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Lattice polytopes with a view toward algebraic geometry

Benjamin Nill
(Case Western Reserve University → Stockholm University)

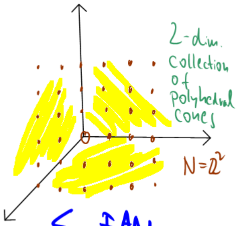
MEGA Frankfurt 2013

What is Toric Geometry?

X_Σ toric variety / $\mathbb{C}$
(normal, abstract)

2-dim.
complex
algebraic
surface

$\leftrightarrow$



Σ FAN

N LATTICE

$X_\Sigma \cong X_{\Sigma'} \iff$ affine lattice automorphism of N maps cones in Σ to cones of Σ'

Why Toric Geometry?

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Why Toric Geometry?

- Shopping mall for algebraic geometers looking for:
  - examples
  - experiments
  - databases
  - conjectures
  - intuition
  - proof ideas

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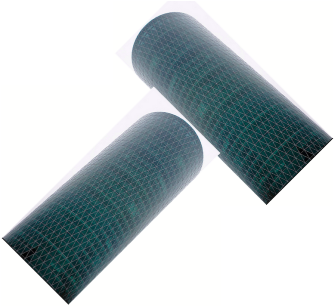
  - proof ideas
- Hammer for geometric combinatorialists to nail proofs (and to provide new viewpoints and interesting headaches!)



Toric Geometry is MEGA!

Toric varieties allow for

- explicit combinatorial description
- often complete classification results
- effective computation of invariants

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This talk:

LATTICE POLYTOPES AND TORIC GEOMETRY

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This talk:

LATTICE POLYTOPES AND TORIC GEOMETRY

Focus on

Classifications and invariants of toric Fano varieties

Polytopes enter the picture ...

Definition

A polytope Q is called **lattice polytope**, if its vertices are lattice points.

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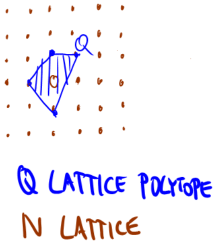

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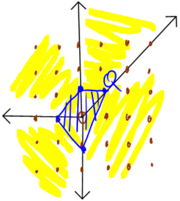

X_{Σ_Q} PROJECTIVE TORIC VARIETY/$\mathbb{C}$ $\leftrightarrow$ Σ_Q FAN SPANNED BY Q

N LATTICE

Different lattice polytopes may define same toric variety!

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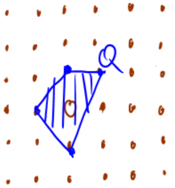
... Fano polytopes ...

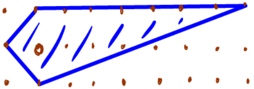


Definition

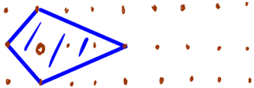
A polytope Q is called **Fano polytope**, if

- the origin is in the interior
- every vertex is a primitive lattice point

Yes:



No:



... and toric Fano varieties!

Definition

Projective X is **Fano variety**, if $-K_X$ is ample $\mathbb{Q}$-Cartier divisor.

Correspondence:

$$\text{toric Fano varieties } X_{\Sigma_Q} \quad \longleftrightarrow \quad \text{Fano polytopes } Q$$

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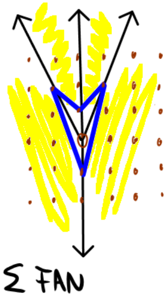

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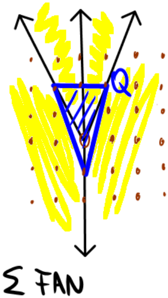
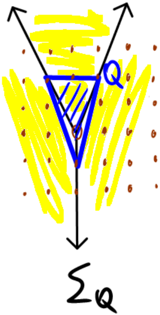
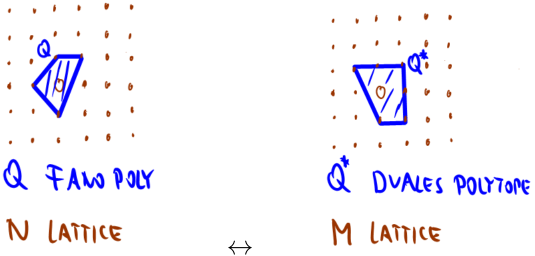




The M-side of things

Let M be the dual lattice of N.



