Reflexive polytopes, **Gorenstein polytopes**, and combinatorial mirror symmetry

Benjamin Nill

U Kentucky 10/04/10

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Convince you that (reflexive &) Gorenstein polytopes

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2. consist of interesting examples
3. have fascinating and not yet understood properties

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I. Reflexive polytopes

Combinatorial polytopes and duality

Combinatorial types of polytopes

Isomorphisms: combinatorially isomorphic face posets

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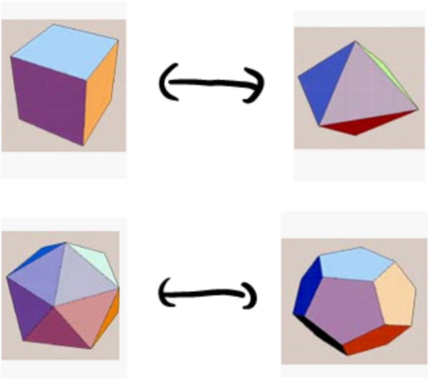


Realized polytopes and duality

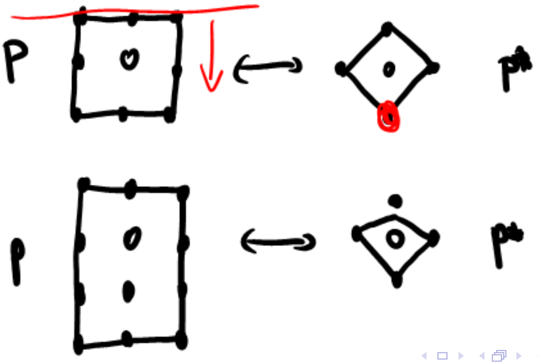
Embedded polytopes: $P \subset \mathbb{R}^d$
Isomorphisms: affine isomorphisms

Realized polytopes and duality

Embedded polytopes: $P \subset \mathbb{R}^d$
Isomorphisms: affine isomorphisms

$P \subset \mathbb{R}^d$ d-polytope with interior point 0 $\Longrightarrow$

$$P^* := \{y \in (\mathbb{R}^d)^* \; : \; \langle y, x \rangle \geq -1 \; \forall \, x \in P\}$$



Lattice polytopes and duality

Lattice polytopes: $P = \mathrm{conv}(m_1, \ldots, m_k)$ for $m_i \in \mathbb{Z}^d$
isomorphisms: affine lattice isomorphisms of $\mathbb{Z}^d$ (unimodular equivalence)

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Definition (Batyrev '94)

A **reflexive polytope** is a lattice polytope P with $0 \in \mathrm{int}(P)$ such that P^* is also a lattice polytope.

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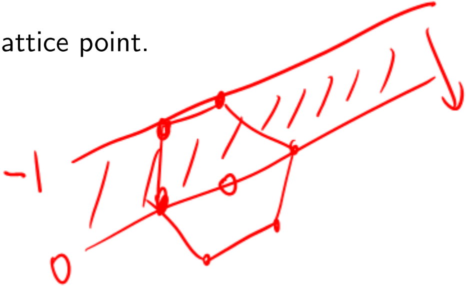
$\rightsquigarrow$ origin *only* interior lattice point.



Reflexive polytopes

Facts

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d	2	3	4
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d	2	3	4
vertices $\leq$	6	14	36

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Reflexive polytopes of higher index! (Joint work with A. Kasprzyk)

Let P be a lattice polytope with 0 in its interior.

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ℓP^* ℓ-reflexive

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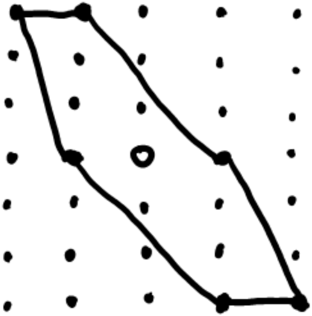
$$\ell P^* \ \ell\text{-reflexive} \quad \text{and} \quad P = \ell(\ell P^*)^*.$$

Duality of ℓ-reflexive polytopes!

Examples of ℓ-reflexive polygons?!

$\ell = 2$: No?!

$\ell = 3$:



Classification of ℓ-reflexive polygons (Joint work with A. Kasprzyk)

Theorem

P ℓ-reflexive polygon; $\Lambda := < \partial P \cap \mathbb{Z}^2 >_{\mathbb{Z}} \implies$
P is 1-reflexive w.r.t. Λ.

