FANO POLYTOPES

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Abstract. Fano polytopes are the convex-geometric objects corresponding to toric Fano varieties. We give a brief survey of classification results for different classes of Fano polytopes.

1. Fano polytopes

A normal projective variety X (over an algebraically closed field) with log terminal singularities such that the anticanonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor is called a ($\mathbb{Q}$-)*Fano* variety. Fano surfaces are more usually referred to as *log del Pezzo surfaces*. The classification of smooth del Pezzo surfaces dates back to the late 19th century: there are exactly ten cases, given by $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^2$ blown up in at most eight points (in general position). Smooth Fano threefolds have also been classified: there are 17 families with Picard number one [22, 23], and 88 other families [35, 36].

By restricting to the toric setting, a great deal more can be said. Recall that X is said to be *toric* if it contains a dense algebraic torus $(\mathbb{C}^\times)^n$ which acts in a natural way on X. If $M \cong \mathbb{Z}^n$ is the lattice of characters of $(\mathbb{C}^\times)^n$, then every toric variety has a combinatorial description in terms of a fan Δ in $N_\mathbb{Q} := \mathrm{Hom}(M, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{Q}$. Many geometric properties of X can be rephrased as combinatorial statements about Δ. Let $\{\rho_1, \ldots, \rho_k\}$ be the set of rays of Δ. Each ρ_i is generated by a unique primitive lattice element $v_i \in N$, and X is Fano if and only if $\{v_1, \ldots, v_k\}$ correspond to the vertices of a convex lattice polytope in $N_\mathbb{Q}$. This motivates the following:

Definition 1.1. A convex lattice polytope $P \subset N_\mathbb{Q}$, $\dim P = \dim N$, is called *Fano* if:

(1) The origin is contained in the strict interior of P (we write $\mathbf{0} \in \mathrm{int}(P)$);
(2) Each vertex $v \in \mathcal{V}(P)$ is a primitive lattice point of N.

Of the ten smooth del Pezzo surfaces, the first five are toric. They correspond to the first five polygons depicted in Figure 1. In dimension three there are 18 smooth toric Fano varieties [4, 53]. The degree $(-K_X)^n$ of any smooth Fano variety X of dimension n is bounded, as is the number of deformation types [29]. Similar results are not known for Fano varieties in general, however the number of isomorphism classes of toric Fano varieties of fixed dimension and bounded discrepancy is known to be finite [10, 9]. Equivalently, a result of Lagarias and Ziegler [33] implies that, for fixed dimension n and fixed $l \in \mathbb{Z}_{>0}$,

FIGURE 1. The 16 reflexive polygons, up to isomorphism.

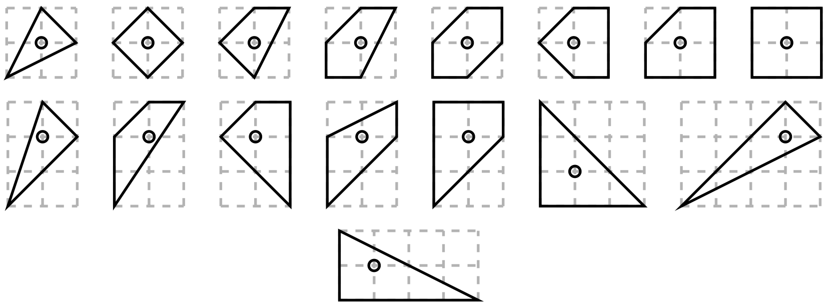

TABLE 1. A summary of the known classifications of Fano polytopes [26].

		Terminal			Canonical		
n	Smooth	Reflex.	Simp.	Total	Reflexive	Simplicial	Total
2	5	5	5	5	16	16	16
3	18 [4, 53]	100	233 [24]	634 [24]	4,319 [31]	12,190 [26]	674,688 [26]
4	124 [6, 52]	166,841[32]			473,800,776 [32]		
5	866 [30, 45]						
6	7,622 [45]						
7	72,256 [45]						
8	749,892 [45]						

there are only finitely many isomorphism classes of n-dimensional Fano polytopes with $\mathrm{int}(P) \cap l\mathbb{Z}^n = \{\mathbf{0}\}$.

Classifications of Fano polytopes of fixed dimension n divide into finite classifications (usually by restricting the resulting singularities), and partial classifications of infinite families (these tend to allow log terminal singularities, but satisfy other combinatorial conditions). The currently known finite classifications are summarised in Table 1, along with references.

