Point-Free Topology and Foundations of Constructive Analysis[1]

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[1] These slides are available at
www.math.su.se/~palmgren/Ljubljana-tutorial-EP.pdf

Covering compactness and choice

A topological space X is *covering compact* if for any family of open sets U_i $(i \in I)$ in X we have

if $\bigcup_{i \in I} U_i = X$, then there are $i_1, \ldots, i_n \in I$ so that $U_{i_1} \cup \cdots \cup U_{i_n} = X$

It is well-known that some basic theorems classical topology use (and require) the full Axiom of Choice:

Tychonov's Theorem (AC): If $(X_i)_{i \in I}$ is a family of covering compact spaces then the product topology

$$\prod_{i \in I} X_i$$

is covering compact.

Special case: The *Cantor space* $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ is compact.

Constructive approaches to topology

To carry out topology on a constructive foundation it is necessary to come to grips with compactness.

Brouwer's solution: The compactness of the Cantor space follows from the nature of choice sequences. This the **Fan Theorem** (in fact, an axiom). Moreover, it implies that the interval $[0, 1]$ is covering compact.

Kleene: The Cantor space is not compact under a recursive realizability interpretation.

E. Bishop: Restricting to metric spaces, covering compactness should be replaced by total boundedness (*Foundations of Constructive Analysis*, 1967).

Bishop (1967) uses the classical definition: A subset K of metric space (X, d) is totally bounded if for every $\varepsilon > 0$, the subset K has an ε-approximation $x_1, \ldots, x_n$. A sequence of points $x_1, \ldots, x_n$ $(n \geq 1)$ in K is an ε-approximation of K if

$$K \subseteq B(x_1, \varepsilon) \cup \cdots \cup B(x_n, \varepsilon)$$

where $B(x, \varepsilon)$ is the open ball of radius ε around x.
Then K is said to be compact, if in addition K is (Cauchy)-complete.

This is motivated by the classical fact: (X, d) is covering compact if and only if (X, d) is totally bounded and complete.

NB: Bishop did not consider $\emptyset$ to be compact. But this can be fixed ...

Example: $[0,1] \subseteq \mathbb{R}$ is compact.
Example: Any finite set $\{x_1, \ldots, x_n\} \subseteq X$ is compact ($n \geq 1$).

However not every subset of a finite set is compact!

Brouwerian counterexample: Let P be any property. If the set

$$\{0\} \cup \{x \in \mathbb{R} : x = 1 \ \& \ P\}$$

is totally bounded, then P or $\neg P$.

Exercise: Both $[-1,0]$ and $[0,1]$ are compact subsets of $\mathbb{R}$. Is $[-1,0] \cup [0,1]$ compact? Show that if the answer is yes, then for every $x \in \mathbb{R}$,

$$x \geq 0 \text{ or } x \leq 0$$

This is a so-called Brouwerian counterexample to closure of compact sets under finite unions.

Exercise: Give a Brouwerian counterexample to : If $A, B \subseteq \mathbb{R}$ are compact, and $A \cap B$ inhabited, then $A \cap B$ is compact.

Exercise: Let $A \subseteq \mathbb{R}^2$ be a compact set that intersects a straight line L. Is $A \cap L$ compact? (Give a constructive proof or a Brouwerian counterexample.)

Some fundamental spaces in analysis are locally compact:

$$\mathbb{R}, \mathbb{R}^n, \mathbb{C}, \ldots$$

The Bishop-Bridges definition is: An inhabited metric space X is locally compact if it is complete, and every bounded subset is included in a totally bounded set.

Prop. Every locally compact space is separable and complete.

NB: This definition is narrower than the classical as it excludes $\mathbb{Q}$, $(0, 1)$ and certain non-separable spaces.

Because of failure of the Heine-Borel theorem, Bishop (1967) took uniform continuity on compact spaces as the fundamental continuity notion.

Definition Let K be a compact metric space. A function f from K to a metric space is *uniformly continuous* if for every $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in K$

$$d(x, y) \leq \delta \Longrightarrow d(fx, fy) \leq \varepsilon.$$

This is the most fruitful notion constructively, since it entails that suprema and integrals of continuous functions may be found.

The notion of continuity can then be extended to larger classes of spaces.

Definition A function f from a locally compact metric space X to a metric space Y is continuous, if f is uniformly continuous on each compact subset of X.

The class of locally compact metric spaces and continuous functions form a category **LCM**.

A general definition of continuity between any metric spaces was suggested by Bishop (see Bridges 1979).

Let X be a metric space. A subset $A \subseteq X$ is a *compact image* if there is a compact metric space K and a uniformly continuous function $\lambda : K \longrightarrow X$ with $A = \lambda[K]$. A function $f : X \longrightarrow Y$ between metric spaces is uniformly continuous near each compact image if for any compact image $A \subseteq X$ and each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$x \in A, y \in X, d(x, y) \leq \delta \Longrightarrow d(fx, fy) \leq \varepsilon.$$

Classically this notion coincides with standard continuity:

Thm.(classically): $f : X \longrightarrow Y$ is uniformly continuous near each compact image iff f is pointwise continuous.

The notion also generalizes continuity on LCMs:

Thm. Let X be locally compact. Then $f : X \longrightarrow Y$ is uniformly continuous near each compact image iff f is continuous.

In point-free topology (locale theory, formal topology) there is **one** standard notion of continuity defined in terms of covering relations between the basic open.

This can help us to find the fruitful notions of continuity between metric spaces or other point-based spaces.

One example is the following.

Continuous maps between open subspaces

Let X and Y be locally compact metric spaces. For open subsets $U \subseteq X$ and $V \subseteq Y$ a suitable definition of continuous map $U \longrightarrow V$ in BISH is:

$f : U \longrightarrow V$ is *continuous* iff for every compact $S \Subset U$,

C1) f is uniformly continuous on S and
C2) $f[S] \Subset V$.

Here $S \Subset U \Leftrightarrow_{\text{def}} (\exists t > 0)(\{x \in X : d(x, S) \leq t\} \subseteq U)$.

These spaces and maps form the category **OLCM** of open subspaces of locally compact metric spaces.

Note that the meaning of $\Subset$ is strongly dependent on the underlying space X:

Write S_r for $\{x \in X : d(x, S) \leq t\}$.

- $X = [0, 1]$: Then $S = [0, 1]$ is totally bounded and $S_r = [0, 1]$ for any r. Hence $S \Subset X$.
  - For $Y = V = \mathbb{R}$, we get the standard notion of (uniform) continuity: $[0, 1] \longrightarrow \mathbb{R}$.
  - For $Y = \mathbb{R}$ and $V = (0, +\infty)$, we get uniformly continuous functions which are uniformly positive.
- $X = \mathbb{R}$, $U = (0, +\infty)$: Note that $S = (0, 1)$ is totally bounded, but we do not have $S \Subset U$. The reciprocal will be continuous $(0, +\infty) \longrightarrow \mathbb{R}$.

Point-free topology

A quite early idea in topology: study spaces in terms of the relation between the open sets. (Wallman 1938, Menger 1940, McKinsey & Tarski 1944, Ehresmann, Benabou, Papert, Isbell,...)

