Categories with families, FOLDS and logic enriched type theory: *Lawvere signatures and theories*

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Categorical Logic and Univalent Foundations
Leeds, July 27–29, 2016

Background

A dependently typed (or sorted) system of first-order logic, FOLDS, was introduced and studied by Makkai (1995, 1998, 2013).

The purpose of FOLDS is to provide a natural logical system for formalizing (higher) category theory.

FOLDS can also be considered as a so-called logic enriched type theory (Maietti-Sambin 2005, Gambino-Aczel 2006) where the underlying type theory is very rudimentary.

Belo (2007, 2008) and Aczel (2004) introduced similar systems which are based on Cartmell's (1986) generalized algebraic theories. Unlike FOLDS they allow function symbols, and may also allow equality, even between sorts.

Lawvere signatures and theories

Lawvere's approach to first-order signatures and theories appears is here extended to dependently sorted first-order logic. Earlier approaches applies only to restricted sort systems (Makkai, no functions) or to limited semantical domains (Belo 2007). The use of categories with families achieves the full generality.

Here we shall present the pertaining parts of the paper P. (2016) *Categories with families, FOLDS and logic enriched type theory* (arXiv:1605.01586).

Dependent signatures

Example: The signature Σ_{Cat} for a category is given by the sequence of declarations below

$$
\begin{array}{ll}
\mathsf{Ob}\ \mathrm{type} & () \\
X \to Y\ \mathrm{type} & (X : \mathsf{Ob}, Y : \mathsf{Ob}) \\
1_X : X \to X & (X : \mathsf{Ob}) \\
g \circ f : X \to Z & (X : \mathsf{Ob}, Y : \mathsf{Ob}, Z : \mathsf{Ob}, g : Y \to Z, f : X \to Y)
\end{array}
$$

How do we assign semantical objects to the components of the signature?

For standard many-sorted (non-dependent) signatures we can do this individually for each symbol and type/sort. A category with finite products can serve as a semantic domain.

Equality on dependent types and families of setoids

No built in equality assumed in DFOL. Let A be a type with an equivalence relation $=_A$. For a dependent type B over A we may introduce an equivalence relation $=_{B,x}$ (write $=_{B(x)}$) as follows

$$
\begin{array}{ll}
u =_{B(x)} v \text{ formula} & (x : A, u, v : B(x)) \\
u =_{B(x)} u & (x : A, u : B(x)) \\
u =_{B(x)} v \Longrightarrow v =_{B(x)} u & (x : A, u, v : B(x)) \\
u =_{B(x)} v \wedge v =_{B(x)} w \Longrightarrow u =_{B(x)} w & (x : A, u, v, w : B(x))
\end{array}
$$

How do elements of $B(x)$ and $B(y)$ compare if $x =_A y$ is true? We associate $=_A$ with a proposition-type E_A as follows

$$
\begin{array}{ll}
E_A(x, y) \text{ type} & (x, y : A) \\
x =_A y & (x, y : A, p : E_A(x, y)) \\
x =_A y \Longrightarrow (\exists p : E_A(x, y))\top & (x, y : A).
\end{array}
$$

Next introduce transport functions tr_p to be able to relate elements of $B(x)$ and $B(y)$ when $x =_A y$

$$
\begin{array}{ll}
\mathrm{tr}_p(u) : B(y) & (x, y : A, p : E_A(x, y), u : B(x)) \\
u =_{B(x)} v \Rightarrow \mathrm{tr}_p(u) =_{B(y)} \mathrm{tr}_p(v) & (x, y : A, p : E_A(x, y), u, v : B(x)) \\
\mathrm{tr}_p(u) =_{B(y)} \mathrm{tr}_q(u) & (x, y : A, p, q : E_A(x, y), u : B(x)) \\
\mathrm{tr}_p(u) =_{B(x)} u & (x : A, p : E_A(x, x), u : B(x)) \\
\mathrm{tr}_q(\mathrm{tr}_p(u)) =_{B(z)} \mathrm{tr}_r(u) & (x, y, z : A, q : E_A(y, z), p : E_A(x, y), \\
& \quad r : E_A(x, z), u : B(x))
\end{array}
$$

Note that second equation says that the transport function is independent of the particular value of the proof object in $E_A(x, y)$. This makes the family B over A proof-irrelevant.
These definitions can be extended to arbitrary contexts of setoids, see (Maietti 2009).