2. Reflexive polytopes

In 1994 Batyrev [5] introduced reflexive polytopes as a key combinatorial tool for constructing topologically mirror-symmetric pairs of Calabi-Yau varieties as hypersurfaces in Gorenstein toric Fano varieties. This initiated an intense study of the geometric and combinatorial properties of reflexive polytopes [14, 19, 31, 32, 40, 42].

Definition 2.1. A Fano polytope $P \subset N_{\mathbb{Q}}$ is called *reflexive* if each facet of P has lattice distance one from $\mathbf{0}$. Equivalently, the *dual polytope* $P^* := \{u \in M_{\mathbb{Q}} \mid \langle u, v \rangle \geq -1 \text{ for all } v \in P\}$ is a lattice polytope.

Reflexive polytopes naturally appear as dual pairs: P is reflexive if and only if P^* is reflexive. As varieties they correspond to Gorenstein toric Fano varieties. Since they contain only one interior lattice point [5], there are finitely many reflexive polytopes up to unimodular equivalence in each dimension n: 16 when $n = 2$, 4319 when $n = 3$, and $473,800,776$ when $n = 4$ [31, 32]. This final value is, of course, the remarkable achievement of Max Kreuzer and Harald Skarke. Classification results for $n \geq 5$ are probably only realistic for subclasses of reflexive polytopes.

There are many open questions in higher dimensions. For example, it is still not known how many vertices a reflexive n-tope can have, although the maximal value is conjectured to be $6^{n/2}$ when n is even, attained by the product of hexagons. Much more is known about *simplicial* reflexive polytopes (corresponding to $\mathbb{Q}$-factorial Gorenstein toric Fano varieties). Their combinatorics is quite restrictive: The vertex-edge-graph has diameter two, and given any vertex $v \in \mathcal{V}(P)$ there exist at most three other vertices of P not contained in a facet containing v [40]. Casagrande [14] showed that the maximal number of vertices is $3n$ if n is even, or $3n - 1$ if n is odd. Equivalently, this bounds the rank of the Picard group of the corresponding variety X, since $\operatorname{rk} \operatorname{Pic} X = |\mathcal{V}(P)| - \dim P$. The simplicial reflexive polytopes achieving these bounds have been classified [42].

As special as reflexive polytopes are, it is interesting to note that in some sense they are rather general [19]:

Proposition 2.2. *Any lattice polytope is isomorphic to the face of a reflexive polytope.*

3. Smooth polytopes

A cone σ of a fan Δ is non-singular if and only if the generators $\{v_1, \ldots, v_k\}$ of the rays of σ form part of a $\mathbb{Z}$-basis for the lattice N. We shall call a Fano polytope $P \subset N_{\mathbb{Q}}$ *smooth* if for each facet $F \in \mathcal{F}(P)$, the vertices $\mathcal{V}(F)$ of F are a $\mathbb{Z}$-basis for N. Clearly any such polytope is necessarily simplicial and reflexive (the supporting hyperplane for F must lie at lattice distance one from $\mathbf{0}$). Note that in geometric combinatorics, the term “smooth” is often used in the dual sense: lattice polytopes whose tangent cones are unimodular.

In dimension two, the classification of smooth polygons is well known – there are exactly five cases. A complete classification in dimension three was obtained by Batyrev [4] and, independently, by Watanabe and Watanabe [53] (18 cases), and in dimension four, the classification was done by Batyrev [6] and Sato [52] (124 cases).

Batyrev [6] showed that the projection of a smooth n-tope P along a vertex $v \in \mathcal{V}(P)$ is a reflexive $(n-1)$-tope. Given the classification of reflexive $(n-1)$-topes, there exists an algorithm [30] “unprojecting” a reflexive polytope P' in all possible ways to generate the smooth n-topes. Using the classification of reflexive 4-topes [32], the smooth polytopes in dimension five were classified. There exists 866 isomorphism classes. In particular [30]:

Proposition 3.1. *Let X be an n-dimensional smooth toric Fano variety, $n \leq 5$. Then there exists precisely one X in each dimension with maximal anticanonical degree $(-K_X)^n$.*

n	2	3	4	5
$(-K_X)^n$	9	64	800	14762
X	$\mathbb{P}^2$	$\mathbb{P}^3$	$\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(3))$	$\mathbb{P}_{\mathbb{P}^4}(\mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(4))$

We conclude this section by discussing the work of Øbro [45], which was used to classify all smooth polytopes up to dimension eight.