For a topological space X the frame of open sets $(\mathcal{O}(X), \subseteq)$ is a complete lattice satisfying an infinite distributive law

$$U \wedge \bigvee_{i \in I} V_i = \bigvee_{i \in I} U \wedge V_i$$

(or, equivalently, is a complete Heyting algebra). The inverse mapping of a continuous function $f : X \longrightarrow Y$ gives rise to a lattice morphism

$$f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$$

which preserves arbitrary suprema $\bigvee$.

Many important properties of topological spaces may be expressed without referring to the points, but only referring to the relation between the open sets.

Def A *frame* (or *locale*) is a complete lattice A which satisfies the *infinite distributive law:*

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} a \wedge b_i$$

for any subset $\{b_i : i \in I\} \subseteq A$.

A frame morphism $h : A \longrightarrow B$ from between frames is a lattice morphism that preserves infinite suprema:

(i) $h(\top) = \top$,

(ii) $h(a \wedge b) = h(a) \wedge h(b)$,

(iii) $h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i)$.

The frames and frame morphisms form a category, **Frm**.

Example

A typical frame morphism is the pre-image operation $f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$, where $f : X \longrightarrow Y$ is any continuous function.

For conceptual reasons one considers the opposite category of the category of frames:

Def Let A and B be two locales. A *locale morphism* $f : A \longrightarrow B$ is a frame morphism $f^* : B \longrightarrow A$.

The composition $g \circ f$ of locale morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is given by

$$(g \circ f)^* = f^* \circ g^*.$$

Denote the category of locales and locale morphisms by **Loc**.

The locale (frame) $(\mathcal{O}(\{\star\}), \subseteq)$ that comes from the one point space $\{\star\}$ is denoted $\mathbf{1}$. It is the terminal object of **Loc**. Here $\top = \{\star\}$ and $\bot = \emptyset$.

Fundamental adjunction

Theorem The functor $\Omega : \mathbf{Top} \longrightarrow \mathbf{Loc}$ is left adjoint to the functor $\mathrm{Pt} : \mathbf{Loc} \longrightarrow \mathbf{Top}$, that is, there is an bijection

$$\theta_{X,A} : \mathbf{Loc}(\Omega(X), A) \cong \mathbf{Top}(X, \mathrm{Pt}(A)),$$

natural in X and A.

Here $\Omega(X) = (\mathcal{O}(X), \subseteq)$ and $\Omega(f)^* = f^{-1} : \Omega(Y) \longrightarrow \Omega(X)$ for $f : X \longrightarrow Y$.

$\mathrm{Pt}(A)$ is the set of points of the locale A, and $\mathrm{Pt}(f)(x) = f \circ x$.

A locale is *spatial* if it has enough points to distinguish elements of the locale i.e.

$$a^* = b^* \implies a = b.$$

A space X is *sober* if every irreducible closed set is the closure of a unique point. (A nonempty closed C is *irreducible*, if for any closed C', C'': $C \subseteq C' \cup C''$ implies $C \subseteq C'$ or $C \subseteq C''$.)

The adjunction induces an equivalence between the category of sober spaces and category of spatial locales:

$$\mathbf{Sob} \simeq \mathbf{Spa}$$

Remark Ω gives a full and faithful embedding of all Hausdorff spaces into locales (in fact of all sober spaces).

Locale Theory — Formal Topology

Standard Locale Theory can be dealt with in an impredicative constructive setting, e.g. in a topos (Joyal and Tierney 1984).

The predicativity requirements of BISH requires a good representation of locales, formal topologies. They were introduced by **Martin-Löf and Sambin** (Sambin 1987) for this purpose.

$(A, \leq, \lhd)$ is a formal topology if $(A, \leq)$ is a preorder, and $\lhd$ is an abstract cover relation which extends $\leq$. It represents a locale $(\mathrm{Sat}(A), \subseteq)$ whose elements are the saturated subsets $U \subseteq A$, i.e.

$$a \lhd U \Longrightarrow a \in U$$

(A formal topology corresponds to Grothendieck's notion of a site on a preordered set.)

More in detail: A *formal topology* X is a pre-ordered set $(X, \leq)$ of so-called *basic neighbourhoods*. This is equipped with a *covering relation* $a \lhd U$ between elements a of X and subsets $U \subseteq X$ and so that $\{a \in X : a \lhd U\}$ is a subset of X. The cover relation is supposed to satisfy the following conditions:

(Ext) If $a \leq b$, then $a \lhd \{b\}$,

(Refl) If $a \in U$, then $a \lhd U$,

(Trans) If $a \lhd U$ and $U \lhd V$, then $a \lhd V$,

(Loc) If $a \lhd U$ and $a \lhd V$, then $a \lhd U \wedge V$.

Here $U \lhd V$ is an abbreviation for $(\forall x \in U)\, x \lhd V$. Moreover $U \wedge V$ is short for the *formal intersection* $U_{\leq} \cap V_{\leq}$, where $W_{\leq} = \{x \in X : (\exists y \in W) x \leq y\}$, the *down set of* W.

A *point* of a formal topology $X = (X, \leq, \lhd\,)$ is a subset $\alpha \subseteq X$ such that

(i) α is inhabited,

(ii) if $a, b \in \alpha$, then for some $c \in \alpha$ with $c \leq a$ and $c \leq b$,

(iii) if $a \in \alpha$ and $a \leq b$, then $b \in \alpha$,

(iv) if $a \in \alpha$ and $a \lhd U$, then $b \in \alpha$ for some $b \in U$.

The collection of points in X is denoted $\mathrm{Pt}(X)$. It has a point-set topology given by the open neighbourhoods:

$$a^* = \{\alpha \in \mathrm{Pt}(X) : a \in \alpha\}.$$

Many properties of the formal topology X can now be defined in terms of the cover directly. We say that X is *compact* if for any subset $U \subseteq X$

$$X \lhd U \Longrightarrow (\exists \text{ f.e. } U_0 \subseteq U)\ X \lhd U_0.$$

For $U \subseteq X$ define

$$U^{\perp} = \{x \in X : \{x\} \wedge U \lhd \emptyset\},$$

the *open complement of U*. It is easily checked that $U^{\perp} \wedge U \lhd \emptyset$, and if $V \wedge U \lhd \emptyset$, then $V \lhd U^{\perp}$.
A basic neighbourhood a is *well inside* another neighbourhood b if $X \lhd \{a\}^{\perp} \cup \{b\}$. In this case we write $a \lll b$. A formal topology X is *regular* if its cover relation satisfies

$$a \lhd \{b \in X : b \lll a\}.$$

(Compact Hausdorff = Compact regular.)

Point-free proofs are more basic

Theorem (Johnstone 1981): Tychonov's Theorem holds for locales, without assuming AC.

A slogan of B. Banaschewski:

choice-free localic argument
+ suitable choice principles = classical result on spaces

In fact, Tychonov's theorem for locales is constructive even in the predicative sense (Coquand 1992).

This suggestive slogan can be generalised

constructive localic argument
+ Brouwerian principle = intuitionistic result on spaces.

However, since the work of the Bishop school (BISH) on constructive analysis it is known that there is often a basic

BISH constructive argument
+ Brouwerian principle = intuitionistic result on *metric spaces.*

A natural question: how does BISH constructive topology and constructive locale theory relate? For instance on metric spaces.