Categories with families

— a standard semantical structure for dependent types, which may be thought of a dependent version of finite product categories (cf Lawvere theories).

A category with families (cwf) consists of the following data

(a) A category $\mathcal{C}$ with a terminal object $\top$.

(b) A functor $\mathrm{Ty} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$. Write: $A\{f\} = \mathrm{Ty}(f)(A)$.

(c) For each $A \in \mathrm{Ty}(\Gamma)$, an object $\Gamma.A$ in $\mathcal{C}$ and a morphism $\mathrm{p}(A) = \mathrm{p}_\Gamma(A) : \Gamma.A \longrightarrow \Gamma$.

(d) For each $A \in \mathrm{Ty}(\Gamma)$, there is a set $\mathrm{Tm}(\Gamma, A)$ — thought of as the terms of type A. It should be such that for $f : \Delta \longrightarrow \Gamma$ there is a function $\mathrm{Tm}(f) : \mathrm{Tm}(\Gamma, A) \to \mathrm{Tm}(\Delta, A\{f\})$, where we write $a\{f\}$ for $\mathrm{Tm}(f)(a)$, satisfying the following

  (i) $a\{1_\Gamma\} = a$ for $a \in \mathrm{Tm}(\Gamma, A)$

  (ii) $a\{f \circ g\} = a\{f\}\{g\}$ for $a \in \mathrm{Tm}(\Gamma, A)$

(e) For each $A \in \mathrm{Ty}(\Delta)$ there is an element $\mathrm{v}_A \in \mathrm{Tm}(\Delta.A, A\{\mathrm{p}(A)\})$.

(f) For any morphism $f : \Gamma \longrightarrow \Delta$ and $a \in \mathrm{Tm}(\Gamma, A\{f\})$, there is

$$\langle f, a\rangle_A : \Gamma \longrightarrow \Delta.A.$$

This construction should satisfy

(i) $\mathrm{p}(A) \circ \langle f, a\rangle_A = f$,

(ii) $\mathrm{v}_A\{\langle f, a\rangle_A\} = a$,

(iii) $\langle \mathrm{p}(A) \circ h, \mathrm{v}_A\{h\}\rangle_A = h$ for any $h : \Gamma \longrightarrow \Delta.A$.

(iv) for any $g : \Theta \longrightarrow \Gamma$,

$$\langle f, a\rangle_A \circ g = \langle f \circ g, a\{g\}\rangle_A$$

□

Cwf morphisms

Dybjer (1996) defines a notion of morphism between cwfs.

A (strict) cwf morphism

$$(\mathcal{C}, \mathrm{Ty}, \mathrm{Tm}) \longrightarrow (\mathcal{C}', \mathrm{Ty}', \mathrm{Tm}')$$

is a triple (F, σ, θ) consisting of a functor $F : \mathcal{C} \to \mathcal{C}'$ such that

$$F(\top_{\mathcal{C}}) = \top_{\mathcal{C}'} \tag{1}$$

and a family of functions $\sigma_\Gamma : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}'(F(\Gamma))$ satisfying the condition for $f : \Delta \to \Gamma$, and $A \in \mathrm{Ty}(\Gamma)$,

$$\begin{array}{ccc} \mathrm{Ty}(\Gamma) & \xrightarrow{\sigma_\Gamma} & \mathrm{Ty}'(F(\Gamma)) \\ {\scriptstyle \_\{f\}} \downarrow & & \downarrow {\scriptstyle \_\{F(f)\}} \\ \mathrm{Ty}(\Delta) & \xrightarrow[\sigma_\Delta]{} & \mathrm{Ty}'(F(\Delta)). \end{array} \tag{2}$$

commutes (that is $\sigma : \mathrm{Ty} \longrightarrow \mathrm{Ty}' \circ F$ is a natural transformation) and ...