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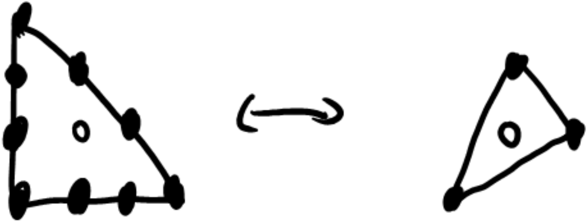
ℓ	1	3	5	7	9	11	13	15	17	$\cdots$
#	16	1	12	29	1	61	81	1	113	$\cdots$

The "number 12"

$\# \partial P \cap \mathbb{Z}^2 = 7$

$\# \partial P \cap \mathbb{Z}^2 = 5$

The “number 12”



$\# \partial P \cap \mathbb{Z}^2 = 9$ $\qquad$ $\# \partial P \cap \mathbb{Z}^2 = 3$

The "number 12"

12-Property

P reflexive polygon $\implies$

$$|\partial P \cap \mathbb{Z}^2| + |\partial P^* \cap \mathbb{Z}^2| = 12.$$

The "number 12" generalizes! (Joint work with A. Kasprzyk)

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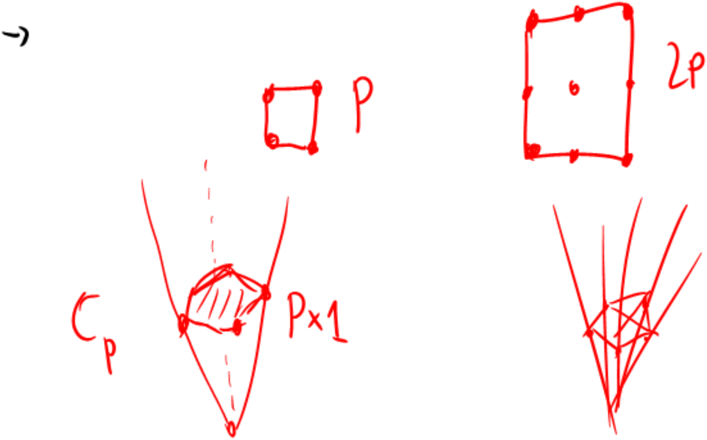
What else can be generalized?
What about higher dimensions?
What about algebro-geometric implications?

II. Gorenstein polytopes

Definition and duality

Def.[Batyrev/Borisov '97] A **Gorenstein polytope of codegree** r is a lattice polytope P such that rP is a reflexive polytope (up to lattice translation).



Definition and duality

Let $C_P := \mathbb{R}_{\geq 0}(P \times 1)$.

Proposition (Batyrev/Borisov '97)

P is a Gorenstein polytope if and only if

$$(C_P)^* \cong C_Q$$

for some lattice polytope Q.

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$$\text{codeg}(P) = \text{codeg}(P^*).$$

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$\rightsquigarrow$ Natural duality of Gorenstein polytopes!

Characterization via Commutative Algebra

Facts (see Bruns & Gubeladze, Miller & Sturmfels)

If P is a lattice d-polytope, then $S_P := \mathbb{C}[C_P \cap \mathbb{Z}^{d+1}]$ is a positively graded normal Cohen-Macaulay ring,

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$$\omega_{S_P} = < x^m \ : \ m \in \mathrm{int}(C_P) \cap \mathbb{Z}^{d+1} >$$

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- S_P Gorenstein ring
- the Hilbert series $H_{S_P}(t)$ satisfies

$$H_{S_P}(t) = (-1)^{d+1} H_{S_P}(t^{-1}).$$

Characterization via Lattice-Point-Enumeration

P lattice d-polytope.

$$\sum_{k \geq 0} |\, C_P \cap (\mathbb{Z}^d \times k) |\; t^k = \frac{h^*(t)}{(1-t)^{d+1}},$$

where $h^*(t)$ is a polynomial with nonnegative integer coefficients of degree $\leq d$.

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Ehrhart theory: $k \mapsto kP \cap \mathbb{Z}^d$ is a polynomial of degree d.

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$$\Longrightarrow \deg(P) = d + 1 - \operatorname{codeg}(P).$$

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Proposition (Stanley)

T.f.a.e.

- P Gorenstein polytope (of codegree $\operatorname{codeg}(P)$)
- h^*-polynomial of P is symmetric (of degree $\deg(P)$)

Finiteness of Gorenstein polytopes

Observation: Lattice pyramids don't change the h^*-polynomial.

