Dual defect toric manifolds and the Cayley polytope conjecture

Benjamin Nill
(joint work with Alicia Dickenstein)

University of Sydney 10/06/10

I. The Cayley polytope conjecture

Building blocks: lattice polytopes without interior lattice points

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Our focus: When do these polytopes have width 1 ("Pancake")?

Invariant No. I: $\operatorname{codeg}(P)$ (measuring "hollowness")

Let $P \subseteq \mathbb{R}^n$ be an n-dimensional lattice polytope.

Def. The *codegree* of P is defined as

$$\operatorname{codeg}(P) := \min\{k \in \mathbb{Z}_{\geq 0} \ : \ \operatorname{int}(kP) \cap \mathbb{Z}^n \neq \emptyset\}.$$

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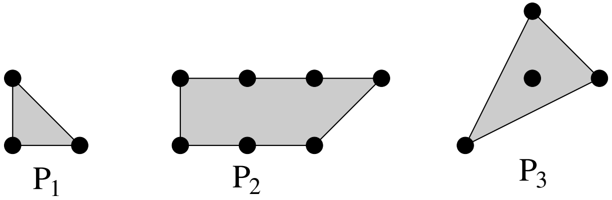

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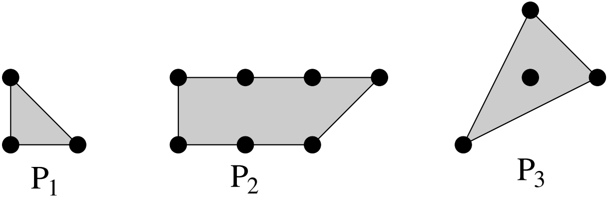

$\operatorname{codeg}(P_1) = 3$ $\operatorname{codeg}(P_2) = 2$ $\operatorname{codeg}(P_3) = 1$

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[Batyrev, N. 07]: Done for $\mathrm{codeg}(P) = n$.

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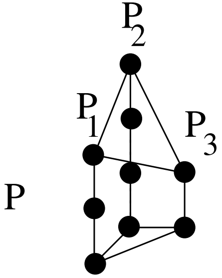
$P = P_1 * P_2 * P_3$ for P_1, P_2, P_3 intervals:



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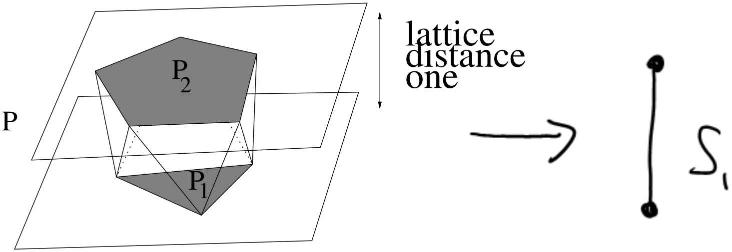

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$\rightsquigarrow \mathrm{codeg}(P) \geq k$ and $\mathrm{width(P)} = 1$.

High codegree $\Longrightarrow$ Cayley polytope?

Thm. [Haase, N., Payne 08]
If

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Not sharp?! $2S_n$ has codegree $\lceil \frac{n+1}{2} \rceil$ and is not a Cayley polytope.

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Thm. [Di Rocco 03; Di Rocco, Dickenstein, Piene 08; Dickenstein, N. 09] The Cayley polytope conjecture holds for *smooth* lattice polytopes.

II. *A*-discriminants and dual defect toric manifolds

The dual variety

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Thm.(Biduality-Theorem)[GKZ 94]

$$(X^\vee)^\vee = X.$$

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Cor. If $X^\vee$ has codimension $r+1$, then X is a union of r-dimensional projective subspaces.

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We say, $X \subseteq \mathbb{P}^{N-1}$ has *has dual defect*, if $r > 0$.

Much work on the classification of dual defect varieties!

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Example: $A := \{(d, 0), (d-1, 1), \ldots, (0, d)\}$
$\rightsquigarrow$ Δ_A classical discriminant of degree d.

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[Gelfand, Kapranov, Zelivinsky 94], [Dickenstein, Feichtner, Sturmfels 05], [Matsui, Takeuchi 08], [Esterov 08].

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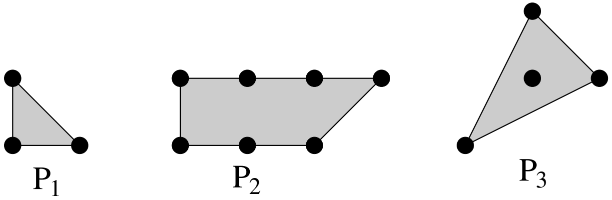

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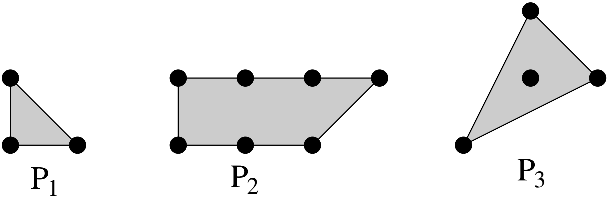

P_1, P_2 smooth, P_3 not smooth.