such that

$$F(\Gamma.A) = F(\Gamma).\sigma_\Gamma(A)$$

and

$$F(\mathrm{p}_\Gamma(A)) = \mathrm{p}_{F(\Gamma)}(\sigma_\Gamma(A)) \tag{3}$$

and as third component a family of functions

$$\theta_{\Gamma,A} : \mathrm{Tm}(\Gamma, A) \to \mathrm{Tm}'(F(\Gamma), \sigma_\Gamma(A))$$

such that for $f : \Delta \to \Gamma$, $A \in \mathrm{Ty}(\Gamma)$ the following diagram commutes

$$\begin{array}{ccc} \mathrm{Tm}(\Gamma, A) & \xrightarrow{\theta_{\Gamma,A}} & \mathrm{Tm}'(F(\Gamma), \sigma_\Gamma(A)) \\ \Big\downarrow {\scriptstyle \_\{f\}} & & \Big\downarrow {\scriptstyle \_\{F(f)\}} \\ \mathrm{Tm}(\Delta, A\{f\}) & \xrightarrow[\theta_{\Delta,A\{f\}}]{} & \mathrm{Tm}'(F(\Delta), \sigma_\Delta(A\{f\})). \end{array} \tag{4}$$

Moreover ...

it is required that for $A \in \mathrm{Ty}(\Gamma)$,

$$\theta_{\Gamma.A,A\{\mathrm{p}(A)\}}(\mathrm{v}_A) = \mathrm{v}_{\sigma_\Gamma(A)} \tag{5}$$

and furthermore it is required that for $f : \Delta \longrightarrow \Gamma$ and $a \in \mathrm{Tm}(\Delta, A\{f\})$,

$$F(\langle f, a\rangle_A) = \langle F(f), \theta_{\Delta,A\{f\}}(a)\rangle_{\sigma_\Gamma(A)}. \tag{6}$$

□

Theorem

The cwfs and cwf morphisms form a category.

Free cwfs from DFOL signatures

(following Belo 2007) Fix a symbol system $\mathcal{S} = ((V, (\varphi, \mathrm{fr})), F, T)$. For any presignature Σ over $\mathcal{S}$, let $\mathcal{J}(\Sigma)$ be the smallest set of judgement expressions closed under the five rules below.

$$\frac{}{\langle\rangle \text{ context}}$$

$$\frac{\Gamma \text{ context} \quad A \text{ type } (\Gamma)}{\Gamma, x : A \text{ context}} \quad x \in \mathrm{Fresh}(\Gamma)$$

$$\frac{x_1 : A_1, \ldots, x_n : A_n \text{ context}}{x_i : A_i \; (x_1 : A_1, \ldots, x_n : A_n)}$$

$$\frac{\bar{a} : \Delta \longrightarrow \Gamma}{S(\bar{a}_{\bar{i}}) \text{ type } (\Delta)} \quad (\Gamma, S, \bar{i}) \text{ in } \Sigma$$

$$\frac{\bar{a} : \Delta \longrightarrow \Gamma \quad U[\bar{a}/\Gamma] \text{ type } (\Delta)}{f(\bar{a}_{\bar{i}}) : U[\bar{a}/\Gamma] \; (\Delta)} \quad (\Gamma, f, \bar{i}, U) \text{ in } \Sigma$$

($\bar{i} = (1, \ldots, n)$, $n = |\Gamma|$ when no hidden variables are considered.)

We define following (Belo 2007) a pre-signature Σ to be a signature if the following *correctness conditions* hold:

- If $(\Gamma, S, \bar{i}) \in \Sigma$, then $(\Gamma \ \mathrm{context}) \in \mathcal{J}(\Sigma)$,
- If $(\Gamma, f, \bar{i}, U) \in \Sigma$, then $(U \ \mathrm{type} \ (\Gamma)) \in \mathcal{J}(\Sigma)$.

Apart from taking consistent unions, we have two possibilities to extend a signature.

Lemma

Let Σ be a signature. Suppose that $(\Gamma \ \mathrm{context}) \in \mathcal{J}(\Sigma)$.

(i) *If $S \in T$ is a type symbol not declared in Σ and $\bar{i} \in \mathrm{DS}(\Gamma)$, then $\Sigma \cup \{(\Gamma, S, \bar{i})\}$ is a signature.*

(ii) *Suppose that $(U \ \mathrm{type} \ (\Gamma)) \in \mathcal{J}(\Sigma)$. If $f \in F$ is a function symbol not declared in Σ and $\bar{i} \in \mathrm{DS}(\Gamma)$, then $\Sigma \cup \{(\Gamma, f, \bar{i}, U)\}$ is a signature.*

□

Theorem

The contexts and context maps for a fixed signature Σ form a category with families $\mathcal{F}_\Sigma$.