$\leftrightarrow$

Definition

The **dual polytope** of a Fano polytope Q is defined as

$$Q^* := \{y \ : \ \langle x, y \rangle \geq -1 \ \ \forall \, x \in Q\}$$

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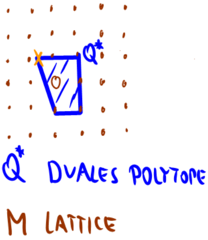

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Vertices of Q^* don't have to be lattice points anymore!

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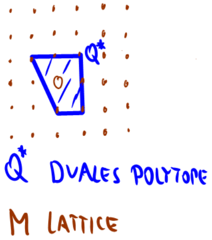

Correspondence

Q is **reflexive**	$\longleftrightarrow$	X_{Σ_Q} is **Gorenstein**
(Q^* is lattice polytope)	$\longleftrightarrow$	($-K_X$ is Cartier divisor)

In this case, $H^0(X_{\Sigma_Q}, O(-K_X))$ has $Q^* \cap M$ as $\mathbb{C}$-basis.

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Conjecture (Alexeev-Batyrev-Borisov-$\cdots$)

For fixed $n \in \mathbb{N}$, $\epsilon > 0$, there are **finitely** many families of n-dimensional Fano varieties with ϵ-logterminal singularities.

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- Fano manifolds
- toric Fano varieties
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Complete classification exists for Fano manifolds up to dimension 3

Let's have a look at the toric case ...

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THE TABLE of toric Fano varieties X of dim. n

n	canonical	Gorenstein	smooth
2	16	16	5
3	674,688	4,319	18
4		473,800,776	124
5			866
6			7,622
7			72,256
8			749,892
9			8,229,721
due to	Kasprzyk	Kreuzer, Skarke	Watanabe, Watanabe; Batyrev; Sato; Kreuzer, Nill; Øbro

$\longrightarrow$ Graded Ring Database (GRDB)

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Current development [Coates, Corti, Galkin, Golyshev, Kasprzyk]

Many (all?) Fano manifolds may be reconstructed from 'mirror' Laurent polynomials found using these polytopes

Excursion: Normal form algorithm

LOTS of Fano polytopes, how to check, if they are isomorphic?

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Solution: Normal form for list of vertices of lattice polytope Q:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & -1 \end{bmatrix} \quad \Rightarrow \quad NF(Q) = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

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Keyword: Hermite-normal form (of permutations of the columns)

Max Kreuzer
(1960-2010)

Implementation in PALP

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Implementation in PALP; MAGMA, SAGE [Grinis, Kasprzyk '13]

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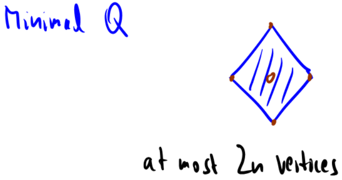
Two classification approaches in a nutshell

Growing [Kasprzyk]

Step 1: Classifying minimal Fano's (of certain type)



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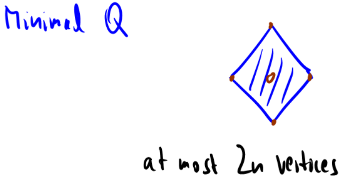
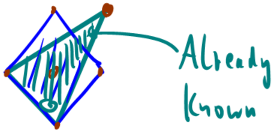


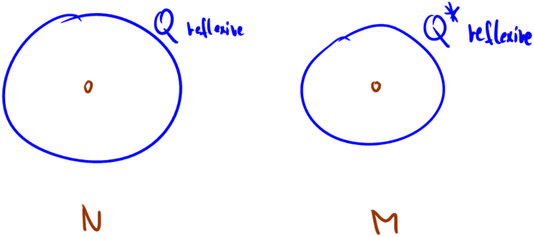
Step 2: Recursively add vertices



Two classification approaches

Duality [Kreuzer, Skarke]

$$Q \text{ reflexive} \Leftrightarrow Q^* \text{ reflexive}$$



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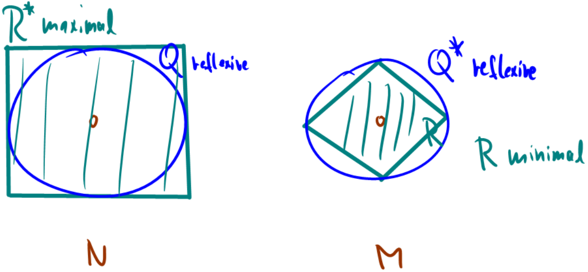

