The degree of lattice polytopes

Benjamin Nill - FU Berlin

San Francisco State University, October 12, 2007

1. Introduction

The h^*-polynomial

Throughout:

- $M \cong \mathbb{Z}^r$ is a lattice, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$.
- $P \subseteq M_{\mathbb{R}}$ is n-dimensional lattice polytope.

Theorem. [Stanley 1980]
There is a polynomial $h^*(t) \in \mathbb{Z}[t]$ of degree $\leq n$ with nonnegative integer coefficients such that

$$\sum_{k \geq 0} |\, kP \cap M \,|\, t^k = \frac{h^*(t)}{(1-t)^{n+1}}.$$

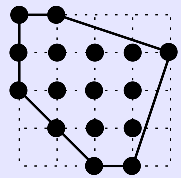
Definition:

- $h^*_P(t) := h^*(t)$ is called h^*-**polynomial** of P.
- $h^*_P := (h^*_0, \ldots, h^*_n)$ is called h^*-**vector** of P.

Observations:

- $h^*_0 = 1$.
- $h^*_1 = |\,P \cap M\,| - (n+1)$.
- $h^*_n = |\,\mathrm{int}(P) \cap M\,|$.
- $h^*_0 + \cdots + h^*_n = \mathrm{Vol}(P) = n!\,\mathrm{vol}(P)$.
  (from now on: volume = normalized volume)

Example: $n = 2$



1+13t+8t²

$$h^*_P(t) = 1 + (|\,P \cap M\,| - 3)\; t + |\,\mathrm{int}(P) \cap M\,|\; t^2.$$

Pick's formula:

$$\mathrm{Vol}(P) = h^*_P(1) = |\,\partial(P) \cap M\,| + 2|\,\mathrm{int}(P) \cap M\,| - 2.$$

The degree

Definition: The **degree** of P is defined as the degree of the h^*-polynomial.

$$\deg(P) := \max(i \,:\, h_i^* \neq 0).$$

Hence

$$0 \leq \deg(P) \leq n = \dim(P).$$

Philosophy: The degree of a lattice polytope may be regarded as its "true" dimension.

The codegree

Definition: The **codegree** of P is defined as:

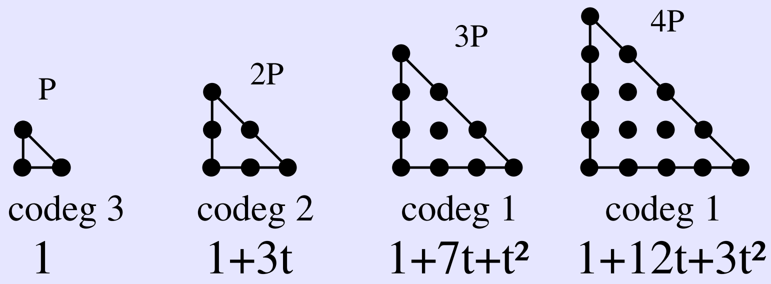
$$\mathrm{codeg}(P) := \min\{k \in \mathbb{Z}_{\geq 0} \ : \ \mathrm{int}(kP) \cap M \neq \emptyset\}$$

Reciprocity-Theorem $\Rightarrow$

$$1 \leq \boxed{\mathrm{codeg}(P) = n + 1 - \deg(P)} \leq n + 1,$$

$$h^*_{\deg(P)} = |\,\mathrm{int}(\mathrm{codeg}(P)P) \cap M\,|.$$

Example: $n = 2$



Observations:

- $\mathrm{codeg}(P) = 1 \Leftrightarrow \deg(P) = n \Leftrightarrow \mathrm{int}(P) \cap M \neq \emptyset$
- $\mathrm{codeg}(P) = n+1 \Leftrightarrow \deg(P) = 0 \Leftrightarrow$ P is unimodular simplex.