Formal reals $\mathcal{R}$

The basic neighbourhoods of $\mathcal{R}$ are $\{(a,b) \in \mathbb{Q}^2 : a < b\}$ given the inclusion order (as intervals), denoted by $\leq$. The cover $\lhd$ is generated by

$$\text{(G1)} \quad (a,b) \vdash \{(a',b') : a < a' < b' < b\} \text{ for all } a < b,$$

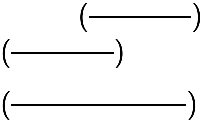
$$\text{(G2)} \quad (a,b) \vdash \{(a,c),(d,b)\} \text{ for all } a < d < c < b.$$

The set of points $\mathrm{Pt}(\mathcal{R})$ of $\mathcal{R}$ form a structure isomorphic to the Cauchy reals $\mathbb{R}$. For a point α with $(a,b) \in \alpha$ we have by (G2) e.g.

$$(a,(a+2b)/3) \in \alpha \text{ or } ((2a+b)/3,b) \in \alpha.$$

G1:

G2:



An elementary characterisation of the cover relation on formal reals is:

$$(a,b) \lhd U \iff (\forall a', b' \in \mathbb{Q})(a < a' < b' < b \Rightarrow (\exists \text{ finite } F \subseteq U)\,(a',b')^* \subseteq F^*)$$

Note that for finite F the pointwise inclusion $(a',b')^* \subseteq F^*$ is decidable, since the end points of the intervals are rational numbers. (Exercise)
This characterization is crucial in the proof of the Heine-Borel theorem (see Cederquist, Coquand and Negri).

For any rational number q there is a point in $\mathrm{Pt}(R)$;

$$\hat{q} = \{(a, b) \in \mathbb{Q}^2 : a < q < b\}.$$

For a Cauchy sequence $x = (x_n)$ of rational numbers we define

$$\hat{x} = \{(a, b) \in \mathbb{Q}^2 : (\exists k)(\forall n \geq k)\ a < x_n < b\}$$

which is a point of R.

Order relations on real numbers $\alpha, \beta \in \mathrm{Pt}(R)$:

$\alpha < \beta \iff_{\mathrm{def}} (\exists (a, a') \in \alpha)(\exists (b, b') \in \beta)\ a' < b$

$\alpha \leq \beta \iff_{\mathrm{def}} \neg(\beta < \alpha) \iff (\forall (b, b') \in \beta)(\forall (a, a') \in \alpha) a \leq b'$

Closed subspaces

Let X be a formal topology. Each subset $V \subseteq X$ determines a closed sublocale $X \setminus V$ whose covering relation $\lhd'$ is given by

$$a \lhd' U \Longleftrightarrow_{\mathrm{def}} a \lhd U \cup V.$$

Note that $V \lhd' \emptyset$ and $\emptyset \lhd' V$, so the new cover relation identifies the open set V with the empty set. What remains is the complement of V.

Theorem If X is compact, and $U \subseteq X$ a set of neighbourhoods, then $X \setminus U$ is compact.

We define he unit interval by $[0,1] = \mathcal{R} \setminus V$ where $V = \{(a,b) : b < 0 \text{ or } 1 < a\}$. Denote the cover relation of $[0,1]$ by $\lhd'$.

Heine-Borel theorem $[0,1]$ is a compact formal topology.

Proof. Suppose $X \lhd' U$, where X is the basic nbhds of $\mathcal{R}$. Thus in particular $(0-2\varepsilon, 1+2\varepsilon) \lhd' U$ and thus $(0-2\varepsilon, 1+2\varepsilon) \lhd V \cup U$. By the elementary characterisation of $\lhd$ there is a finite $F \subseteq V \cup U$ with $(0-\varepsilon, 1+\varepsilon)^* \subseteq F^*$. As F is finite, we can prove $(0-\varepsilon, 1+\varepsilon) \lhd F$ using G1 and G2. Also we find finite $F_1 \subseteq F \cap U \subseteq U$ so that $(0-\varepsilon, 1+\varepsilon) \lhd V \cup F_1$. Hence $(0-\varepsilon, 1+\varepsilon) \lhd' F_1$. But $X \lhd' (0-\varepsilon, 1+\varepsilon)$ so $X \lhd' F_1$. □

Open subspaces

Let X be a formal topology. Each set $V \subseteq X$ of neighbourhoods determines an open sublocale $X_{|V}$ whose covering relation $\lhd'$ is given by

$$a \lhd' U \Longleftrightarrow_{\mathrm{def}} a \wedge V \lhd U.$$

Note that $U_1 \lhd' U_2$ iff $U_1 \wedge V \lhd U_2 \wedge V$. Hence only the part inside V counts when comparing two open sets.

Example

For two points $\alpha, \beta \in \mathcal{R}$ the open interval (α, β) is $\mathcal{R}_{|V}$ where

$$V = \{(a, b) : \alpha < \hat{a} \;\&\; \hat{b} < \beta\}.$$

Continuous maps relate the covers

Let $\mathcal{X} = (X, \leq, \lhd)$ and $\mathcal{Y} = (Y, \leq', \lhd')$ be formal topologies. A relation $F \subseteq X \times Y$ is a *continuous mapping* $\mathcal{X} \longrightarrow \mathcal{Y}$ if

- $U \lhd' V \Longrightarrow F^{-1}U \lhd F^{-1} V$, ("preservation of arbitrary sups")
- $X \lhd F^{-1} Y$, ("preservation of finite infs")
- $a \lhd F^{-1}V,\ a \lhd F^{-1}W \Longrightarrow a \lhd F^{-1}(V \wedge W)$.
- $a \lhd U$, $x\,F\,b$ for all $x \in U \Longrightarrow a\,F\,b$,

Each such induces a continuous point function $f = \mathrm{Pt}(F)$ given by

$$\alpha \mapsto \{b : (\exists a \in \alpha) R(a, b)\} : \mathrm{Pt}(\mathcal{X}) \longrightarrow \mathrm{Pt}(\mathcal{Y})$$

and that satisfies: $a\,F\,b \Rightarrow f[a^*] \subseteq b^*$.

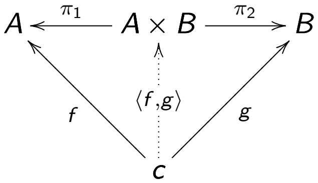
The category of formal topologies.

The inductively generated formal topologies and continuous mappings form a category **FTop**, which is classically equivalent to **Loc**.

They both share many abstract properties with the category of topological spaces which can be expressed in category theoretic language.

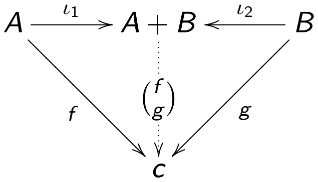
This makes it possible to describe topological constructions without mentioning points.

Recall: A *product* of objects A and B in a category $\mathcal{C}$ is an object $A \times B$ and two arrows $\pi_1 : A \times B \longrightarrow A$ and $\pi_2 : A \times B \longrightarrow B$ in $\mathcal{C}$, so that for any arrows $f : P \longrightarrow A$ and $g : P \longrightarrow B$ there is a unique arrow $\langle f, g \rangle : P \longrightarrow A \times B$ so that $\pi_1 \langle f, g \rangle = f$ and $\pi_2 \langle f, g \rangle = g$.