Definition 3.2. Let P be a Fano polytope. A facet $F \in \mathcal{F}(P)$ is said to be *special* if $\sum_{v \in \mathcal{V}(P)} v \in \operatorname{pos}(F)$.

Clearly any Fano polytope P has at least one special facet F. Since P is smooth we may assume that $\mathcal{V}(F) = \{e_1, \ldots, e_n\}$, the standard basis for N. This is called a *special embedding* for P. Let $v = (a_1, \ldots, a_n) \in \mathcal{V}(P)$ be a vertex of P, so that $\gcd\{a_1, \ldots, a_n\} = 1$, and define $a := a_1 + \ldots + a_n$. Then $-n \leq a \leq 1$ and each a_i satisfies [45]:

$$\left.\begin{array}{r} 0 \\ -1 \\ a \end{array}\right\} \leq a_i \leq \left\{\begin{array}{ll} 1, & \text{if } a = 1, \\ n-1, & \text{if } a = 0, \\ n+a, & \text{if } a < 0. \end{array}\right.$$

Already this is a finite problem. In fact it is possible to avoid repetitions by carefully defining an order on the smooth polytopes. Let A and B be finite subsets of N. We recursively define $A \preceq B$ if and only if $A = \emptyset$, or $B \neq \emptyset$ and $\min A \prec \min B \vee (\min A = \min B \wedge A \setminus \{\min A\} \preceq B \setminus \{\min B\})$.

Definition 3.3. Let P be a smooth polytope. The *order of P* is given by

$$\operatorname{ord}(P) := \min\{\mathcal{V}(Q) \mid Q \text{ is a special embedding of } P\}.$$

If P_1 and P_2 are two smooth polytopes, we say that $P_1 \leq P_2$ if and only if $\operatorname{ord}(P_1) \preceq \operatorname{ord}(P_2)$. This defines a total order on the set of isomorphism classes of smooth n-topes.

A classification algorithm using this order can avoid expensive isomorphism testing by insisting that $\mathcal{V}(P) = \operatorname{ord}(P)$. In particular, the classification can be generated in such a way that if $P_1 \leq P_2$ then P_1 is discovered before P_2. This is very efficient, and has been used to classify all smooth polytopes up to dimension eight. Applications include new examples of Einstein-Kähler manifolds [43], and the study of Riemannian polytopes [21].

4. Gorenstein polytopes

Definition 4.1. A lattice polytope $Q \subset M_{\mathbb{Q}}$ is called *Gorenstein of index r* if there exists an integer $r \in \mathbb{Z}_{>0}$ and lattice point $m \in rQ \cap M$ such that $rQ - m$ is reflexive.

Minkowski summands of reflexive polytopes define complete intersections (CYCIs) in Gorenstein toric Fano varieties. Their stringy Hodge numbers can be computed from the combinatorial and geometric data of the associated *Cayley polytope* [7, 12], which is a Gorenstein polytope [8]. In order to obtain a duality for this construction, one needs to consider special Minkowski summands called *nef-partitions* [11]. Not all Gorenstein polytopes arise as Cayley polytopes, however stringy Hodge numbers of Gorenstein polytopes are always well-defined [44] and satisfy the mirror-symmetry property [12].

Gorenstein polytopes can be characterised in terms of the lattice distance of the supporting hyperplanes in much the same way as reflexive polytopes. A lattice polytope Q is Gorenstein of index r if and only if there exists a rational point $x \in \mathrm{int}(Q) \cap (1/r)M$ having lattice distance $1/r$ from any facet $F \in \mathcal{F}(Q)$. Another characterisation can be given in terms of reflexive Gorenstein cones.

Given an $(n+1)$-dimensional lattice $\overline{M}$ with dual lattice $\overline{N}$, recall that a cone $\sigma \subset \overline{M}_{\mathbb{Q}}$ is *Gorenstein* if it is generated by finitely many lattice points contained in an affine hyperplane $H_{u_\sigma} := \{x \in \overline{M}_{\mathbb{Q}} \mid \langle x, u_\sigma \rangle = 1\}$, for some primitive vector $u_\sigma \in \overline{N}$. In particular, $u_\sigma \in \mathrm{int}(\sigma^\vee)$ is uniquely determined, and $\mathrm{int}(\sigma^\vee) \cap \overline{N} = u_\sigma + \sigma^\vee \cap \overline{N}$.