Volume bounds on canonical polytopes

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Theorem [Hensley '83; Lagarias, Ziegler '91; Pikhurko '01]

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Theorem [Averkov, Krümpelmann, Nill '13]

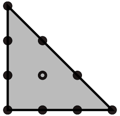
The volume of n-dimensional canonical polytopes is at most

$$2^{n2^n}$$

Volume bounds on canonical polytopes

Sharp bounds on the maximal volume

- $n = 2$: 4.5, attained *only* by



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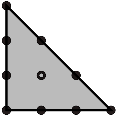

- $n = 3$ [Kasprzyk '08]: 12, attained by *two* canonical tetrahedra

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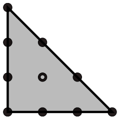

- $n = 3$ [Kasprzyk '08]: 12, attained by *two* canonical tetrahedra
- $n \geq 4$: Conjecture [Zaks, Perles, Wills '82; Hensley '83]:

$$2(s_n - 1)^2/n!,$$

where

$$s_1 = 2, \quad s_2 = 3, \quad s_3 = 7, \quad s_4 = 43, \ldots, \qquad s_k := s_1 \cdots s_{k-1} + 1.$$

Bound attained *only* for canonical simplex

$$Q_n := \mathrm{conv}(0, s_1 e_1, \ldots, s_{n-1} e_{n-1}, 2(s_n - 1) e_n)$$

Volume bounds on canonical polytopes

Theorem [Averkov, Krümpelmann, Nill '13]

Conjecture holds for canonical *simplices*.

Applications

- [Kasprzyk '13] 35,947 4-dimensional canonical simplices
- [Kasprzyk, N. '13] Conjecture holds for reflexive polytopes

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Applications

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Challenging problems:

- Conjecture still open for canonical *polytopes*
- Sharp bounds on number of lattice points?!

NEXT:

- Proof sketch
- Translation into algebraic geometry

Proof of volume bound

Let Q be simplex with origin only interior lattice point.
Let $\beta_0 \geq \cdots \geq \beta_n > 0$ be the barycentric coordinates of 0:

$$\sum_{i=0}^{n} \beta_i v_i = \mathbf{0}, \quad \sum_{i=0}^{n} \beta_i = 1$$

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Lemma [Pikhurko '01]

$$n! \operatorname{vol}(Q) \leq \frac{1}{\beta_0 \cdots \beta_{n-1}}$$

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Lemma [Pikhurko '01]

$$n! \operatorname{vol}(Q) \leq \frac{1}{\beta_0 \cdots \beta_{n-1}}$$

Theorem [Averkov '11]

Let Q be *lattice* simplex. Then

for $j = 0, \ldots, n-1$ we have

$$\beta_0 \cdots \beta_j \leq \beta_{j+1} + \cdots + \beta_n$$

Proof of volume bound

Lemma [N.'07]

Let $x_0 \geq \cdots \geq x_n > 0$ such that $x_0 + \cdots + x_n = 1$ and

$$x_0 \cdots x_j \leq x_{j+1} + \cdots + x_n$$

for $j = 0, \ldots, n-1$.

Proof of volume bound

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$$x_0 \cdots x_j \leq x_{j+1} + \cdots + x_n$$

for $j = 0, \ldots, n-1$. If $n \geq 4$, then

$$x_0 \cdots x_{n-1} \geq \frac{1}{2(s_n - 1)^2}$$

Proof of volume bound

Lemma [N.'07]

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for $j = 0, \ldots, n-1$. If $n \geq 4$, then

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Proof finish

Q canonical simplex $\implies$

$$n! \operatorname{vol}(Q) \leq \frac{1}{\beta_0 \cdots \beta_{n-1}} \leq 2(s_n - 1)^2$$

□

Why does the sequence $2, 3, 7, 43, \ldots$ appear?