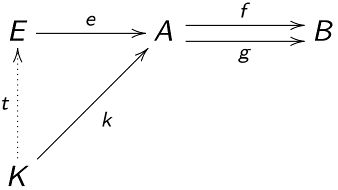


The *dual notion* of product is coproduct or sum.

A *coproduct* of objects A and B in a category $\mathcal{C}$ is an object $A+B$ and two arrows $\iota_1 : A \longrightarrow A+B$ and $\iota_2 : B \longrightarrow A+B$ in $\mathcal{C}$, so that for any arrows $f : A \longrightarrow P$ and $g : B \longrightarrow P$ there is a unique arrow $\binom{f}{g} : A+B \longrightarrow P$ so that $\binom{f}{g}\iota_1 = f$ and $\binom{f}{g}\iota_2 = g$.



A *equalizer* of a pair of arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in a category $\mathcal{C}$ is a object E and an arrow $e : E \longrightarrow A$ with $fe = ge$ so that for any arrow $k : K \longrightarrow A$ with $fk = gk$ there is a unique arrow $t : K \longrightarrow E$ with $et = k$.



In **Top**: $E = \{x \in A : f(x) = g(x)\}$.

Dual notion: A *coequalizer* of a pair of arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in a category $\mathcal{C}$ is a object Q and an arrow $q : B \longrightarrow Q$ with $qf = qg$ so that for any arrow $k : B \longrightarrow Q$ with $kf = kg$ there is a unique arrow $t : Q \longrightarrow K$ with $tq = k$.

Categorical topology. As the category of topological spaces has limits and colimits, many spaces of interest can be built up using these universal constructions, starting from the real line and intervals.

The circle

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$$

is an equaliser of the constant 1 map and $(x, y) \mapsto x^2 + y^2$.

It can also be constructed as a coequaliser of $s, t : \{\star\} \longrightarrow [0, 1]$ where $s(\star) = 0, t(\star) = 1$. (Identifying ends of a compact interval.)

The categorical properties of the category **FTop** of set-presented formal topologies ought to be same as that of the category of locales **Loc**.

Theorem. **Loc** has small limits and small colimits.

However, since we are working under the restraint of predicativity (as when the meta-theory is Martin-Löf type theory) this is far from obvious. (Locales are *complete* lattices with an infinite distributive law.)

FTop has ...

Limits:

- Products
- Equalisers
- (and hence) Pullbacks

Colimits:

- Coproducts (Sums)
- Coequalisers
- (and hence) Pushouts

Moreover:

• certain exponentials (function spaces): X^I, when I is locally compact.

Further constructions using limits and colimits

1. The torus may be constructed as the coequaliser of the followings maps $\mathbb{R}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{R}^2$

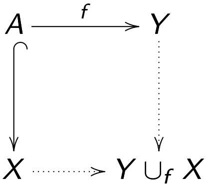
$$(\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x} \qquad (\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x} + \mathbf{n}.$$

2. The real projective space $\mathbb{RP}^n$ may be constructed as coequaliser of two maps

$$\mathbb{R}^{n+1} \times \mathbb{R}_{\neq 0} \longrightarrow \mathbb{R}^{n+1}$$

$$(\mathbf{x}, \lambda) \mapsto \mathbf{x} \qquad (\mathbf{x}, \lambda) \mapsto \lambda \mathbf{x}.$$

3. For $A \hookrightarrow X$ and $f : A \longrightarrow Y$, the pushout gives the attaching map construction:



4. The special case of 3, where $Y = 1$ is the one point space, gives the space X/A where A in X is collapsed to a point.

Products of formal topologies

Let $A = (A, \leq_A, \lhd_A)$ and $B = (B, \leq_B, \lhd_B)$ be inductively generated formal topologies. The product topology is $A \times B = (A \times B, \leq', \lhd')$ where

$$(a, b) \leq' (c, d) \iff_{\text{def}} a \leq_A c \,\&\, b \leq_B d$$

and $\lhd'$ is the smallest cover relation on $(A \times B, \leq_{A \times B})$ so that

- $a \lhd_A U \implies (a, b) \lhd' U \times \{b\}$,
- $b \lhd_B V \implies (a, b) \lhd' \{a\} \times V$.

The projection $\pi_1 : A \times B \longrightarrow A$ is defined by

$$(a, b)\, \pi_1\, c \iff_{\text{def}} (a, b) \lhd' A \times \{c\}$$

(Second projection is similar.)

Example

The formal real plane $\mathcal{R}^2 = \mathcal{R} \times \mathcal{R}$ has by this construction formal rectangles $((a,b),(c,d))$ with rational vertices for basic neighbourhoods (ordered by inclusion). The cover relation may be characterized in an elementary, non-inductive way as

$$\begin{aligned} &((a,b),(c,d)) \lhd' U \iff \\ &\quad (\forall u,v,x,y)[a < u < v < b, c < x < y < d \Rightarrow \\ &\qquad (\exists \text{ finite } F \subseteq U)(u,v) \times (x,y) \subseteq F^*] \end{aligned}$$

Formal topological manifolds

A formal topology M is an *n-dimensional manifold* if there are two families of sets of open neighbourhoods $\{U_i \subseteq M\}_{i \in I}$ and $\{V_i \subseteq \mathcal{R}^n\}_{i \in I}$ and a family of isomorphisms in **FTop**

$$\varphi_i : M_{|U_i} \longrightarrow (\mathcal{R}^n)_{|V_i} \qquad (i \in I)$$

so that

$$M \lhd \{U_i : i \in I\}.$$

Subspaces defined by inequations

Let $V \subseteq \mathcal{R}^2$ be the set of open neighbourhoods above the graph $y = x$ i.e.

$$V = \{((a, b), (c, d)) \in \mathcal{R}^2 : b < c\}$$

Then $L = (\mathcal{R}^2)_{|V} \hookrightarrow \mathcal{R}^2$ has for points pairs (α, β) with $\alpha < \beta$.
Let $f, g : X \longrightarrow \mathcal{R}$ be continuous maps. Then form the pullback:

$$\begin{array}{ccc} S & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \langle f, g\rangle} \\ L & \hookrightarrow & \mathcal{R}^2 \end{array}$$

Then the open subspace $S \hookrightarrow X$ has for points those $\xi \in \mathrm{Pt}(X)$ where $f(\xi) < g(\xi)$.

Subspaces defined by inequations (cont.)

Similarly we may define subspaces by non-strict inequalities. Define $K = (\mathcal{R}^2 \setminus V) \hookrightarrow \mathcal{R}^2$, then K has for points $(\alpha, \beta) \in \mathcal{R}^2$ such that $\alpha \geq \beta$.

Replacing L by K is the previous pullback diagram we get that $S \hookrightarrow X$ has for points those $\xi \in \mathrm{Pt}(X)$ where $f(\xi) \geq g(\xi)$.

Coequalizers in formal topology

Coequalizers are used to built quotient spaces, and together with sums, to attach spaces to spaces.

As the category **Loc** is opposite of the category **Frm** the coequalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in **Loc** can be constructed as the equalizer of the pair $B \underset{g^*}{\overset{f^*}{\rightrightarrows}} A$ in **Frm**.