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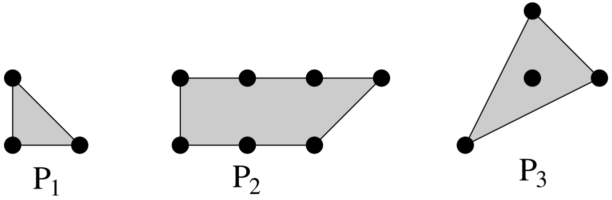

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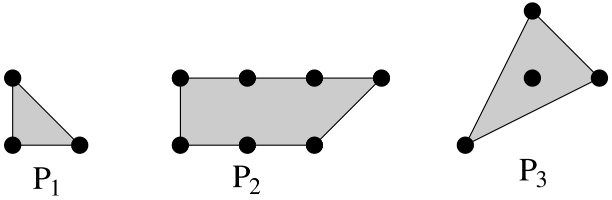

$\mathrm{Vol}_{\mathbb{Z}}(P_1) = 1$ $\mathrm{Vol}_{\mathbb{Z}}(P_2) = 5$ $\mathrm{Vol}_{\mathbb{Z}}(P_3) = 3$

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$\rightsquigarrow$ $c(P) = 0$ if and only if X_P has dual defect.

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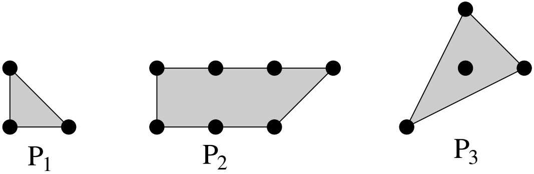

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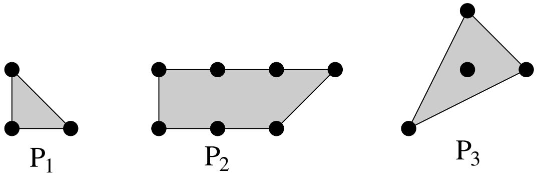

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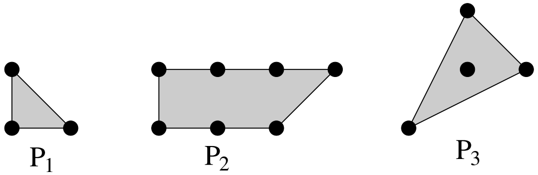

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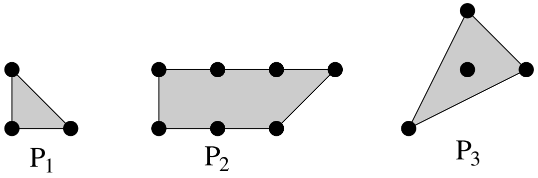

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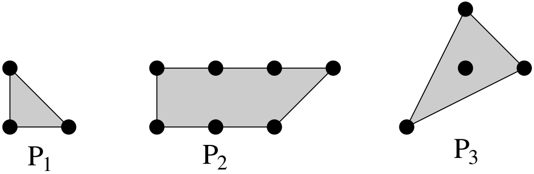

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[Dickenstein, N. 10]: True for lattice simplices.

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A clever lemma yields:

$$c(P) = 0 \quad \Longleftrightarrow \quad m \neq l.$$

An example from GKZ 94

Let S_n be the n-dimensional unimodular simplex.
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We will give an easier argument at the end.

III. The relation between $c(P)$ and $\mathrm{codeg}(P)$

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Main thm. [Di Rocco 06, Dickenstein, N. 10]
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Thm. [Di Rocco 06] In this case, P is a (strict) Cayley polytope.

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Bold guess:

$$c(P) = \sum_{p=d+1}^{n} \sum_{i=1}^{p-d} \quad \textbf{???} \quad \left(\sum_{G \leq P, \dim(G) = p} |\mathrm{int}(iG) \cap \mathbb{Z}^n| \right).$$

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By Stanley's monotonicity theorem:

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Based on lower-dimensional computer calculations we guessed:

$$c(P) = \sum_{p=d+1}^{n} \sum_{i=1}^{p-d} (-1)^{d-i} i \binom{p+1}{p-d-i} \left(\sum_{G \leq P, \dim(G)=p} |\operatorname{int}(iG) \cap \mathbb{Z}^n| \right)$$

A magic identity

Ehrhart theory reduces proof to:

Lemma. For $k < n - d$ and $j \in \{k, \ldots, n\}$:

$$\sum_{i=0}^{n-d} (-1)^{n-d-i} i \binom{i+j-k}{j} \binom{j+1}{n-d-i} = j+1.$$

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$\rightsquigarrow$ Zeilberger's algorithm comes to the rescue! $\square$

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Application: A conjecture in adjunction theory

Two invariants from [Beltrametti, Sommese 95]

The adjunction theory of complex projective varieties.

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