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$$

$$\beta_0 > \beta_1 > \beta_2 > \beta_3 > \beta_4 > 0$$

$$\beta_0 \leq \beta_1 + \beta_2 + \beta_3 + \beta_4$$
$$\beta_0 \cdot \beta_1 \leq \beta_2 + \beta_3 + \beta_4$$
$$\beta_0 \cdot \beta_1 \cdot \beta_2 \leq \beta_3 + \beta_4$$
$$\beta_0 \cdot \beta_1 \cdot \beta_2 \cdot \beta_3 \leq \beta_4$$

with minimal

$$\beta_0 \cdot \beta_1 \cdot \beta_2 \cdot \beta_3$$

Why does the sequence $2, 3, 7, 43, \ldots$ appear?

Assume

$$\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 = 1$$

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$$\beta_0 < \beta_1 + \beta_2 + \beta_3 + \beta_4$$
$$\beta_0 \cdot \beta_1 \leq \beta_2 + \beta_3 + \beta_4$$
$$\beta_0 \cdot \beta_1 \cdot \beta_2 \leq \beta_3 + \beta_4$$
$$\beta_0 \cdot \beta_1 \cdot \beta_2 \cdot \beta_3 \leq \beta_4$$

with minimal

$$\beta_0 \cdot \beta_1 \cdot \beta_2 \cdot \beta_3$$

Why does the sequence $2, 3, 7, 43, \ldots$ appear?

$$\beta_0 + \epsilon \quad + \quad \beta_1 - \epsilon \quad + \quad \beta_2 \quad + \quad \beta_3 \quad + \quad \beta_4 \quad = 1$$

$$\beta_0 + \epsilon \quad > \quad \beta_1 - \epsilon \quad > \quad \beta_2 \quad > \quad \beta_3 \quad > \quad \beta_4 \quad > 0$$

$$\begin{array}{ccccccccc} \beta_0 + \epsilon & < & \beta_1 - \epsilon & + & \beta_2 & + & \beta_3 & + & \beta_4 \\ (\beta_0 + \epsilon) & \cdot & (\beta_1 - \epsilon) & \leq & \beta_2 & + & \beta_3 & + & \beta_4 \\ (\beta_0 + \epsilon) & \cdot & (\beta_1 - \epsilon) & \cdot & \beta_2 & \leq & \beta_3 & + & \beta_4 \\ (\beta_0 + \epsilon) & \cdot & (\beta_1 - \epsilon) & \cdot & \beta_2 & \cdot & \beta_3 & \leq & \beta_4 \end{array}$$

with

$$(\beta_0 + \epsilon) \quad \cdot \quad (\beta_1 - \epsilon) \quad \cdot \quad \beta_2 \quad \cdot \quad \beta_3$$

Why does the sequence $2, 3, 7, 43, \ldots$ appear?

$$\beta_0 + \epsilon \quad + \quad \beta_1 - \epsilon \quad + \quad \beta_2 \quad + \quad \beta_3 \quad + \quad \beta_4 \quad = 1$$

$$\beta_0 + \epsilon \quad > \quad \beta_1 - \epsilon \quad > \quad \beta_2 \quad > \quad \beta_3 \quad > \quad \beta_4 \quad > 0$$

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contradiction, since

$$(\beta_0 + \epsilon)(\beta_1 - \epsilon) = \beta_0\beta_1 - \epsilon(\beta_0 - \beta_1) - \epsilon^2 < \beta_0\beta_1.$$

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$$\begin{aligned} \beta_0 &= \beta_1 + \beta_2 + \beta_3 + \beta_4 \\ \beta_0 \cdot \beta_1 &\leq \beta_2 + \beta_3 + \beta_4 \\ \beta_0 \cdot \beta_1 \cdot \beta_2 &\leq \beta_3 + \beta_4 \\ \beta_0 \cdot \beta_1 \cdot \beta_2 \cdot \beta_3 &\leq \beta_4 \end{aligned}$$

with minimal

$$\beta_0 \cdot \beta_1 \cdot \beta_2 \cdot \beta_3$$

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Corollary

Let $X := X_{\Sigma_Q}$ toric Fano variety for canonical simplex Q.

n	$n = 2$	$n = 3$	$n \geq 4$
$(-K_X)^n \leq$	9	72	$2(s_n - 1)^2$

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$(-K_X)^n \leq$	9	72	$2(s_n - 1)^2$
equality	$\mathbb{P}^2$	$\mathbb{P}(3,1,1,1)$, $\mathbb{P}(6,4,1,1)$	$\mathbb{P}\left(\frac{2(s_n-1)}{s_1}, \ldots, \frac{2(s_d-1)}{s_{n-1}}, 1, 1\right)$