As **Frm** is "algebraic" the equalizer can be constructed as

$$E = \{b \in B : f^*(b) = g^*(b)\} \hookrightarrow B \underset{g^*}{\overset{f^*}{\rightrightarrows}} A$$

Coequalizers in formal topology (cont.)

A direct translation into **FTop** yields the following suggestion for a construction:

$$\{U \in \mathcal{P}(B) : \widetilde{F^{-1}U} = \widetilde{G^{-1}U}\}$$

for a pair of continuous mappings $F, G : A \longrightarrow B$ between formal topologies. Here $\widetilde{W} = \{a \in A : a \lhd W\}$.

Difficulty: the new basic neighbourhoods U do not form a set.

It turns out that can one find, depending on F and G, a set of subsets $\mathcal{R}(B)$ with the property that

$$\widetilde{F^{-1}U} = \widetilde{G^{-1}U}, b \in U \Longrightarrow$$
$$(\exists V \in \mathcal{R}(B))(b \in V \subseteq U \,\&\, \widetilde{F^{-1}V} = \widetilde{G^{-1}V})$$

Now the formal topology, whose basic neighbourhoods are

$$Q = \{V \in \mathcal{R}(B) : \widetilde{F^{-1}V} = \widetilde{G^{-1}V}\},$$

and where

$U \lhd_Q W$ iff $U \lhd_B \cup W$

for $W \subseteq \mathcal{R}(B)$, defines a coequaliser. Moreover the coequalising morphism $P : B \longrightarrow Q$ is given by:

$a\, P\, U$ iff $a \lhd_B U$.

Exercise: Construct the standard topological n-simplex

$$\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 + \cdots + x_n = 1, x_0 \geq 0, \ldots, x_n \geq 0\}$$

in a point-free way. (Combine the above constructions.)

Exercise: For a simplicial set S, construct its geometric realization $|S|$ in a point-free way.

Cohesiveness of formal spaces: joining paths

Def The space Y has the *Path Joining Principle (PJP)* if
$f : [a, b] \longrightarrow Y$ and $g : [b, c] \longrightarrow Y$
are continuous functions with $f(b) = g(b)$, then there exists a unique continuous function $h : [a, c] \longrightarrow Y$, with $h(t) = f(t)$ for $t \in [a, b]$, and $h(t) = g(t)$ for $t \in [b, c]$.

Theorem Each complete metric space Y satisfies PJP.

For general metric spaces PJP fails constructively:

Proposition There is a metric space Y $(=[-1,0]\cup[0,1])$, such that if the PJP is valid for Y, then for any real x

$$x \leq 0 \text{ or } x \geq 0.$$

Proof : Exercise.

However:

Proposition In **FTop** the PJP holds for any Y.

There is a stronger version (HJP) which allows for the composition of homotopies:

Proposition: Let $\mathcal{X}$ be a formal topology. For $\alpha \leq \beta \leq \gamma$ in $\mathrm{Pt}(\mathcal{R})$, the diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\langle 1_{\mathcal{X}}, \hat{\beta} \rangle} & \mathcal{X} \times [\beta, \gamma] \\ {\scriptstyle \langle 1_{\mathcal{X}}, \hat{\beta} \rangle} \downarrow & & \downarrow {\scriptstyle 1_{\mathcal{X}} \times E_2} \\ \mathcal{X} \times [\alpha, \beta] & \xrightarrow[1_{\mathcal{X}} \times E_1]{} & \mathcal{X} \times [\alpha, \gamma] \end{array} \tag{1}$$

is a pushout diagram in **FTop.** Here E_1 and E_2 are the obvious embeddings of subspaces.

The following lemma is used in the proof. It may be regarded as a (harmless) point-free version of the trichotomy principle for real numbers.

Lemma For any point β of the formal reals $\mathcal{R}$ let

$$T_\beta = \{(a,b) \in \mathcal{R} : b < \beta \text{ or } (a,b) \in \beta \text{ or } \beta < a\}.$$

Then for any $U \subseteq \mathcal{R}$ we have

$$U \sim_{\mathcal{R}} U_{\leq} \cap T_\beta.$$

Here $V \sim_{\mathcal{R}} W$ means that $V \lhd_{\mathcal{R}} W$ and $W \lhd_{\mathcal{R}} V$.

Proof. The covering $U_{\leq} \cap T_\beta \lhd_{\mathcal{R}} U$ is clear by the axioms (Tra) and (Ext).

To prove the converse covering, suppose that $(a, b) \in U$. Then it suffices by axiom (G1) to show $(c, d) \lhd_{\mathcal{R}} U_{\leq} \cap T_\beta$ for any $a < c < d < b$. For any such c, d we have $\beta < c$ or $a < \beta$, by the co-transitivity principle for real numbers. In the former case, $(c, d) \in U_{\leq} \cap T_\beta$, so suppose $a < \beta$. Similarly comparing d, b to β we get $d < \beta$, in which case $(c, d) \in U_{\leq} \cap T_\beta$, or we get $\beta < b$. In the latter case we have $(a, b) \in U_{\leq} \cap T_\beta$, so in particular $(c, d) \lhd U_{\leq} \cap T_\beta$. □

Exercise/Research problem: For a formal topology $\mathcal{X}$, what is a reasonable notion of fundamental group or fundament groupoid?

Constructivity of the Fundamental Adjunction ?

By the adjunction

$$\Omega : \textbf{Top} \longrightarrow \textbf{Loc} \qquad \dashv \qquad \mathrm{Pt} : \textbf{Loc} \longrightarrow \textbf{Top}$$

the functor Ω gives a full and faithful embedding of all Hausdorff spaces into locales.

However, the functor Ω yields only point-wise covers in **Loc**, so $\Omega(\{0,1\}^{\omega})$ is not compact, (unless e.g. the Fan Theorem is assumed). The localic version of the Cantor is compact, as we have seen.

To embed the locally compact metric spaces used in BISH we need a more refined functor.

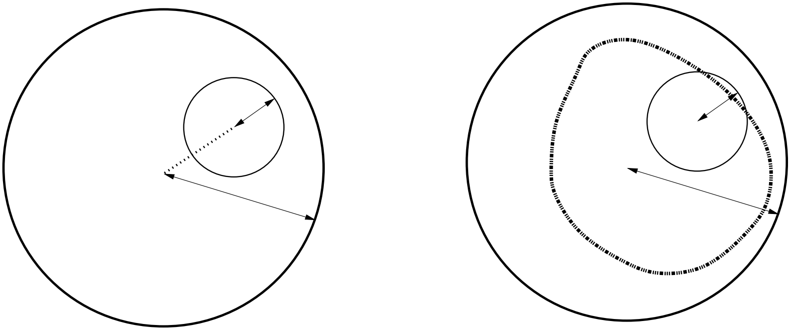
Localic completion

For any metric space (X, d) its **localic completion** (Vickers 2005, 2009) $\mathcal{M} = \mathcal{M}_X$ is a formal topology $(M, \leq_{\mathcal{M}}, \lhd_{\mathcal{M}})$ where M is the set of *formal ball symbols*

$$\{\mathrm{b}(x, \delta) : x \in X, \delta \in \mathbb{Q}_+\}.$$

These symbols are ordered by formal inclusion

- $\mathrm{b}(x, \delta) \leq_{\mathcal{M}} \mathrm{b}(y, \varepsilon) \iff d(x, y) + \delta \leq \varepsilon$
- $\mathrm{b}(x, \delta) <_{\mathcal{M}} \mathrm{b}(y, \varepsilon) \iff d(x, y) + \delta < \varepsilon.$



The cover is generated by

(M1) $p \lhd \{q \in M : q < p\}$, for any $p \in M$,

(M2) $M \lhd \{\mathrm{b}(x, \delta) : x \in X\}$ for any $\delta \in \mathbb{Q}_+$,

The points of $\mathcal{M}_X$ form a metric completion of X. There is a metric embedding

$$j_X = j : X \longrightarrow \mathrm{Pt}(\mathcal{M}_X)$$

given by

$$j(u) = \{\mathrm{b}(y, \delta) : d(u, y) < \delta\}.$$

Moreover, j_X is an isometry in case X is already complete.