The height one slice $\sigma \cap H_{u_\sigma}$ defines an n-dimensional lattice polytope called the *support* of σ. Conversely, given any n-dimensional lattice polytope $P \subset M_{\mathbb{Q}}$, one can associate a Gorenstein cone σ in $\overline{M} := M \oplus \mathbb{Z}$ simply by taking $\mathrm{cone}(P \times \{1\})$. A Gorenstein cone σ is called *reflexive* if $\sigma^\vee$ is also a Gorenstein cone. The value $r := \langle u_{\sigma^\vee}, u_\sigma \rangle$ is called the *index* of σ.

Batyrev and Borisov showed that reflexive Gorenstein cones correspond to Gorenstein polytopes [8]. Whilst Gorenstein polytopes do not possess interior lattice points when $r > 1$, they still satisfy a beautiful duality. Here, the dual Gorenstein polytope Q^* is defined to be the support of the dual cone $\sigma^\vee$. When $r \geq 1$ it is not generally true that the reflexive polytopes rQ and rQ^* are dual to each other: one must move to a sublattice [3]. Nevertheless, $(rQ)^*$ and Q^* have the same set of boundary lattice points. Analogous behaviour occurs in the study of l-reflexive polygons (see Section 7).

There are 5363 Gorenstein 4-topes of index two, but only 36 Gorenstein polytopes of index three. Whilst a classification of reflexive polytopes in dimension $n \geq 5$ is impractical, it is a more tractable task for Gorenstein polytopes of relatively large index[2]. We refer to the article of Harald Skarke in this volume for how this can be achieved in several cases up to dimension seven.

5. Terminal and canonical polytopes

We say that a fan Δ is *terminal* if each cone $\sigma \in \Delta$ satisfies the following:

(1) The generators $\{v_1, \dots, v_k\}$ of the rays of σ are contained in an affine hyperplane $H_{u,l} := \{v \in N_{\mathbb{Q}} \mid \langle u, v\rangle = l\}$ for some primitive vector $u \in M$ and integer $l \in \mathbb{Z}_{>0}$;
(2) The only other lattice point in the cone σ on or below $H_{u,l}$ is the origin, i.e. $\{v \in \sigma \cap N \mid \langle u, v\rangle \leq l\} = \{\mathbf{0}, v_1, \dots, v_k\}$.

A toric variety X has at worst terminal singularities if and only if the cones of Δ are all terminal. Relaxing the definition slightly to allow lattice points on $H_{u,l}$, one obtains the definition of a *canonical* cone, and X has canonical singularities [50].

Definition 5.1. A Fano polytope P is called *terminal* if $P \cap N = \mathcal{V}(P) \cup \{\mathbf{0}\}$. If $\mathrm{int}(P) \cap N = \{\mathbf{0}\}$ then P is said to be *canonical*.

Terminal singularities play an important role in birational geometry [34, 50, 38]. In dimension two, a consequence of Castelnuovo's Contractibility Criterion is that a normal surface has only terminal singularities if and only if it is smooth. Mori [37] proved that, with two exceptions, isolated canonical cyclic quotient singularities in dimension three are all either Gorenstein or terminal, whilst Reid [51] addressed the issue of classifying 3-fold terminal singularities.

The only empty polygons (i.e. polygons Q such that $Q \cap \mathbb{Z}^2 = \mathcal{V}(Q)$) are the triangle and the square – these are the possible facets of a terminal 3-tope P. Furthermore, if P is also simplicial and reflexive, then P must be smooth. There are 634 three-dimensional terminal polytopes [24], of which 233 are simplicial and 100 are reflexive.

Consider now a canonical 3-tope. Up to isomorphism, there are $674,688$ possibilities [26]. In this case, there is no known *a priori* description of the facets that can occur, although inspection gives 4248 distinct facets, ranging from triangles (of which there are 97 choices), through to a unique facet with 9 vertices and 7 interior points.

The approach to classification is essentially the same in both the terminal and canonical case; we will describe the canonical setting.