$\mathbb{P}(q_0, \ldots, q_n)$ is *weighted projective space*, given as quotient of $(\mathbb{C}^*)^{n+1}$ via

$$(\lambda_0, \ldots, \lambda_n) \cdot (x_0, \ldots, x_n) := (\lambda_0^{q_0} x_0, \ldots, \lambda_n^{q_n} x_n).$$

Back on track: Toric Fano manifolds

Definition

A Fano polytope Q is called **unimodular polytope**, if the vertex set of every facet of Q forms a lattice basis.

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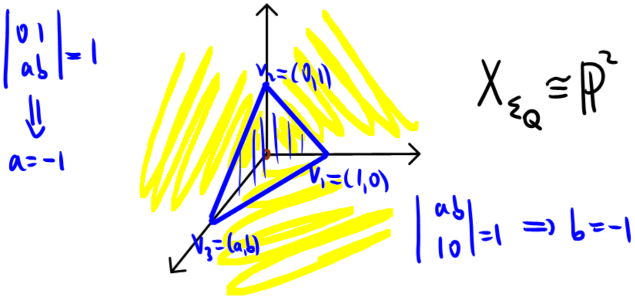
Up to lattice isomorphisms only ONE unimodular simplex:



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Picard number of toric Fano manifolds

Picard number is most important invariant:

$$\rho_X = \text{rank Pic}(X) = |\mathrm{Verts}(Q)| - n$$

$$\rho_X = 1 \iff X \cong \mathbb{P}^n$$

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Still open: Sharp upper bound on $\chi(X) = |\mathrm{facets}(Q)|$

Toric Fano manifolds with large Picard number

Let S_2 be $\mathbb{P}^2$ blown-up in two torus-invariant points:

X toric Fano manifold with $\rho_X = 2n - k$

- k=0: [N. 05; Casagrande '07] n even, $S_2 \times \cdots \times S_2$

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- k=1: [Øbro '07] 1 type, if n is even; 2 types, if n is odd
- k=2: [Assarf, Joswig, Paffenholz '13] 11 types, if $n \geq 6$ is even; 5 types, if $n \geq 6$ is odd

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Conjecture [Assarf, Joswig, Paffenholz '13]

If $k \leq n/3$, then

$$X \cong X' \times S_2 \times \cdots \times S_2$$

for $X' \leq 3k + 1$-dimensional toric Fano manifold.

Excursion: Einstein-Kähler-Manifolds (EKM)

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Minkowski's lattice point theorem

If K is n-dimensional convex body such that

1. origin is only interior lattice point,
2. K is centrally-symmetric with respect to origin,

then $\mathrm{vol}(K) \leq 2^n$.

Excursion: Einstein-Kähler-Manifolds (EKM)

Ehrhart's conjecture ['64]

If K is n-dimensional convex body such that

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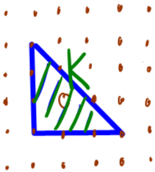

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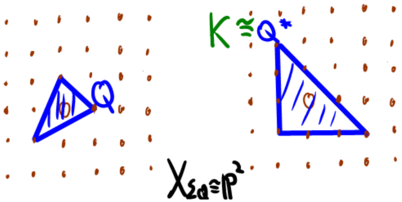

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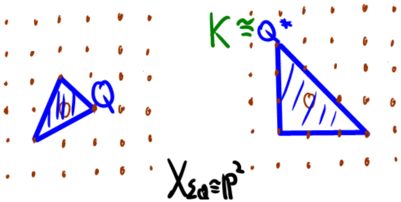

Criterion [Wang, Zhu '03]

2. holds for dual of unimodular polytope $Q \iff X_{\Sigma_Q}$ is EKM

Excursion: Einstein-Kähler-Manifolds (EKM)

[N., Paffenholz '09]:
Found counterexample to conjecture on EKM's in Øbro's database in dimension 6.

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Theorem [Berman, Berndtsson '12]

- $\mathbb{P}^n$ has largest anticanonical degree among toric EKM's.
- Ehrhart's conjecture holds for duals of Fano polytopes.