For $X = \mathbb{Q}$, we get the formal real numbers modulo notation for intervals $\mathrm{b}(x, \delta) = (x - \delta, x + \delta)$.

Remarks Countable choice is used to prove the isometry.

The covering relation is, in the case when X is the Baire space, given by an infinite wellfounded tree. Must allow generalised inductive definitions in the meta theory.

Theorem (Vickers) Let X be a complete metric space. Then: X is compact iff $\mathcal{M}_X$ is compact.

Theorem (P.) If $X = (X, d)$ is locally compact, then $\mathcal{M}_X$ is locally compact as a formal topology, i.e. $p \lhd \{q \in M : q \ll p\}$ for any p.

Here the way below relation is

$q \ll p$ iff for every U: $p \lhd U$ implies there is some finitely enumerable (f.e.) $U_0 \subseteq U$ with $q \lhd U_0$.

Elementary characterisation of the cover of $\mathcal{M}_X$

Refined cover relations

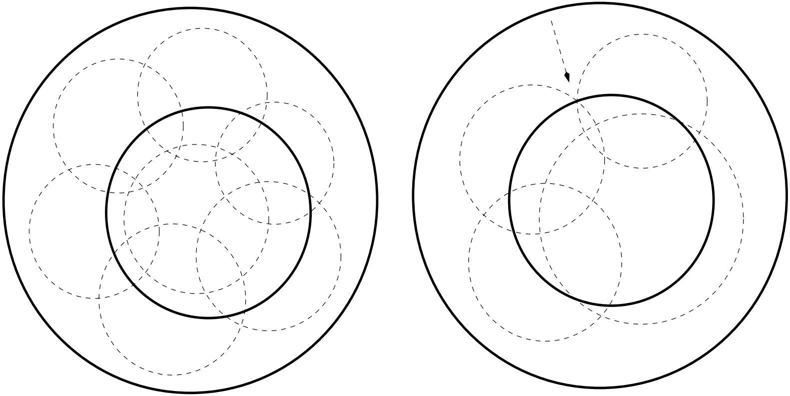
$$U \leq V \Longleftrightarrow_{\mathrm{df}} (\forall p \in U)(\exists q \in V) p \leq q$$

Note: If $U \leq V$, then $U \lhd V$. (Similar extension for $<$)

$$p \sqsubseteq_{\varepsilon} U \Leftrightarrow_{\mathrm{df}} (\forall q \leq p)[\mathrm{radius}(q) \leq \varepsilon \Rightarrow \{q\} \leq U]$$

Write $p \sqsubseteq U$ iff $p \sqsubseteq_{\varepsilon} U$ for some $\varepsilon \in \mathbb{Q}_{+}$.

Note: If $p \sqsubseteq V$, then $p \lhd V$.



Any sufficiently small disc included in the inner left disc “slips” into one of the dashed discs.

Elementary characterisation of the cover of $\mathcal{M}_X$

Lemma 4 Let X be locally compact. Let $\delta \in \mathbb{Q}_+$. For any formal balls $p < q$ there is a finitely enumerable C such that

$$p \sqsubseteq C < q$$

where all balls in C have radius $< \delta$.

Remark: In fact, this is as well a sufficient condition for local compactness.

Let $A(p, q) = \{C \in \mathcal{P}_{\text{fin.enum.}}(M_X) : p \sqsubseteq C < q\}$.

Elementary characterisation of the cover of $\mathcal{M}_X$

The following cover relation is generalising the one introduced by Vermeulen and Coquand for $\mathbb{R}$:

$$a \lessdot U \Leftrightarrow_{\mathrm{df}} (\forall b < c < a)\,(\exists U_0 \in A(b,c))\, U_0 < U.$$

Theorem If X is a locally compact metric space, then

$$a \lhd U \iff a \lessdot U$$

($\Leftarrow$ holds for any metric X space.)
Proof of $\Rightarrow$ goes by verifying that $\lessdot$ is a cover relation satisfying (M1) and (M2).

Let $f : X \longrightarrow Y$ be a function between complete metric spaces. Define the relation $A_f \subseteq M_X \times M_Y$ by

$$a\, A_f\, b \iff_{\mathrm{df}} a \lhd \{p : (\exists q < b) f[p_*] \subseteq q_*\}$$

Here $\mathrm{b}(x, \delta)_* = B(x, \delta)$.

Lemma $A_f : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ is a continuous morphism between formal topologies, whenever X is locally compact and f is continuous.

Define thus $\mathcal{M}(f) : \mathcal{M}_X \longrightarrow \mathcal{M}_Y$ as A_f

Remark Classically, $a\, A_f\, b$ is equivalent to $f[a_*] \subseteq b_*$. However, this definition does not work constructively.

Embedding Theorem

Embedding Theorem: The functor $\mathcal{M} : \mathbf{LCM} \longrightarrow \mathbf{FTop}$ is a full and faithful functor from the category of locally compact metric spaces to the category of formal topologies. It preserves finite products.

Explanation: Suppose that X and Y are locally compact metric spaces.
$\mathcal{M}$ *faithful* means: For $f, g : X \longrightarrow Y$ continuous functions

$$f = g \Longleftrightarrow \mathcal{M}(f) = \mathcal{M}(g)$$

$\mathcal{M}$ *full* means: any continuous mapping $F : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$ is the comes from some (unique) continuous function $f : X \longrightarrow Y$:

$$F = \mathcal{M}(f).$$

That $\mathcal{M}$ preserves binary products means e.g. that if

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

is a product diagram, then so is:

$$\mathcal{M}(X) \xleftarrow{\mathcal{M}(\pi_1)} \mathcal{M}(X \times Y) \xrightarrow{\mathcal{M}(\pi_2)} \mathcal{M}(Y).$$

Hence

$$\mathcal{M}(X \times Y) \cong \mathcal{M}(X) \times \mathcal{M}(Y).$$

Since we have $\mathcal{M}(\mathbb{R}) \cong \mathcal{R}$ we may then lift any continuous operation $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ from Cauchy reals to formal reals $\mathcal{M}(f) : \mathcal{R}^n \longrightarrow \mathcal{R}$.