By restricting the fresh variable providers (fvps) (φ, fr) we get even a contextual category:

Theorem

The contexts and context maps for a fixed signature Σ, where the fvp is of de Bruijn type, form a contextual category with families $\mathcal{F}_\Sigma$. $\square$

Models for type systems

A model of a type system over the signature Σ is a cwf morphism $(F, \sigma, \theta) : \mathcal{F}_\Sigma \to \mathcal{C}$ into some cwf $\mathcal{C}$. Such models are determined by their values on the components of the signature. However it is not obvious how to construct such a cwf morphism.

The extension problem is given a cwf morphism

$$G = (G, \sigma, \theta) : \mathcal{F}_\Sigma \to \mathcal{C},$$

a signature $\Sigma' \supseteq \Sigma$, and some suitable data in $\mathcal{C}$, find a cwf morphism

$$G' = (G', \sigma', \theta') : \mathcal{F}_{\Sigma'} \to \mathcal{C}$$

mapping the new syntactic entities to this data and such that $G' \circ E = G$, where E is the canonical cwf embedding $\mathcal{F}_\Sigma \to \mathcal{F}_{\Sigma'}$.

Fix a signature Σ and a cwf morphism $G = (G, \sigma, \theta) : \mathcal{F}_\Sigma \to \mathcal{C}$.

Theorem

Suppose that S is a type symbol not in Σ and $(\Gamma_S \text{ context}) \in \mathcal{J}(\Sigma)$, so that $\Sigma' = \Sigma \cup \{(\Gamma_S, S)\}$ is a signature. For $A \in \mathrm{Ty}_\mathcal{C}(G(\Gamma_S))$, there is a unique cwf morphism $G' = (G', \sigma', \theta') : \mathcal{F}_{\Sigma'} \to \mathcal{C}$ with $G' \circ E = G$ and

$$\sigma'(\Gamma_S, S(\bar{x})) = A,$$

where $\bar{x} = \mathrm{OV}(\Gamma_S)$.

Theorem

Suppose that f is a function symbol not in Σ, and $(U_f \text{ type } (\Gamma_f)) \in \mathcal{J}(\Sigma_f)$ so that $\Sigma' = \Sigma \cup \{(\Gamma_f, f, U_f)\}$ is a signature. For $a \in \mathrm{Tm}_\mathcal{C}(G(\Gamma_f), \sigma_{\Gamma_f}(\Gamma_f, U_f))$, there is a unique cwf morphism $G' = (G', \sigma', \theta') : \mathcal{F}_{\Sigma'} \to \mathcal{C}$ with $G' \circ E = G$ and

$$\theta'(\Gamma_f, U_f, f(\bar{x})) = a,$$

where $\bar{x} = \mathrm{OV}(\Gamma_f)$.

Hyperdoctrines over CwFs

A natural generalization of hyperdoctrines to dependent types: semantics of first-order logic over a cwf is given by the following data

- A category with families $\mathcal{C}$.
- A functor $\mathrm{Pr} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Heyting}$ into the category of Heyting algebras (preorder formulation). For $f : \Delta \longrightarrow \Gamma$ we write

$$R\{f\} =_{\mathrm{def}} \mathrm{Pr}(f)(R).$$

- For any $\Gamma \in \mathrm{Ob}(\mathcal{C})$ and $S \in \mathrm{Ty}(\Gamma)$ monotone operations $\forall_S, \exists_S : \mathrm{Pr}(\Gamma.S) \longrightarrow \mathrm{Pr}(\Gamma)$ such that
  1. $$Q \leq \forall_S(R) \Longleftrightarrow Q\{\mathrm{p}(S)\} \leq R$$
  2. $$\exists_S(R) \leq Q \Longleftrightarrow R \leq Q\{\mathrm{p}(S)\}.$$

- (Beck-Chevalley) For the pullback square

$$
\begin{CD}
\Delta.S\{f\} @>{f.S}>> \Gamma.S \\
@V{\mathrm{p}(S\{f\})}VV @VV{\mathrm{p}(S)}V \\
\Delta @>>{f}> \Gamma
\end{CD}
\tag{7}
$$

we have for $R \in \mathrm{Pr}(\Gamma.S)$,

$$\forall_S(R)\{f\} = \forall_{S\{f\}}(R\{f.S\}) \qquad \exists_S(R)\{f\} = \exists_{S\{f\}}(R\{f.S\}).$$

The resulting structure $\mathcal{H} = (\mathcal{C}, \mathrm{Pr}, \forall, \exists)$ is called a first-order hyperdoctrine over $\mathcal{C}$. Write $\mathcal{C}^{\mathcal{H}} = \mathcal{C}$ and $\mathrm{Pr}^{\mathcal{H}} = \mathrm{Pr}$.