2. Classification of lattice polytopes of degree one

Dimension two

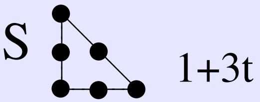
Let $P \subseteq M_{\mathbb{R}}$ be a lattice polygon.

$$\deg(P) \leq 1 \;\Leftrightarrow\; P \text{ has no interior lattice points.}$$

Theorem. [Arkinstall 1980; Koelman 1991; Khovanskii 1997; Schicho 2003]
There are precisely *two* cases, in which P has no interior lattice points:

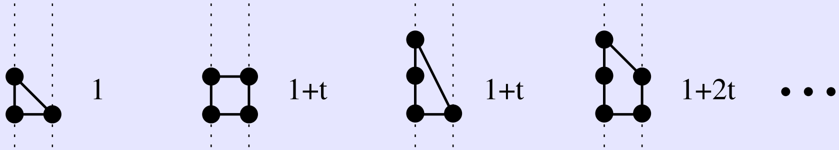
either

P is the **exceptional triangle** S:



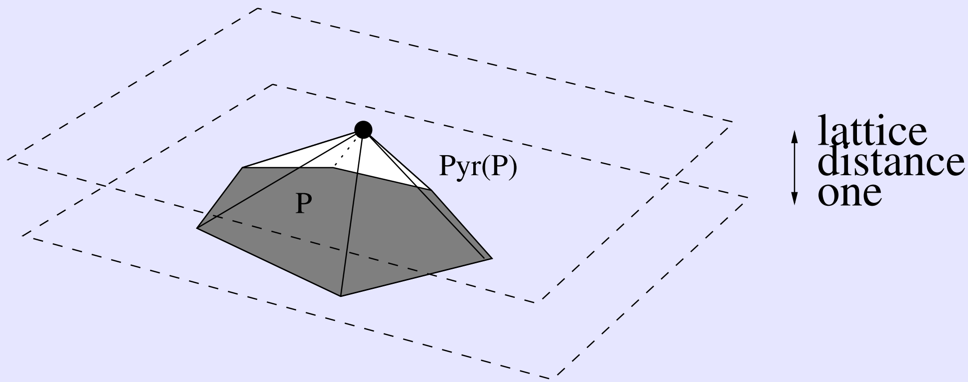
or

P lies between two parallel hyperplanes of integral distance one:



Lattice pyramid construction

Definition: The **lattice pyramid** over P is defined as $\mathrm{Pyr}(P) := \mathrm{conv}(\{0\},\ P \times \{1\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$.



Recursively: k-**fold pyramid** over P:

$$\mathrm{Pyr}^k(P) := \mathrm{Pyr}(\mathrm{Pyr}^{k-1}(P)).$$

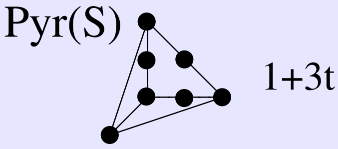
Proposition: P and $\mathrm{Pyr}(P)$ have same h^*-polynomial.

$\Rightarrow$ Degree and volume is unchanged!

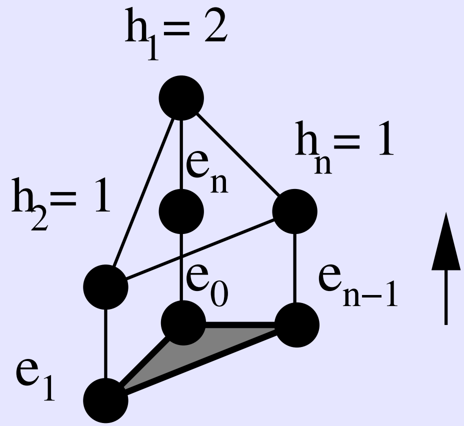
Higher dimensions

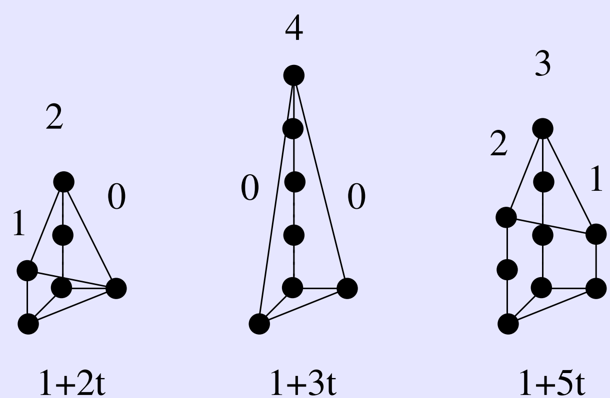
Definition: An n-dimensional **exceptional simplex** is an $(n-2)$-fold lattice pyramid over S.