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Theorem (Batyrev/N. '08; Haase/N./Payne '09; Batyrev/Juny '09)

There exist only *finitely* many Gorenstein polytopes of degree s that are not lattice pyramids.

s, $h^*_s = h^*_0 = 1$

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s	0	1	2
#	1	1	37

Gorenstein polytopes in algebraic and polyhedral combinatorics

- Toric ideals of Gorenstein polytopes are "classical"; "Nice initial complexes on some classical ideals" (Conca/Hosten/Thomas '06).

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- dimension: $(n-1)^2$

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Lattice distance of x equals 1 from any facet $\Longrightarrow$ nB_n is reflexive.

'The' example of a Gorenstein polytope: the Birkhoff polytope

Facts

$B_n \subseteq \mathbb{R}^{n^2}$ is a **Gorenstein** polytope of codegree n

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- number of facets: n^2 (inequalities: $x_{i,j} \geq 0$)

A point in $kB_n \cap \mathbb{Z}^{n^2}$ is $n \times n$-matrix with entries in $\{0, \ldots, n\}$ and row and column sum k: **semi-magic square with magic number** k.

A semi-magic square is in the interior of kB_n if and only if any entry is $\neq 0$.

Semi-magic square x in interior with smallest magic number:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Lattice distance of x equals 1 from any facet $\Longrightarrow$ nB_n is reflexive.

Athanasiadis' proof of Stanley's conjecture

Def.: The coefficient vector $(h_0^*, \dots, h_s^*)$ is **unimodal**, if

$$h_0^* \le h_1^* \le \cdots \ge \cdots \ge h_s^*.$$

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Proof relies on the notion of *special simplices*.

Special simplices

Let P be a Gorenstein d-polytope of codegree r.

Proposition (Batyrev/N. '07)

S is a **special** $(r-1)$-**simplex**, if the vertices of S are r affinely independent lattice points of P such that

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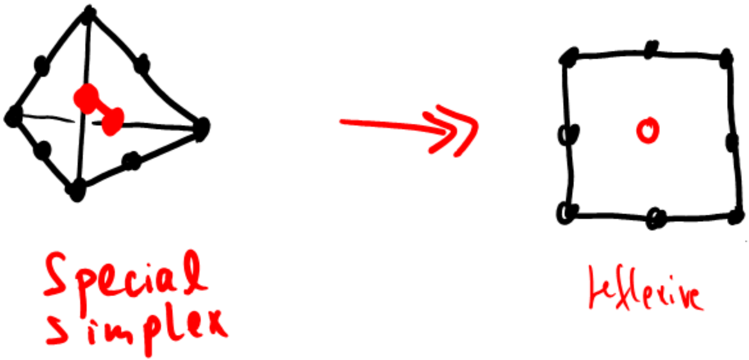
Then S is unimodular.

Example: B_n contains special $(n-1)$-simplex: permutation matrices corresponding to elements in cyclic subgroup generated by $(1\ 2\ \cdots\ n)$.

Special simplices

Proposition (Bruns/Roemer '05; Batyrev/N.'07)

Projecting P along a special $(r-1)$-simplex yields a reflexive polytope with the same h^*-polynomial.



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Question: P normal Gorenstein polytope $\Longrightarrow$ h^*_P unimodal ?

III. Combinatorial mirror symmetry

Philosophy

Gorenstein polytopes are combinatorial models of Calabi-Yau varieties.

Mirror symmetry

Y **Calabi-Yau** n-fold, if its canonical divisor is trivial.

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Example: Let P be reflexive polygon. For generic coefficients $c_{(a,b)} \in \mathbb{C}^*$

$$Y := \{(x,y) \in (\mathbb{C}^*)^2 \; : \; \sum_{(a,b)\in P\cap\mathbb{Z}^2} c_{(a,b)} \, x^a y^b \; = \; 0\}$$

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$$Y := \overline{\{(x,y) \in (\mathbb{C}^*)^2 \ : \ \sum_{(a,b) \in P \cap \mathbb{Z}^2} c_{(a,b)}\, x^a y^b \ = \ 0\}}$$

is an elliptic curve (Calabi-Yau 1-fold).

Mirror symmetry

String Theory proposes mirror pairs of CY-n-folds Y, Y^*!

Topological mirror symmetry test

$$h^{p,q}(Y) = h^{p,n-q}(Y^*)$$

for Hodge numbers $h^{p,q} = h^q(Y, \Omega^p_Y)$.