Propositions-as-types interpretation into a category with families and type constructions

Let $\mathcal{C}$ be a cwf. Define for $\Gamma \in \mathcal{C}$,

$$\mathrm{Pr}_{\mathcal{C}}(\Gamma) = (\mathrm{Ty}(\Gamma), \leq)$$

where

$$A \leq B \Longleftrightarrow_{\mathrm{def}} \mathrm{Tm}(\Gamma.A, B\{\mathrm{p}(A)\}) \text{ is inhabited.}$$

For $\sigma : \Delta \longrightarrow \Gamma$, define

$$\mathrm{Pr}_{\mathcal{C}}(f)(A) =_{\mathrm{def}} \mathrm{Ty}(f)(A).$$

Theorem

If $\mathcal{C}$ is a cwf which admits the type constructions $\Sigma, \Pi, +, \mathrm{N}_0$ and N_1 then $(\mathcal{C}, \mathrm{Pr}_{\mathcal{C}})$ is a cwf with first-order doctrine.

Morphisms of hyperdoctrines

Suppose that $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a cwf morphism. Assume that $\mathcal{H} = (\mathcal{C}, \mathrm{Pr}, \forall, \exists)$ and $\mathcal{H}' = (\mathcal{C}', \mathrm{Pr}', \forall', \exists')$ are two hyperdoctrines over the respective cwfs. An *F-based morphism of hyperdoctrines*

$$G : \mathcal{H} \longrightarrow \mathcal{H}'$$

is a natural transformation $G : \mathrm{Pr} \longrightarrow \mathrm{Pr}' \circ F$

$$\begin{array}{ccc} \mathrm{Pr}(\Gamma) & \xrightarrow{\;G_\Gamma\;} & \mathrm{Pr}'(F\Gamma) \\ {\scriptstyle \_\{f\}}\downarrow & & \downarrow{\scriptstyle \_\{Ff\}} \\ \mathrm{Pr}(\Delta) & \xrightarrow[\;G_\Delta\;]{} & \mathrm{Pr}'(F\Delta) \end{array}$$

such that for $R \in \mathrm{Pr}(\Gamma.S)$,

1. $G_\Gamma(\forall_S(R)) = \forall'_{\sigma_\Gamma(S)}(G_{\Gamma.S}(R))$,
2. $G_\Gamma(\exists_S(R)) = \exists'_{\sigma_\Gamma(S)}(G_{\Gamma.S}(R))$.

Dependently typed first-order logic

Here we consider versions dependently typed first-order logic based on (Belo 2007) which generalizes the standard presentation of many-sorted first-order logic (Johnstone 2002). The first version DFOL uses capture avoiding substitutions and standardized contexts, and is suitable for an easy completeness proof. The second version DFOL* is closer to (Belo 2007) and is conservative over DFOL.

(F1) For each predicate declaration $(\Delta, \bar{i}, R)$ in Π, and $(\bar{a} : \Gamma \longrightarrow \Delta) \in \mathcal{J}(\Sigma)$ we have

$$\frac{}{R(\bar{a}_i) \text{ form } (\Gamma)}$$

(F2) For $(\Gamma \text{ context}) \in \mathcal{J}(\Sigma)$,

$$\frac{}{\bot \text{ form } (\Gamma)} \qquad \frac{}{\top \text{ form } (\Gamma)}$$

(F3) For the connectives $\bigcirc = \wedge, \vee, \rightarrow$:

$$\frac{\phi \text{ form } (\Gamma) \quad \psi \text{ form } (\Gamma)}{(\phi \bigcirc \psi) \text{ form } (\Gamma)}$$

(F4) For the quantifiers $Q = \forall, \exists$, and for $(A \text{ type } (\Gamma)) \in \mathcal{J}(\Sigma)$,

$$\frac{\phi \text{ form } (\Gamma, x : A)}{(Qx : A)\phi \text{ form } (\Gamma)}$$

Capture avoiding substitution:

We define for $(\phi \text{ form } (\Gamma)) \in \mathrm{Form}(\Sigma, \Pi)$ and $(\bar{a} : \Delta \longrightarrow \Gamma) \in \mathcal{J}(\Sigma)$ the capture avoiding substitution instance

$$\phi\{(\Delta, \Gamma, \bar{a})\}$$

by induction on formulas:

- if $(\phi \text{ form } (\Gamma)) \in \mathrm{Form}(\Sigma, \Pi)$ and ϕ is atomic, then
$$\phi\{(\Delta, \Gamma, \bar{a})\} = \phi[\bar{a}/\Gamma]$$
(syntactic substitution).
- if $(\phi \text{ form } (\Gamma)), (\psi \text{ form } (\Gamma)) \in \mathrm{Form}(\Sigma, \Pi)$, then
$$(\phi \bigcirc \psi)\{(\Delta, \Gamma, \bar{a})\} = \phi\{(\Delta, \Gamma, \bar{a})\} \bigcirc \psi\{(\Delta, \Gamma, \bar{a})\}$$
for $\bigcirc = \wedge, \vee, \rightarrow$.
- if $(\theta \text{ form } (\Gamma, x : A)) \in \mathrm{Form}(\Sigma, \Pi)$, then for $Q = \forall, \exists$,
$$((Qx : A)\theta)\{(\Delta, \Gamma, \bar{a})\} =$$
$$(Qy : A[\bar{a}/\Gamma])\big(\theta\{(\langle \Delta, y : A[\bar{a}/\Gamma]\rangle, \langle \Gamma, x : A\rangle, (\bar{a}, y))\}\big)$$
where $y = \mathrm{fresh}(\Delta)$.

Let T be a theory with respect to the signature (Π, Σ). Let $\mathrm{Thm}(\Pi, \Sigma, T)$ denote the smallest set of sequents containing T and closed under the propositional and quantificational rules, and the substitution rules below.

Propositional rules. For $(\phi \text{ form } (\Gamma)), (\psi \text{ form } (\Gamma)), (\theta \text{ form } (\Gamma)) \in \mathrm{Form}(\Sigma, \Pi)$

(ref)

$$\frac{}{\phi \stackrel{\Gamma}{\Longrightarrow} \phi}.$$

(cut)

$$\frac{\phi \stackrel{\Gamma}{\Longrightarrow} \theta \quad \theta \stackrel{\Gamma}{\Longrightarrow} \psi}{\phi \stackrel{\Gamma}{\Longrightarrow} \psi}$$

(conj)

$$\frac{}{\theta \wedge \psi \stackrel{\Gamma}{\Longrightarrow} \theta} \quad \frac{}{\theta \wedge \psi \stackrel{\Gamma}{\Longrightarrow} \psi} \quad \frac{\phi \stackrel{\Gamma}{\Longrightarrow} \theta \quad \phi \stackrel{\Gamma}{\Longrightarrow} \psi}{\phi \stackrel{\Gamma}{\Longrightarrow} \theta \wedge \psi} \qquad \frac{}{\phi \stackrel{\Gamma}{\Longrightarrow} \top}$$

etc ....

Write for the canonical projection

$$\mathbf{p}_\Gamma(x : A) = (\langle \Gamma, x : A \rangle, \Gamma, \mathrm{OV}(\Gamma)). \tag{8}$$

Quantificational rules. For
$(\phi \text{ form } (\Gamma)), (\psi \text{ form } (\Gamma, x : A)) \in \mathrm{Form}(\Sigma, \Pi)$:

(univ)

$$\frac{\phi\{\mathbf{p}_\Gamma(x : A)\} \xRightarrow{\Gamma, x:A} \psi}{\phi \xRightarrow{\Gamma} (\forall x : A)\psi} \qquad \frac{\phi \xRightarrow{\Gamma} (\forall x : A)\psi}{\phi\{\mathbf{p}_\Gamma(x : A)\} \xRightarrow{\Gamma, x:A} \psi}$$

(exis)

$$\frac{\psi \xRightarrow{\Gamma, x:A} \phi\{\mathbf{p}_\Gamma(x : A)\}}{(\exists x : A)\psi \xRightarrow{\Gamma} \phi} \qquad \frac{(\exists x : A)\psi \xRightarrow{\Gamma} \phi}{\psi \xRightarrow{\Gamma, x:A} \phi\{\mathbf{p}_\Gamma(x : A)\}}$$

Substitution rule. For $(\phi \text{ form } (\Gamma)), (\psi \text{ form } (\Gamma)) \in \mathrm{Form}(\Sigma, \Pi)$,
and $(\bar{a} : \Delta \longrightarrow \Gamma) \in \mathcal{J}(\Sigma)$,