Example: $n = 3$



Definition: An n-dimensional **Lawrence prism** with heights $h_1, \ldots, h_n$:





Classification result

Theorem. [Batyrev, N. 2006]
A lattice polytope has degree ≤ 1 if and only if it is an exceptional simplex or a Lawrence prism.

Proof by induction on the number of lattice points, using the monotonicity property of the degree:

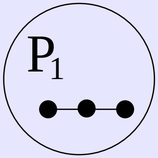
Theorem. [Stanley 1993]
$Q \subseteq P$ lattice polytopes $\Rightarrow$ $h^*_Q \leq h^*_P$ coefficientwise.

3. How many lattice polytopes with given h^*-polynomial?

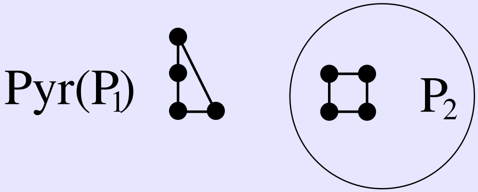
Case of degree one

Example: $h^* = 1 + t \quad (\deg = 1$ and $\mathrm{Vol} = 2)$:

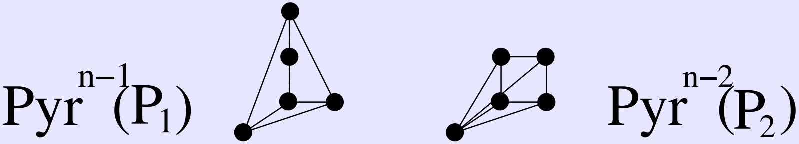
- $n = 1$:



- $n = 2$:



- $n \geq 3$:



Proposition: An n-dimensional lattice polytope P of degree one is a **lattice pyramid**, if $n > \mathrm{Vol}(P)$.

The general case

Theorem. [Batyrev 2006]
A lattice polytope of dimension n, volume V, and degree d is a **lattice pyramid**, if

$$n \geq 4d \binom{2d + V - 1}{2d}.$$

Proof relies heavily on commutative algebra.

Definition: Let V, d be fixed.
$C(n) :=$ number of isomorphism classes of n-dimensional lattice polytopes of volume V and degree d.

Then $C(n)$ is

- *finite* (theorem of Lagarias and Ziegler),
- *upper monotone* in n (pyramid construction),
- *becomes constant* for n large (Batyrev's result).

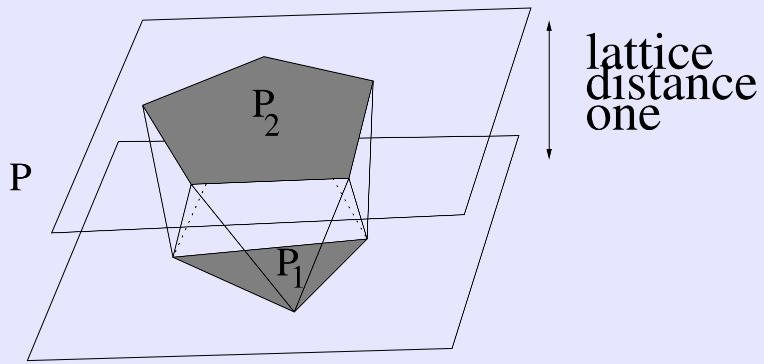
Corollary: There are only **finitely** many lattice polytopes with the same h^*-polynomial up to isomorphisms and pyramid constructions.

4. What about lattice polytopes of given degree?

The Cayley polytope conjecture

Definition: A lattice polytope P is a **Cayley polytope**

$$P = P_1 * P_2, \quad \text{if}$$



On the other hand, let $P_1, \ldots, P_s \subseteq M_{\mathbb{R}}$ be given.

Definition: $P_1 * \cdots * P_s$ is defined as

$$\operatorname{conv}(P_1 \times \{e_1\}, \ldots, P_s \times \{e_s\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{Z}^s,$$

where $e_1, \ldots, e_s$ is a lattice basis of $\mathbb{Z}^s$.

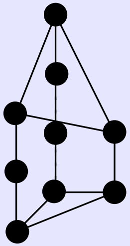
Conjecture I [Batyrev, N.]: Let d be fixed.
There exists N such that any lattice polytope of degree d and dimension n is a **Cayley polytope**, if

$$n \geq N.$$

Example: Conjecture I holds for $d = 1$ with $N = 3$.

