$$

such that the following equations hold for any substitution $f : B \longrightarrow \Gamma$:

(λ-comp) $\mathsf{App}_{\sigma,\tau}(\lambda_{\sigma,\tau}(P), N) = P\{N\}$,

(Π-subst) $\Pi(\sigma, \tau)\{f\} = \Pi(\sigma\{f\}, \tau\{\mathsf{q}(f, \sigma)\})$,

(λ-subst) $\lambda_{\sigma,\tau}(P)\{f\} = \lambda_{\sigma\{f\},\tau\{\mathsf{q}(f,\sigma)\}}(P\{\mathsf{q}(f, \sigma)\})$,

(App-subst) $\mathsf{App}_{\sigma,\tau}(M, N)\{f\} = \mathsf{App}_{\sigma\{f\},\tau\{\mathsf{q}(f,\sigma)\}}(M\{f\}, N\{f\})$.