WHAT IS THE MAXIMAL NUMBER OF VERTICES OF A REFLEXIVE POLYTOPE?

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Introduction

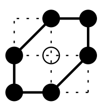
Reflexive polytopes are lattice polytopes containing the origin in their interior such that also the dual polytope is a lattice polytope. So they always appear as dual pairs. The notion of reflexive polytopes was introduced by Batyrev in 1994 to provide a combinatorial framework for constructing mirror symmetric pairs of Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties, [Bat94]. In fixed dimension n only a finite number of reflexive polytopes exist up to unimodular isomorphisms. This follows from results in [LZ91], since any reflexive polytope contains no lattice points in its interior except the origin. Using heavy computer calculations Kreuzer and Skarke succeeded in classifying reflexive polytopes up to dimension four, [KS98, KS00]. They found up to unimodular isomorphisms 16 reflexive polytopes for $n = 2$, 4319 for $n = 3$, and 473800776 for $n = 4$. Contrasting these finiteness results, Haase and Melnikov showed that *any* lattice polytope is isomorphic to a face of some (possibly much higher-dimensional) reflexive polytope, [HM04].

In recent years progress has been made towards a better understanding of the combinatorics and geometry of reflexive polytopes. Still, many elementary questions are open. Here we focus on the number of vertices.

Reflexive polytopes with the maximal number of vertices

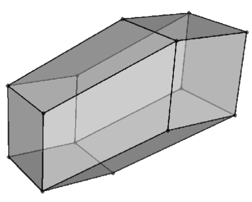
Using the database of reflexive polytopes [KS06] we find the following reflexive polytopes with the maximal number of vertices in dimension $n \leq 4$:

- $n = 2$: The maximum is 6 vertices; realized only by the following hexagon $\mathcal{H}$:



$\mathcal{H}$

- $n = 3$: The maximum is 14 vertices; realized only by the reflexive polytope having vertices with coordinates *either* all in $\{0, 1\}$ *or* in $\{0, -1\}$:



- $n = 4$: The maximum is 36 vertices; realized only by $\mathcal{H} \times \mathcal{H}$.

These observations motivate the following question (note that $14 = \lfloor 6^{3/2} \rfloor$):

Problem: Are there n-dimensional reflexive polytopes having more than $6^{n/2}$ vertices? If not, is $\mathcal{H}^{n/2}$, for n even, the only one with this number of vertices?

It suffices to solve the problem for *even* n, since products of reflexive polytopes are reflexive. We remark that in odd dimension we cannot even state a conjectural sharp bound. One should also note that for $n \geq 6$ there are non-reflexive lattice polytopes with only one interior lattice point and more than $6^{n/2}$ vertices. We thank Günter Ziegler for this observation.

Actually, the problem should be seen as a quest to find, if possible, intersting examples, or even better, explicit constructions of higher-dimensional reflexive polytopes having many vertices that are not simply products of lower-dimensional ones.

Implications

If the bound in the problem would hold, then we would also get sharp upper bounds on the following invariants:

- The number of vertices of an n-dimensional Gorenstein polytope, i.e., of a lattice polytope with some multiple being a reflexive polytope (up to translation). A Gorenstein polytope can also be characterized as a lattice polytope with symmetric δ-vector (sometimes also called h^*-, or h-vector), [BR05]. An example is the Birkhoff polytope.
- The class number of a Gorenstein toric Fano variety, [Nil05a, Sect.5].
- The number of facets of an n-dimensional reflexive polytope, simply due to duality.
- The topological Euler characteristic of a Gorenstein toric Fano variety, [Ful93, p.59].

Results

The following cases have been settled:

- For $n = 2$ a simple explanation for the above bound can be given, [Nil05a, Cor.4.2(2)]. However, for $n = 3, 4$ no direct proof is known apart from the computer classification of Kreuzer and Skarke, [KS98].
- An n-dimensional simplicial reflexive polytope has at most $3n$ vertices, where equality holds only for the dual of $\mathcal{H}^{n/2}$, [Cas06].
- An n-dimensional reflexive polytope where the vertices are the only lattice points on the boundary has $\leq 2^{n+1} - 2$ vertices, and equality implies central-symmetry, [Nil05a, Cor.6.3].
- The problem is solved affirmatively for simple reflexive polytopes whose dual contains a centrally symmetric pair of facets, [Nil05b]. In particular, $\mathcal{H}^{n/2}$ is, for n even, the only centrally symmetric simple reflexive polytope with the maximal number of $6^{n/2}$ vertices.

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