- Pyramids are Cayley polytopes.
- Lawrence prisms are Cayley polytopes of segments.

[0,2]*[0,3]*[0,1]



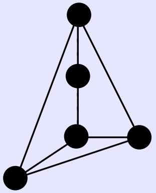
A surprise ...

Theorem I. [N. 2007]
A lattice simplex of degree d and dimension n is a **lattice pyramid**, if

$$n \geq 4d - 1.$$

Example: A lattice simplex of degree $d = 1$ is a lattice pyramid, if $n \geq 3$.

- Exceptional simplices are lattice pyramids for $n \geq 3$.
- Simplices that are Lawrence prisms are lattice pyramids for $n \geq 2$.



... and its generalization

Theorem II. [N. 2007]
A lattice polytope of degree d and dimension n is a **lattice pyramid**, if

$$n \geq (f_0 - n - 1)(2d + 1) + 4d - 1,$$

where f_0 equals the number of vertices of P.

Proofs of Theorems I and II are purely combinatorial.

Back to h^*-polynomial

$$f_0 - n - 1 \leq |\,P \cap M\,| - n - 1 = h_1^* \quad \Rightarrow$$

Corollary: A lattice polytope of degree d and dimension n is a **lattice pyramid**, if

$$n \geq h_1^*(2d+1) + 4d - 1.$$

Fine tuning of Theorem II $\rightsquigarrow$

Corollary: A lattice polytope of dimension n, volume V, and degree d is a **lattice pyramid**, if

$$n \geq (V-1)(2d+1) \approx O(V\,d).$$

5. Proof of Theorem I

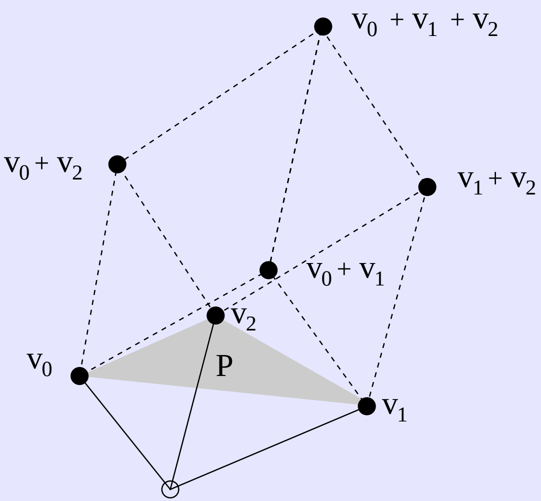
The setup

Let P be an n-dimensional lattice simplex of degree d.

$$P := \operatorname{conv}(v_0, \ldots, v_n) \subseteq M_{\mathbb{R}} \times \{1\}, \quad \overline{M} := M \oplus \mathbb{Z}.$$

Definition: (half-open) **parallelepiped**

$$\Pi(P) := \{\sum_{i=0}^{n} \lambda_i v_i \ : \ 0 \leq \lambda_i < 1\}.$$



The support

1. Let $m \in M_{\mathbb{R}} \oplus \mathbb{R}$. Then

$$m = \sum_{i=0}^{n} \lambda_i v_i \quad \text{for} \quad \lambda_i \in \mathbb{R},$$

$$\text{supp}(m) := \{i \in \{0, \dots, n\} \; : \; \lambda_i \neq 0\}.$$

2. $$\text{supp}(P) := \bigcup_{m \in \Pi(P) \cap \overline{M}} \text{supp}(m).$$

A simple reformulation

Lemma:

$$P \text{ lattice pyramid } \Leftrightarrow \text{supp}(P) \subsetneq \{0, \ldots, n\}$$

Proof: Follows from

$$\text{Vol}(P) = |\,\Pi(P) \cap \overline{M}\,|.$$

T.f.a.e. for $P' := \text{conv}(v_0, \ldots, v_{n-1})$:

- P is a lattice pyramid over P' (with apex v_n)
- $\text{Vol}(P) = \text{Vol}(P')$
- $\Pi(P) \cap \overline{M} = \Pi(P') \cap \overline{M}$.
- $\text{supp}(P) \subseteq \{0, \ldots, n-1\}$

$\square$

Line of argument

Goal:

$$|\text{supp}(P)| \leq 4d - 1.$$

Idea: Cover the support of P in a greedy manner.

