STOCKHOLMS UNIVERSITET
Matematiska institutionen
Erik Palmgren

Course information
Realizability 7.5 cp,
a graduate course
in mathematics,
spring term 2017
2017-01-16

Realizability: computational interpretations of logic - a graduate course in mathematics

Realizability (or recursive realizability) was introduced by Kleene in 1945 as a method for giving computational interpretations of mathematical proofs, more specifically formal proofs carried out from Peano's axioms of arithmetic using intuitionistic logic. The method has since then been extended in various directions. For instance, set theory based on intuitionstic logic (IZF and CZF) can be given such interpretation. The method can also be extended to allow of extraction of other kinds information from proofs, such as estimates in mathematical analysis. This idea is exploited in so called proof mining where explicit bounds are obtained by performing formal interpretations of pure existence proofs.

Realizability techniques are also very important for establishing independence results for intuitionistic theories.

The course is suitable for students in mathematical logic, mathematics, computer science and philosophy with background knowledge in basic computability theory and some advanced courses in logic.

Lectures

The course consists of 14 lectures and student seminars. First lecture Monday, January 16, 15.00-16.45 in room 306, building 6, Kräftriket.

Examination

The examination is by homework problems, and a seminar and/or essay on some selected topic.

Litterature

[TvD88] A.S. Troelstra and D. van Dalen: Constructivism in Mathematics, vol I. North-Holland 1988.

[vO08] J. van Oosten. Realizability: An Introduction to its Categorical Side. North-Holland 2008.

[K07] U. Kohlenbach. Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer 2007.

[SW12] H. Schwichtenberg and S.S. Wainer. Proofs and Computations. Perspectives in Logic. Cambridge University Press 2012.

Research Articles (to be specified).

The course literature (books and articles) can be found as electronic resources at the Stockholm University Library. See http://su.se/biblioteket/ and get a library/university card to login.

Contents

The course is divided in roughly the following parts, not necessarily presented in strict chronological order. More details will be given as the course unfolds, and the course can, to some extent, be adapted according to the interests of the participants.

Basic Kleene realizability of first-order arithmetic

As background we assume familiarity with basic computability theory including primitive recursive functions, partial computable functions, universal functions, the S-m-m theorem, existence of non-computable functions. Some of this material can be found in [TvD88], but there are more basic texts on this, for instance N.J. Cutland, Recursion Theory: An Introduction to Computability theory, Cambridge University Press 1980. Also assumed is familiarity with the basics of intuitionistic first-order logic and its BHK-interpretation. Again this can be found in [TvD88], but there are more elementary texts.

The first part of the course will concern the following chapters of [TvD88]: 3.1 – 3.4, 4.1 – 4.5: Primitive recursive arithmetic, Heyting arithmetic, Church Thesis as a non-classical axiom, Kleene realizability, characterization of realizability. Markov's principle and its use in Russian constructive mathematics.

Realizability from a categorical viewpoint

We will make a selection of topics of chapter 1-3 in [vO08], in particular the following will be covered.

Abstract realizers: partial combinatory algebras. First and second Kleene algebras. Triposes, toposes, geometric morphisms and their connection. Realizability toposes. The effective topos. Exact completions. Mathematical principles and non-classical axioms valid in the effective topos.

Applications of realizability in proof mining and program extraction

This part covers applications of realizability and proof transformations to proofs in mathematics and computer science, in order to extract computational content or other information from proofs, such as effective bounds in analysis.

We use selected parts of [K07] and [SW12], in particular we aim to cover the following topics: Gödel's T. Arithmetic in all finite types. Modified realizability. Majorizability. The Dialectica interpretation. Gödel-Gentzen negative translation and its refinements. Applications to concrete mathematical proofs.

— — — — —