Theorem For $F, G : \mathcal{R}^n \longrightarrow \mathcal{R}$ and $f = \mathrm{Pt(F)}, g = \mathrm{Pt(G)} : \mathbb{R}^n \longrightarrow \mathbb{R}$ we have

$$F \leq G \iff (\forall \bar{x}) f(\bar{x}) \leq g(\bar{x}).$$

Here $F \leq G$ is defined via factorization through a closed sublocale, as follows: For $F, G : A \longrightarrow \mathcal{R}$ continuous we say that $F \leq G$ iff $\langle G, F \rangle : A \longrightarrow \mathcal{R}^2$ factorizes through

$$\mathcal{R}^2 \setminus V \hookrightarrow \mathcal{R}^2$$

where $V = \{((a, b), (c, d)) \in \mathcal{R}^2 : b < c\}$.

There is a characterisation of maps into closed subspaces. For a set of neighbourhoods $W \subseteq Y$ let $Y \setminus W$ denote the corresponding closed sublocale, and let $E_{-W} : Y \setminus W \hookrightarrow Y$ be the embedding.

Let $F : X \longrightarrow Y$ be a continuous morphism. TFAE:

- F factors through E_{-W}.
- $F^{-1}W \lhd_X \emptyset$.
- $F : X \longrightarrow (Y \setminus W)$ is continuous.

Def $F : X \longrightarrow \mathcal{R}$ is **non-negative** if it factorises through $\mathcal{R} \setminus N$, where N is the set of $\mathrm{b}(x, \delta)$ with $x < \delta$.

The previous theorem may be used prove equalities and inequalities ($\leq$) for formal real numbers by standard point-wise proofs. Results of constructive analysis are readily reused.
More generally we have

Theorem Let X be a locally compact metric space. For $F, G : \mathcal{M}(X) \longrightarrow \mathcal{R}$ and $f = \mathrm{Pt(F)}, g = \mathrm{Pt(G)} : X \longrightarrow \mathbb{R}$ we have

$$F \leq G \iff (\forall x \in X) f(x) \leq g(x).$$

It would also be desirable to reuse results about strict inequalities ($<$). There is however no such straightforward "transfer theorem" as for $\leq$.

This is connected to the problem of localising infima of continuous functions $[0, 1] \longrightarrow \mathbb{R}$, which we consider as a preparation.

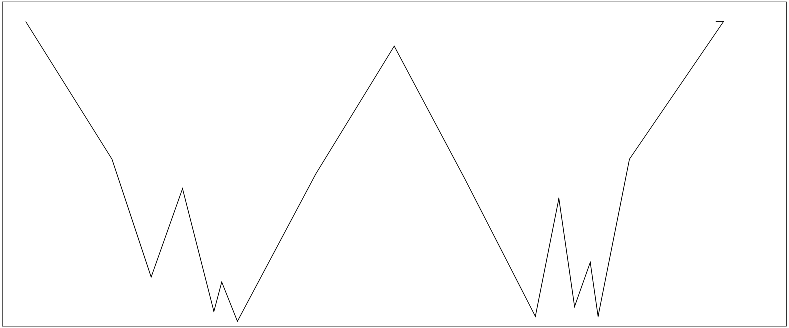
The infimum problem on compact sets

Problem: In BISH find a positive uniform lower bound of a uniformly continuous function $f : [0,1] \longrightarrow (0,\infty)$.

This is impossible by results of Specker 1949, Julian & Richman 1984 using a recursive (computable) interpretation.

(Classically, f attains its minimum.)

However in Brouwerian intuitionism (INT), a uniform lower bound can be found using the FAN theorem.



In Bishop's constructive analysis this phenomenon is serious problem. (P. Schuster: What is continuity, constructively?) We ought to have
(*) The composition of two continuous functions is continuous.

Consider, however, the composition of f with the reciprocal:

$$[0,1] \xrightarrow{f} (0,\infty) \xrightarrow{1/(\cdot)} \mathbb{R}.$$

If $1/f$ is uniformly continuous, we can find an upper bound of the function on the compact the compact interval $[0,1]$. But this means that we can also find a uniform lower bound of f.

One concludes that (*) fails. The two versions of continuity are incompatible.

Point-free topology has a solution.

Open subspaces

$\mathcal{X}_{|G}$ — the open subspace of $\mathcal{X}$ given by a set of formal nbds G.
$E_{|G} : \mathcal{X}_{|G} \longrightarrow \mathcal{X}$ — the associated embedding.

Lemma Let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ be a continuous morphism between formal topologies. Let $G \subseteq Y$ be a set of nbds. TFAE:

- F factorises through $E_{|G} : \mathcal{Y}_{|G} \longrightarrow \mathcal{Y}$
- $X \lhd F^{-1}[G]$
- $F : \mathcal{X} \longrightarrow \mathcal{Y}_{|G}$ is continuous.

Def $F : \mathcal{X} \longrightarrow \mathcal{R}$ is **positive** if it factorises through $\mathcal{R}_{|P}$, where P is the set of $\mathrm{b}(x, \delta)$ with $x > \delta$.

The above Lemma gives several ways to prove positivity.

Locally uniformly positive maps

Let X be a metric space. A function $f : X \longrightarrow \mathbb{R}$ is *locally uniformly positive* (l.u.p) if for every $x \in X$ and every $\delta > 0$ there is some $\varepsilon > 0$ so that for all $y \in Y$

$$d(y, x) < \delta \Longrightarrow f(y) > \varepsilon.$$

Theorem Let X be a locally compact metric space, and suppose $f : X \longrightarrow \mathbb{R}$ is continuous. Then f is l.u.p. if, and only if, $\mathcal{M}(f) : \mathcal{M}(X) \longrightarrow \mathcal{R}$ is positive.

Corollary $\mathcal{M}(f) < \mathcal{M}(g) \quad \Longleftrightarrow \quad g - f$ is l.u.p.

Proof of Theorem ($\Leftarrow$): Suppose $\mathcal{M}(f)$ is positive. Using Theorem 3 we obtain

$$M(X) \lessdot \mathcal{M}(f)^{-1}[G].$$

Let $x \in X$ and $\delta > 0$. Then consider neighbourhoods $s = \mathrm{b}(x, \delta) < s' < p$. Thus we have some $p_1, \ldots, p_n < s'$ with

$$s \sqsubseteq \{p_1, \ldots, p_n\} < \mathcal{M}(f)^{-1}[G]$$

Thus there are $q_i > p_i$ and $r_i \in G$, $i = 1, \ldots, n$, satisfying $f[(q_i)^*] \subseteq (r_i)^*$. Thereby

$$f[s^*] = f[\mathrm{b}(x, \delta)^*] \subseteq f[p_1^*] \cup \cdots \cup f[p_n^*] \subseteq (r_1)^* \cup \cdots \cup (r_n)^*.$$

From which follows

$$f[B(x, \delta)] = f[\mathrm{b}(x, \delta)^*] \subseteq (\varepsilon, \infty),$$

so f is indeed l.u.p. $\square$

Thm Let X and Y be locally compact metric spaces. For formal open (and saturated) $A \subseteq \mathcal{M}_X$ and $B \subseteq \mathcal{M}_Y$, a continuous morphism

$$F : (\mathcal{M}_X)_{|A} \longrightarrow (\mathcal{M}_Y)_{|B}$$

induces a point function $f : A_* \longrightarrow B_*$ which satisfies the following conditions: for every compact $S <_* A$

K1) f is uniformly continuous on S and
K2) $f[S] <_* B$.