Choose recursively lattice points

$$m_0, m_1, m_2, \ldots \in \Pi(P) \cap \overline{M}$$

such that

$$|\text{supp}(m_0)| \text{ maximal},$$

$$|\text{supp}(m_0) \cup \text{supp}(m_1)| \text{ maximal},$$

$$|\text{supp}(m_0) \cup \text{supp}(m_1) \cup \text{supp}(m_2)| \text{ maximal},$$

$$\ldots$$

Claim:

$$|\,\mathsf{supp}(m_0)\,| \leq \mathbf{1} \cdot 2d,$$

$$|\,\mathsf{supp}(m_0) \cup \mathsf{supp}(m_1)\,| \leq (\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}}) \cdot 2d,$$

$$|\,\mathsf{supp}(m_0) \cup \mathsf{supp}(m_1) \cup \mathsf{supp}(m_2)\,| \leq (\mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{4}}) \cdot 2d,$$

$$\dots$$

Finiteness yields

$$|\,\mathsf{supp}(P)\,| = |\bigcup_{j=0}^{\infty} \mathsf{supp}(m_j)\,| < \mathbf{2} \cdot 2d = 4d.$$

Therefore

$$|\,\mathsf{supp}(P)\,| \leq 4d - 1.$$

□

Proof of claim by induction.

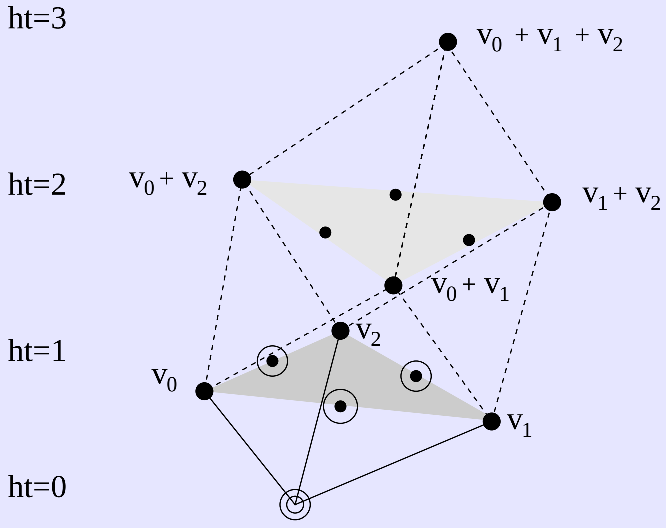
A well-known observation

In the case of a *simplex* the following equation holds:

$$h_i^* = |\{m \in \Pi(P) \cap \overline{M} \,:\, \mathrm{ht}(m) = i\}|,$$

where $\mathrm{ht}(m)$ equals its last coordinate.

Example: $P = S$ the exceptional triangle, $h^* = 1 + 3t$.



Proof of claim for m_0

We have to show

$$\boxed{m \in \Pi(P) \cap \overline{M} \quad \Rightarrow |\mathsf{supp}(m)| \leq 2d.}$$

The previous slide yields

$$d = \deg(P) \geq \mathrm{ht}(m).$$

Let

$$\mathsf{supp}(m) = \{0, \dots, s\},$$
$$P' := \mathrm{conv}(v_0, \dots, v_s)$$
$$\Rightarrow \quad m \in \mathrm{int}(\Pi(P')) \cap \overline{M}.$$

This yields

$$d \geq \mathrm{ht}(m) \geq \mathrm{codeg}(P') = s + 1 - \deg(P')$$
$$\geq s + 1 - \deg(P) = s + 1 - d.$$

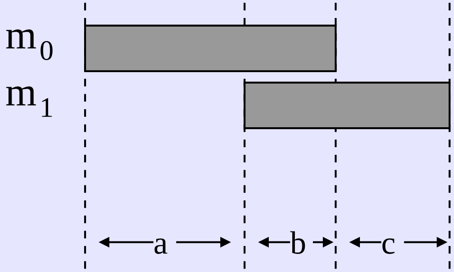
Hence

$$|\mathsf{supp}(m)| = s + 1 \leq 2d.$$

$\square$

Proof of induction step $m_0 \to m_1$

Schematic figure of $\mathsf{supp}(m_0) \cup \mathsf{supp}(m_1) \subseteq \{0, \ldots, n\}$:



$$b + c = |\,\mathsf{supp}(m_1)\,| \leq |\,\mathsf{supp}(m_0)\,| = a + b. \qquad (1)$$