(subs)

$$\frac{\phi \xRightarrow{\Gamma} \psi}{\phi\{(\Delta, \Gamma, \bar{a})\} \xRightarrow{\Delta} \psi\{(\Delta, \Gamma, \bar{a})\}.}$$

Lindenbaum-Tarski algebra

We assume here that the variable system of Σ has the de Bruijn property. We show that the Lindenbaum-Tarski algebra T over (Σ, Π) forms a first-order hyperdoctrine over $\mathcal{F}_\Sigma$.
For $\Gamma \in \mathcal{F}_\Sigma$, define

$$\mathrm{Pr}_{\Sigma,\Pi,T}(\Gamma) = \mathrm{Pr}(\Gamma) =_{\mathrm{def}} \{(\Gamma, \phi) : (\phi \text{ form } (\Gamma)) \in \mathrm{Form}(\Sigma, \Pi)\}.$$

and define its order relation $(\leq_{\mathrm{Pr}(\Gamma)}) = (\leq)$ as follows

$$(\Gamma, \phi) \leq (\Gamma, \psi) \text{ iff } (\phi \overset{\Gamma}{\Longrightarrow} \psi) \in \mathrm{Thm}(\Sigma, \Pi, T)$$

It is clear that $(\mathrm{Pr}(\Gamma), \leq)$ is a Heyting algebra. For an arrow $(\Delta, \Gamma, \bar{a})$ from Δ to Γ in $\mathcal{F}_\Sigma$, define for $(\Gamma, \phi) \in \mathrm{Pr}(\Gamma)$,

$$\mathrm{Pr}((\Delta, \Gamma, \bar{a}))(\Gamma, \phi) = (\Delta, \phi\{(\Delta, \Gamma, \bar{a})\})$$

Theorem

(Completeness: Universal model) The Lindenbaum-Tarski algebra for a theory T over (Σ, Π),

$$\mathcal{H}_{\Sigma,\Pi,T} =_{\text{def}} (\mathcal{F}_\Sigma, \mathrm{Pr}_{\Sigma,\Pi,T}, \forall, \exists)$$

is a hyperdoctrine over $\mathcal{F}_\Sigma$. It has the universal model property, i.e. that

$$(\Gamma, \phi) \leq (\Gamma, \psi) \text{ if, and only if, } (\phi \xrightarrow{\Gamma} \psi) \in \mathrm{Thm}(\Sigma, \Pi, T).$$

□

Let T be a dependent first-order theory over (Σ, Π). Let $\mathcal{H}$ be the hyperdoctrine $\mathcal{H}_{\Sigma,\Pi,T} = (\mathcal{F}_\Sigma, \mathrm{Pr}_{\Sigma,\Pi,T}, \forall, \exists)$. A dependent first-order model of T consists of a Σ-structure given by a cwf morphism

$$F : \mathcal{F}_\Sigma \longrightarrow \mathcal{C}$$

together with a hyperdoctrine $\mathcal{D} = (\mathcal{C}, \mathrm{Pr}^{\mathcal{D}}, \forall', \exists')$ and an F-based hyperdoctrine morphism

$$G : \mathcal{H} \longrightarrow \mathcal{D}.$$

Extension by new predicates is easier:

Theorem

Let Σ be a signature on standard form and $F : \mathcal{F}_\Sigma \longrightarrow \mathcal{C}$ a cwf morphism. Suppose that Π is a predicate signature on standard form over Σ. Let $\mathcal{D}$ be a hyperdoctrine based on $\mathcal{C}$, and suppose that for each $R = (\Gamma_R, P_R) \in \Pi$, there is $R^* \in \mathrm{Pr}_{\mathcal{C}}(\Gamma_R)$. Then there is a unique F-based hyperdoctrine morphism $G : \mathcal{H}_{\Sigma,\Pi,\emptyset} \longrightarrow \mathcal{D}$ with

$$G_{\Gamma_R}(P_R(\mathrm{OV}(\Gamma_R))) = R^*$$

for each $R \in \Pi$.