Definition 5.2. Let P be a canonical n-tope. We say that P is *minimal* if, for all $v \in \mathcal{V}(P)$, the polytope $\mathrm{conv}(P \cap N \setminus \{v\})$ obtained by subtracting v from P is not a canonical n-tope.

Given a canonical polytope P and a lattice point $v \in N$, take the convex hull $P' := \mathrm{conv}(P \cup \{v\})$; if P' fails to be canonical then discard it. If one starts with the minimal canonical polytopes, one will achieve a complete classification using this technique. Assume that P' is obtained by adding the vertex v to P. The ray passing through the origin and $-v$ will intersect ∂P in a point x on some face F not containing v. Take the smallest subset $S \subset \mathcal{V}(F)$ such that $x \in \mathrm{conv}(S)$; the simplex $\mathrm{conv}(S \cup \{v\})$ contains the origin strictly in its (relative) interior, hence is a canonical simplex of dimension $|S| \leq \dim P$.

Since there are only finitely many such simplices [10, 9], there are only finitely many choices for v.

All that remains is to describe the minimal polytopes. Again, we state the result only in the canonical case.

Proposition 5.3. *Any minimal canonical n-tope P is either a simplex, or can be written as $P = \mathrm{conv}(S \cup P')$ for some S a minimal canonical k-simplex and P' a minimal canonical $(n-k+r)$-tope, where $0 \le r < k < n$. Moreover, $\dim(S \cap P') \le r$, and r equals the number of common vertices of S and P'.*

By definition of minimality, given any canonical polytope Q there exists a minimal polytope P such that $P \subset Q$, hence $\mathrm{Vol}(P^*) \ge \mathrm{Vol}(Q^*)$. In dimension three there are 26 minimum canonical polytopes, giving:

Theorem 5.4. *Let X be a toric Fano threefold with at worst canonical singularities. Then $(-K_X)^3 \le 72$, with equality if and only if X is isomorphic to $\mathbb{P}(1,1,1,3)$ or $\mathbb{P}(1,1,4,6)$.*

6. Fano simplices

Definition 6.1. Let $P := \mathrm{conv}\{v_0, \dots, v_n\} \subseteq N_{\mathbb{Q}}$ be a Fano simplex, and let $(\lambda_0, \dots, \lambda_n)$ be a positive collection of weights $\lambda_0 \le \dots \le \lambda_n$ such that $\gcd\{\lambda_0, \dots, \lambda_n\} = 1$ and $\lambda_0 v_0 + \dots + \lambda_n v_n = \mathbf{0}$. The rank one $\mathbb{Q}$-factorial variety X associated with the spanning fan of P is called a *fake weighted projective space* with weights $(\lambda_0, \dots, \lambda_n)$.

Besides being compelling combinatorial objects, Fano simplices arise naturally in toric Mori theory [49, 51, 10, 13]. Let $\Lambda_{\mathcal{V}(P)}$ denote the sublattice generated by the vertices $\mathcal{V}(P)$ of P, and define the *multiplicity* of P to be the index $\mathrm{mult}\, P := [N : \Lambda_{\mathcal{V}(P)}]$. Then[10, 13, 15, 25]:

Theorem 6.2. *Let X be a fake weighted projective space with weights $(\lambda_0, \dots, \lambda_n)$, and let P be the associated simplex in $N_{\mathbb{Q}}$. Let $Q \subseteq N_{\mathbb{Q}}$ be the simplex corresponding to $Y := \mathbb{P}(\lambda_0, \dots, \lambda_n)$.*

(1) *$X \cong Y$ if and only if $\mathrm{mult}\, P = 1$.*
(2) *X is the quotient of the weighted projective space Y by the action of the finite group $N/\Lambda_{\mathcal{V}(P)}$ acting free in codimension one. In particular $\pi_1^1(X) = N/\Lambda_{\mathcal{V}(P)}$.*
(3) *There exists a Hermite normal form H with determinant $\mathrm{mult}\, P$ such that $P = QH$ (up to the action of $GL_n(\mathbb{Z})$).*
(4) *If P is canonical (resp. terminal) then Q is canonical (resp. terminal) and*

$$\mathrm{mult}\, P \le \frac{h^{n-1}}{\lambda_1 \lambda_2 \dots \lambda_n} = \frac{\lambda_0}{h}(-K_Y)^n, \qquad \textit{where } h := \sum_{i=0}^{n} \lambda_i.$$

(5) *If P is reflexive then Q is reflexive and $\mathrm{mult}\, P \mid \mathrm{mult}\, Q^*$.*

If P is canonical, Pikhurko [46] gives an upper bound on the sum of the weights $h \leq 2^{3n-2}15^{(n-1)2^{n+1}}$. In dimensions two and three this is far from sharp (the maximum values are $1+2+3=6$ and $5+6+22+33=66$, respectively). There also exists a bound [25] $\lambda_i \leq h/(n-i+2)$, for $2 \leq i \leq n$.