We are going to show

$$c \leq d = \deg(P).$$

Since then

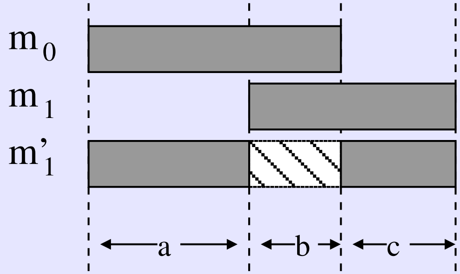
$$|\,\mathsf{supp}(m_0) \cup \mathsf{supp}(m_1)\,| = a + b + c \leq 2d + d = (1 + \frac{1}{2})2d.$$

$$m_0 = \sum_{i=0}^{n} \lambda_i v_i, \quad m_1 = \sum_{i=0}^{n} \mu_i v_i.$$

Then

$$m_1' := \sum_{i=0}^{n} \{\lambda_i + \mu_i\} v_i \quad \in \Pi(P) \cap \overline{M},$$

where $\{\gamma\} \in [0,1[$ is the fractional part of a real number γ.



$$a + c \leq \mathsf{supp}(m_1') \leq \mathsf{supp}(m_0) = a + b,$$

so

$$c \leq b. \tag{2}$$

Combining

$$b + c \leq a + b \tag{1}$$

and

$$c \leq b \tag{2}$$

yields

$$c \leq b \leq a + b - c,$$

so

$$2c \leq a + b = \mathsf{supp}(m_0) \leq 2d,$$

$$\Rightarrow \quad c \leq d.$$

□

The general induction step works similarly.

This finishes the proof of Theorem I.

6. Outlook

The leading term conjecture

Conjecture: Let $i \in \{1, \ldots, d\}$.
There exists a constant N such that any n-dimensional lattice polytope of degree d and with coefficient h_i^* is a **lattice pyramid**, if

$$n \geq N.$$

For $i = d$ the conjecture is equivalent to:

Conjecture II [Batyrev]:
There exists a **uniform upper bound** on the volume of lattice polytopes of degree d and leading coefficient h_d^*.

Holds for $d \leq 2$ due to Treutlein (2007).

The Cayley conjecture - refined

Conjecture I': Let d be fixed.
There exists N such that any lattice polytope P of degree d and dimension $n \geq N$ is a **Cayley polytope** $P_1 * \cdots * P_s$ of non-empty lattice polytopes for $s = n + 2 - N$.

Holds for $d = 1$ with $N = 3$ by the classification.

Theorem. [Haase, N., Payne 2007]
Conjecture I' implies Conjecture II.

Conjecture I' also implies the qualitative statements of Batyrev's result and its generalization Theorem II.

Gorenstein polytopes

Definition: P is **Gorenstein**, if h^*_P is symmetric.
In particular, $h^*_d = h^*_0 = 1$.

Proposition [N.]:
Conjecture I' holds for Gorenstein polytopes (with $N = 2d$).

Corollary:
There exists a uniform bound on the volume of Gorenstein polytopes of degree d.

Proposition follows from combining

- [Batyrev, Borisov 1997] criterion for Gorenstein polytopes in terms of **Gorenstein cones**,
- [Batyrev, N. 2007] characterization of Cayley polytopes by **special simplices**.

References

- V.V. Batyrev: *Lattice polytopes with a given h^*-polynomial*. Preprint, math.CO/0602593, 2006. Appeared in: Contemp. Math. (Proceedings of Euroconference on combinatorics at Anogia 2005) **423**, 269–282, 2007.
- V.V. Batyrev, B. Nill: *Multiples of lattice polytopes without interior lattice points*. Preprint, math.CO/0602336, 2006. Appeared in: Moscow Math. J. **7**, 195–207, 2007.
- V.V. Batyrev, B. Nill: *Combinatorial aspects of mirror symmetry*. Preprint, arXiv:math/0703456, 2007. To appear in Contemporary Mathematics (Proceedings of AMS-conference on integer points at Snowbird 2006).
- B. Nill: *Lattice polytopes having h^*-polynomials with given degree and linear coefficient*. Preprint, arXiv:0705.1082, 2007.