Theorem

(Soundness) Let Σ *be a signature on standard form and* $F : \mathcal{F}_\Sigma \longrightarrow \mathcal{C}$ *a cwf morphism. Suppose that* Π *is a predicate signature on standard form over* Σ*. Let* $\mathcal{D}$ *be a hyperdoctrine based on* $\mathcal{C}$*. Suppose that* $G : \mathcal{H}_{\Sigma,\Pi,\emptyset} \longrightarrow \mathcal{D}$ *is an* F*-based hyperdoctrine morphism. Let* T *be a theory over* Σ, Π *and assume that for every sequent* $(\phi \overset{\Gamma}{\Longrightarrow} \psi) \in T$,

$$G_\Gamma(\Gamma, \phi) \leq G_\Gamma(\Gamma, \psi).$$

Then $G : \mathcal{H}_{\Sigma,\Pi,T} \longrightarrow \mathcal{D}$ *is also model of* T*.*

The standard formulation of soundness follows since the theorems of T are automatically true in $\mathcal{H}_{\Sigma,\Pi,T}$, so by they will be true in $\mathcal{D}$.

Variable free and variable full formulations of DFOL

Let V be an infinite discrete set. A fresh variable provider (fvp) for V is a pair of functions (φ, fr) which to each finite subset $X \subseteq V$ ('already used variables') assigns an inhabited subset

$$\varphi(X) \subseteq V \setminus X,$$

and an element

$$\mathrm{fr}(X) \in \varphi(X).$$

Example 1: Variable full/unrestricted.

$$\varphi_\infty(\{x_1, \ldots, x_n\}) = \{y \in V : \neg y = x_i \text{ for all } i = 1, \ldots, n\}, \quad (9)$$

and $\mathrm{fr}_\infty(\{x_1, \ldots, x_n\})$ selects some element in the set.

Example 2: Variable free/de Bruijn style. Let $V = \mathbb{N}_+$,
$\mathrm{fr}_1(\{x_1, \ldots, x_n\}) = \max\{1, x_1 + 1, \ldots, x_n + 1\}$,

$$\varphi_1(\{x_1, \ldots, x_n\}) = \{\mathrm{fr}_1(\{x_1, \ldots, x_n\})\}$$

DFOL* — formulation with α-conversion

Syntactic substitution rule. For $(\phi \text{ form } (\Gamma)), (\psi \text{ form } (\Gamma)) \in \mathrm{Form}^*(\Sigma, \Pi)$, and any $(\bar{a} : \Delta \longrightarrow \Gamma) \in \mathcal{J}(\Sigma)$

(sub*)

$$\frac{\phi \overset{\Gamma}{\Longrightarrow} \psi}{\phi[\bar{a}/\Gamma] \overset{\Delta}{\Longrightarrow} \psi[\bar{a}/\Gamma].}$$

Quantificational rules. For $(\phi \text{ form } (\Gamma)), (\psi \text{ form } (\Gamma, x : A)) \in \mathrm{Form}^*(\Sigma, \Pi)$:

(univ*)

$$\frac{\phi \overset{\Gamma, x:A}{\Longrightarrow} \psi}{\phi \overset{\Gamma}{\Longrightarrow} (\forall x : A)\psi} \qquad \frac{\phi \overset{\Gamma}{\Longrightarrow} (\forall x : A)\psi}{\phi \overset{\Gamma, x:A}{\Longrightarrow} \psi}$$

(exis*)

$$\frac{\psi \overset{\Gamma, x:A}{\Longrightarrow} \phi}{(\exists x : A)\psi \overset{\Gamma}{\Longrightarrow} \phi} \qquad \frac{(\exists x : A)\psi \overset{\Gamma}{\Longrightarrow} \phi}{\psi \overset{\Gamma, x:A}{\Longrightarrow} \phi}$$

DFOL vs DFOL*

Let (Σ, Π) be a signature with the de Bruijn property, and let $\Sigma^{(}\infty)$ be Σ but with unrestricted fvp.

Theorem

Let T be a theory on standard form over (Σ, Π). If $(\phi \xrightarrow{\Gamma} \psi) \in \mathrm{Thm}(\Sigma, \Pi, T)$, then $(\phi \xrightarrow{\Gamma} \psi) \in \mathrm{Thm}^*(\Sigma^{(\infty)}, \Pi, T)$.

Theorem

Let T be a theory over (Σ, Π). If $(\phi \xrightarrow{\Gamma} \psi) \in \mathrm{Thm}^*(\Sigma^{(\infty)}, \Pi, T)$, then $(\phi^\sigma \xrightarrow{\Gamma^\sigma} \psi^\sigma) \in \mathrm{Thm}(\Sigma, \Pi, T^\sigma)$, where $\sigma = \sigma_\Sigma$.

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