Batyrev's construction

Theorem (Batyrev '94)

P, P^* dual reflexive polytopes $\rightsquigarrow$ Calabi-Yau hypersurfaces Y_P, Y_{P^*} in Gorenstein toric Fano varieties whose stringy Hodge numbers satisfy the topological mirror symmetry test.

Batyrev-Borisov-construction

Theorem (Batyrev/Borisov '96)

Dual *nef-partitions* $\rightsquigarrow$ Calabi-Yau *complete intersections* in Gorenstein toric Fano varieties whose stringy Hodge numbers satisfy the topological mirror symmetry test.

Nef-partitions

Families of lattice polytopes $\rightsquigarrow$ complete intersections Y.

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Y is Calabi-Yau, if $Q_1 + \cdots + Q_r$ is reflexive.

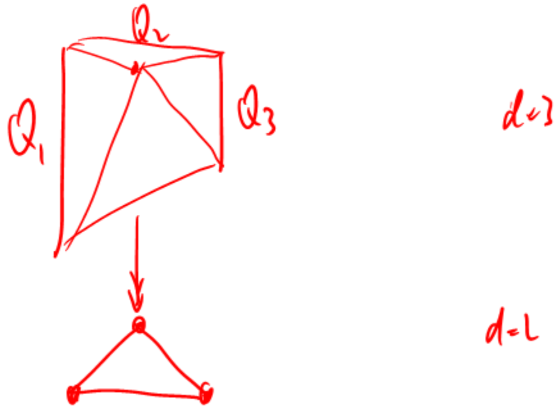
$Q_1, \ldots, Q_r$ **nef-partition**, if $0 \in Q_1$, ..., $0 \in Q_r$.

Gorenstein polytopes enter the picture

$Q_1 + \cdots + Q_r$ reflexive

$\rightsquigarrow$

Cayley-polytope is Gorenstein of codegree r!



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P Gorenstein polytope of codegree r:

Cayley structures of length r on P $\longleftrightarrow$ Special $(r-1)$-simplices of P^*

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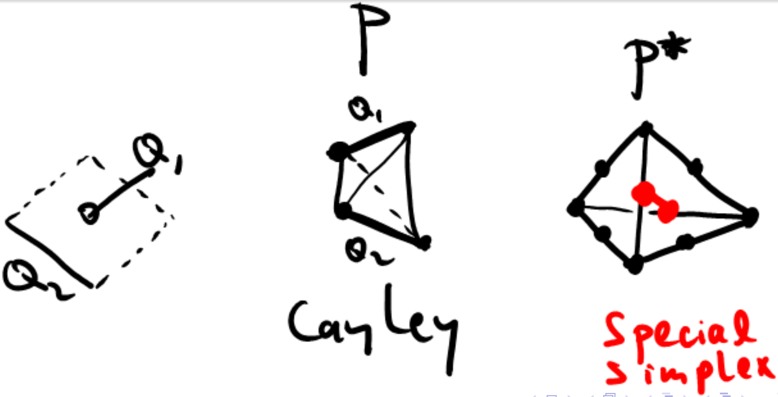

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$\iff$

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The stringy E-polynomial of Y

Def.: Stringy E-polynomial:

$$E_{st}(Y; u, v) := \sum_{p,q} (-1)^{p+q} h_{\mathrm{st}}^{p,q}(Y)\ u^p\ v^q.$$

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Theorem (Batyrev/Borisov '96; Borisov/Mavlyutov '03)

Given Gorenstein polytope P as Cayley polytope of length r and CY complete intersection Y:

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Beautiful facts (Batyrev/Borisov '96; Borisov/Mavlyutov '03)

- "Hodge duality": $E_{st}(P; u, v) = E_{st}(P; v, u)$.
- "Poincare duality": $E_{st}(P; u, v) = (uv)^n \, E_{st}(P; u-1, v-1)$.
- "Mirror symmetry": $E_{st}(P; u, v) = (-u)^n \, E_{st}(P^*; u-1, v)$.

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A priori just a Laurent polynomial!

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Open: Is the degree of $E_{st}(P; u, v) \neq 0$ equal to $2n$?

Finally, the main challenge

Conjecture (Batyrev/N. '08)

There exist only *finitely* many stringy E-polynomials of Gorenstein polytopes with fixed Calabi-Yau dimension n and fixed constant coefficient.

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Question (Yau):
Only finitely many topological types of irreducible CY-3-folds?