Here $S <_* A \Leftrightarrow_{\mathrm{def}} (\exists \mathrm{f.e.}\ F \subseteq \mathcal{M}_X)\ S \subseteq F_* \,\&\, F < A$.

Remark: The continuity condition is equivalent to: *for every* $a < A$*: K1 & K2 where* $S = a_*$*.*

The continuity conditions K1-2 generalize the standard C1-2.

For any open $U \subseteq X$, there is a largest formal open saturated $H(U) = \{a \in \mathcal{M}_X : a_* \subseteq U\}$. We have

Lemma: $S <_* H(U) \Longleftrightarrow S \Subset U$.

We have a converse to the previous theorem.

Thm If $f : A_* \longrightarrow B_*$ is K-continuous then $A_f : (\mathcal{M}_X)_{|A} \longrightarrow (\mathcal{M}_Y)_{|B}$ is a continuous mapping.

Theorem (P. 2010) There is a full and faithful functor from **OLCM** the category of open subspaces of locally compact metric spaces to **FTop** induced by

$$(X, U) \mapsto (\mathcal{M}_X)_{|H(U)}$$

Remark: For metric complements $U = \{x \in X : d(x, S) > 0\}$ we have $H(U) = \{\mathrm{b}(x, \delta) : x \in X, d(x, S) \geq \delta\}$.

Via the localic completion and the metric embedding $j_X : X \longrightarrow \mathrm{Pt}(\mathcal{M}(X))$, which is an isometry when X is complete, one gets a canonical notion of continuity between metric spaces. A function $f : X \longrightarrow Y$ between complete metric spaces is then called *FT-continuous*, or *formally continuous*, if there is a (continuous) morphism of formal topologies $F : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Pt}(\mathcal{M}(X)) & \xrightarrow{\mathrm{Pt}(F)} & \mathrm{Pt}(\mathcal{M}(Y)) \\ \uparrow{\scriptstyle j_X} & & \uparrow{\scriptstyle j_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

In general F is not uniquely determined by f.

This results shows that FT-continuity is stronger than Bishop's general notion of continuity

Thm (P 2012) An FT-continuous function $f : X \longrightarrow Y$ between complete metric spaces is uniformly continuous near each compact image.

Overtness and subspaces of localic completions

We must require something more than compactness of $M \setminus U$ to guarantee that $\mathrm{Pt}(M \setminus U)$ is compact in the metric sense.

The notion of overtness may be used to get a localic counterpart to Bishop's notion of compactness (P. Taylor 2000). See also T. Coquand and B. Spitters (2007) *Computable sets: located and overt locales*.

A formal topology $X = (X, \leq, \lhd)$ is *overt* if there is a set $P \subseteq X$ of neighbourhoods that are *positive*, which means

- P is inhabited,
- $a \lhd U$, $a \in P \Rightarrow U \cap P$ is inhabited,
- $U \lhd U \cap P$.

This P is necessarily unique if it exists.
P is usually called a *positivity predicate* for X.

Remark. In classical set theory, any formal topology X is overt, provided it hasa least one neighbourhood which is not covered by $\emptyset$. We may take

$$P = \{a \in X : \neg a \lhd \emptyset\}.$$

The notion is thus exclusively of constructive interest.

The following was already implicit in (Coquand and Spitters 2007). See (Coquand, P., and Spitters 2011) for a presentation.

Theorem Let X be a metric space and let $M = \mathcal{M}(X)$ be its localic completion. Let $U \subseteq M$ be such that $M \setminus U$ is compact and overt. Then $\mathrm{Pt}(M \setminus U)$ is compact in the metric sense.

To prove it we use:

Lemma: Let $U \subseteq M$ be such that $M \setminus U$ is overt. If a is a positive neighbourhood of $M \setminus U$, then $\alpha \in a^*$ for some $\alpha \in \mathrm{Pt}(M \setminus U)$.

Proof of Lemma: Suppose that P is the positivity predicate of $M \setminus U$. Denote the cover relation of $M \setminus U$ by $\lhd'$.
Suppose $a = \mathrm{b}(x, \delta) \in P$. Let $a_1 = a$. Suppose we have constructed in P:

$$a_1 \geq a_2 \geq \cdots \geq a_n,$$

so that $\mathrm{radius}(a_{k+1}) \leq \mathrm{radius}(a_k)/2$.
By (M1) and localisation we get

$$a_n \lhd' \{a_n\} \wedge \{b(y, \rho) : y \in X\}$$

where $\rho = \mathrm{radius}(a_n)/2$. Since $a_n \in P$ we obtain some $b \in \{a_n\} \wedge \{b(y, \rho) : y \in X\}$ with $b \in P$. Clearly $\mathrm{radius}(b) \leq \mathrm{radius}(a_n)/2$. Let $a_{n+1} = b$.

Let

$$\alpha = \{p \in M : (\exists n) a_n \leq p\}$$

Since the radii of a_n are shrinking, this defines a point in $\mathrm{Pt}(M)$. (Note that we used Dependent Choice.)

We claim that $\alpha \in \mathrm{Pt}(M \setminus U) = \mathrm{Pt}(M) \setminus U^*$. Suppose that $\alpha \in U^*$, i.e. for some $c \in U$: $c \in \alpha$. Hence there is n with $a_n \leq c$. Thus $a_n \lhd U$, that is $a_n \lhd' \emptyset$. But since a_n is positive, this is impossible! So $\alpha \in \mathrm{Pt}(M \setminus U)$. $\square$

Proof of Theorem: Let P and $\lhd'$ be as in the proof of Lemma. Let $\varepsilon > 0$ be given. Then by axiom M2 and positivity

$$M \lhd' \{\mathrm{b}(x, \varepsilon/2) : x \in X\} \lhd' \{\mathrm{b}(x, \varepsilon/2) : x \in X\} \cap P.$$

By compactness there is some finitely enumerable $F = \{\mathrm{b}(x_1, \varepsilon/2), \ldots, \mathrm{b}(x_n, \varepsilon/2)\} \subseteq \{\mathrm{b}(x, \varepsilon/2) : x \in X\} \cap P$ so that

$$M \lhd' F. \tag{2}$$

Since each $\mathrm{b}(x_i, \varepsilon/2)$ is positive there is by Lemma some $\alpha_i \in \mathrm{b}(x_i, \varepsilon/2)^*$ which is in $\mathrm{Pt}(M \setminus U)$. By (2) it follows that each point in $\mathrm{Pt}(M \setminus U)$ has smaller distance than ε to some point α_i. Thus $\{\alpha_1, \ldots, \alpha_n\}$ is the required ε-net. □

Nice open sets - metric complements

Let X be a metric space and let $S \subseteq X$ be a located subset. The *metric complement of S* is defined by

$$-S = \{x \in X : d(x, S) > 0\}$$

Theorem: Let X be locally compact. For formal open set $U \subseteq M(X)$, $\mathrm{Pt}(U)$ is a metric complement of a located subset in case the closed sublocale $M(X) \setminus U$ is overt.

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