Reflexive simplices have been studied in some detail [5, 15, 41]. The crucial observation is:

Proposition 6.3. *Let $Y = \mathbb{P}(\lambda_0, \ldots, \lambda_n)$. Then Y is Gorenstein if and only if $\lambda_i \mid h$ for all i.*

Recall that a family of positive natural numbers $(k_0, k_1, \ldots, k_n)$ is called a *unit partition* if $1/k_0 + 1/k_1 + \ldots + 1/k_n = 1$. Clearly Gorenstein weighted projective spaces and unit partitions are in bijection via $k_i = h/\lambda_i$. Closely associated with unit partitions is the *Sylvester sequence*

$$y_0 := 2, \qquad y_j := 1 + y_0 \cdots y_{j-1}.$$

Then $\max\{k_0, k_1, \ldots, k_n\} \leq y_n$, $h \leq t_n := y_n - 1$, and $\lambda_i \geq h/((i+1)t_{n-i})$.

We conclude with the following (c.f. Theorem 5.4) [41]:

Theorem 6.4. *Suppose that X is a Gorenstein fake weighted projective space.*

(1) *If $n = 3$ then $(-K_X)^3 \leq 72$, with equality if and only if X is isomorphic to $\mathbb{P}(1,1,1,3)$ or $\mathbb{P}(1,1,4,6)$;*
(2) *If $n \geq 4$ then $(-K_X)^n \leq 2t_{n-1}^2$, with equality if and only if X is isomorphic to $\mathbb{P}(1,1,2t_{n-1}/y_{n-2}, \ldots, 2t_{n-1}/y_0)$.*

The results of Theorem 6.4 are conjectured to hold more generally for any Gorenstein Fano variety with canonical singularities. In dimension three this is known as the Fano–Iskovskikh conjecture, and was proven by Prokhorov [48].

7. Fano polygons

Log del Pezzo surfaces have been extensively studied by Nukulin, Alexeev, and Nakayama [1, 39]. In the toric setting they correspond to the Fano polygons, which we usually refer to as *LDP-polygons*. The LDP-triangles were first studied by Dais [16, 17], followed by study of the LDP-polygons and a classification algorithm [18, 27].

Definition 7.1. Let $F \in \mathcal{F}(P)$ be a facet of an LDP-polygon P. There exists a unique primitive lattice vector $u_F \in M \setminus \{\mathbf{0}\}$ such that $\langle u_F, F \rangle = \{l_F\}$, where l_F is a positive integer called the *local index of* F. The *index* of P is defined by $l := \operatorname{lcm}\{l_F \mid F \in \mathcal{F}(P)\}$.

The value l_F is the lattice distance of F from $\mathbf{0}$. Notice that l is the smallest positive integer such that lP^* is a lattice polygon; equivalently, l is the smallest integer such that $-lK_X$ is a Cartier divisor, and is often referred to as the *Gorenstein index*.

TABLE 2. The classification of LDP-polygons [27] with index $l \leq 17$. Here $n(l)$ is the total number of LDP-polygons, $m(l)$ is the number of LDP-triangles (i.e. rank one toric log del Pezzo surfaces), $n_T(l)$ is the number of LDP-polygons with T-singularities [20], and $m_T(l)$ is the number of LDP-triangles with T-singularities.

l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$n(l)$	16	30	99	91	250	379	429	307	690	916	939	1279	1142	1545	4312	1030	1892
$m(l)$	5	7	18	13	33	26	45	27	51	51	67	53	69	74	133	48	89
$n_T(l)$	16	30	11	11	1	56	0	1	2	20	0	66	0	5	28	1	0
$m_T(l)$	5	7	5	2	1	8	0	0	0	4	0	4	0	1	4	0	0

For fixed index l, it is possible to classify all LDP-polygons [27]. The algorithm relies on the notion of a special facet, and was sufficient to allow all LDP-polygons up to index 17 to be classified (see Table 2).

Definition 7.2. A Fano polytope P is called *l-reflexive* if, for some $l \in \mathbb{Z}_{>0}$, the local index l_F equals l for every facet $F \in \mathcal{F}(P)$.

The 1-reflexive polytopes are precisely the reflexive polytopes introduced by Batyrev [5]. In fact they generalise many important combinatorial properties [28]. For example, P is l-reflexive if and only if lP^* is l-reflexive. It is also tempting to regard Gorenstein polytopes as being "$\frac{1}{r}$-reflexive".

In dimension two the l-reflexive polygons form a special subclass of the LDP-polygons of index l. It is unusual for an LDP-polygon to be l-reflexive; for example, there are no l-reflexive polygons of even index. They satisfy a very restrictive condition [28]:

Proposition 7.3. *Let $P \subseteq N_{\mathbb{Q}}$ be an l-reflexive polygon (or more generally an l-reflexive loop), and let $\Lambda_P \subset N$ denote the sublattice generated by the boundary points $\partial P \cap N$. Then $\Lambda_{lP^*} = l{\Lambda_P}^*$. Moreover, $\Lambda_P \subset N$ and $l{\Lambda_P}^* \subset M$ are both sublattices of index l.*

As a corollary, P restricted to the lattice Λ_P is a reflexive polygon Q, and Q^* is isomorphic to lP^* with respect to Λ_{lP^*}. From this observation there follows an efficient classification algorithm. We also see why there exist no l-reflexive polygons of even index. For assume otherwise. Without loss of generality let $F := \mathrm{conv}\{(a,l),(b,l)\} \in \mathcal{F}(P)$ be a facet of P. Since the vertices are primitive, both a and b must be odd, hence the midpoint between the two is a non-vertex lattice point on F. By symmetry this is true for every facet of P, and similarly for lP^*. This property must also hold for the reflexive polytope Q given by restricting to Λ_P, and for Q^*. However, a brief glance at Figure 1 shows that this is impossible.

We summarise the key results in the following theorem (c.f. Theorem 6.2 (1)-(3)).

TABLE 3. The classification of l-reflexive polygons [28] with index $l \leq 80$. Here $n(l)$ is the total number of l-reflexive polygons.

l	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
$n(l)$	16	1	12	29	1	61	81	1	113	131	2	163	50	2	215	233	2	34	285	3
l	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79
$n(l)$	317	335	2	367	182	3	419	72	4	469	489	3	93	539	4	571	591	3	185	641

Theorem 7.4. *Let $P \subseteq N_{\mathbb{Q}}$ be an l-reflexive polygon, and let Q be the restriction of P to Λ_P. Let $X(P)$ denote the toric variety generated by the spanning fan of P.*

(1) *$X(P)$ is Gorenstein if and only if $[N : \Lambda_P] = 1$.*
(2) *$X(P)$ is the quotient of the Gorenstein surface $X(Q)$ by the action of the finite group N/Λ_P.*
(3) *There exists a Hermite normal form H with determinant $[N : \Lambda_P]$ such that $P = QH$ (up to the action of $GL_2(\mathbb{Z})$).*

Finally, we conclude with an intriguing combinatorial result. Recall that for any reflexive polygon Q, the sum $|\partial Q \cap N| + |\partial Q^* \cap M|$ is twelve [47]. This can be proved combinatorially, or in terms of the associated toric variety using Noether's formula. Since any property of the boundary points of reflexive polygons lifts to l-reflexive polygons, we have:

Corollary 7.5. *Let $P \subset N_{\mathbb{Q}}$ be an l-reflexive polygon (or more generally an l-reflexive loop). Then $|\partial P \cap N| + |\partial(lP^*) \cap M| = 12$.*

8. About Max Kreuzer (by the second author)

Max had a sincere interest in the lattice polytope community to which he made lasting contributions. He also had a wonderfully approachable and generous personality. I recall a little incident when I first met him in January 2003 at snowy Oberwolfach, whilst I was still a PhD student. Max invited me to a snowball fight, however I was reluctant to throw a ball at this famous professor from Vienna. But he insisted, declaring, "Ich bin nicht aus Zuckerwatte."

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