$\longrightarrow$ [N., Paffenholz, '12]:
'On the equality case in Ehrhart's volume conjecture'

What about higher dimensions?

n	canonical	Gorenstein	smooth
2	16	16	5
3	674,688	4,319	18
4		473,800,776	124
5		Guess: $\sim 10^{18}$	866
6			7,622
7			72,256
8			749,892
9			8,229,721
due to	Kasprzyk	Kreuzer, Skarke	Watanabe, Watanabe; Batyrev; Sato; Kreuzer, Nill; Øbro

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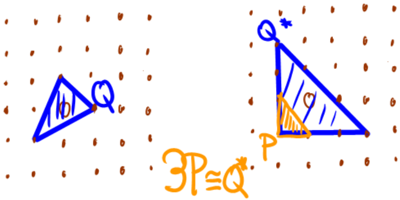
Definition

Index of Gorenstein toric Fano variety X:

$$i_X := \max(r \; : \; -K_X/r \text{ Cartier divisor})$$

The index



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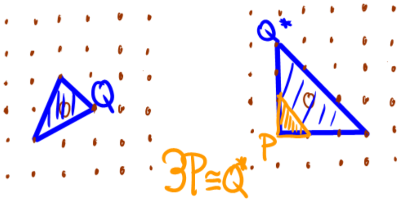
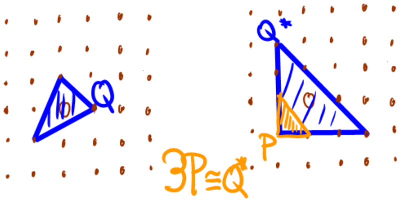


Index of X_{Σ_Q} is maximal r such that exists such P.

Example: $i_{\mathbb{P}^2} = 3$.

The index



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Definition

P lattice polytope.

- P is called **Gorenstein polytope of index** r, if rP is reflexive (up to affine-lattice isomorphisms)
- P is called **smooth Gorenstein polytope**, if rP is dual of unimodular polytope (up to affine-lattice isomorphisms)

A little step further into higher dimensions

Toric Fano manifolds with large index [Lorenz, N. '13]

Algorithm for smooth Gorenstein polytopes of large index yields

$n \setminus i_X$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	3	1	1											
3	15	2		1										
4	118	4	1		1									
5	853	11	1			1								
6	7,590	27	3	1			1							
7	72,167	83	4	1				1						
8	749,620	256	12	2	1				1					
9	8,228,801	891	23	4	1					1				
10	*	*	63	6	2	1					1			
11	*	*	*	13	3	1						1		
12	*	*	*	*	6	2	1						1	
13	*	*	*	*	*	3	1							1

A pattern explained ...

Let $i_X > \frac{n+3}{3}$.

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Mukai-Conjecture [Casagrande '05]

$$\rho_X(i_X - 1) \leq n$$

implies

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[Kleinschmidt '88] implies

- $X \cong \mathbb{P}^n$ $(i_X = n+1)$ or
- $X \cong \mathbb{P}^{\frac{n}{2}} \times \mathbb{P}^{\frac{n}{2}}$ $(i_X = \frac{n+2}{2}$, n even$)$ or
- $$X \cong \mathbb{P}_{\mathbb{P}^{n+1-r}}(O(a_1) \oplus \cdots \oplus O(a_t) \oplus O^{r-t}),$$

  for $a_1 + \cdots + a_t = n + 2 - 2r > 0$.

... and there's more to this than meets the eye

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\blackbox{Cayley polytope associated to}
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$(n + 1 - 2r)$-dimensional

generic Calabi-Yau complete intersection Y

of r hypersurfaces of degrees

$$a_1 + 1,\ \ldots,\ a_t + 1,\ 1,\ \ldots,\ 1$$

in $\mathbb{P}^{n+1-r}$.

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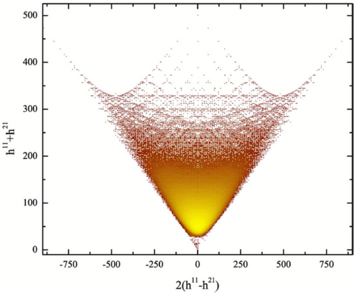
in $\mathbb{P}^{n+1-r}$.

Isomorphism type of Y only depends on $a_1, \ldots, a_t$, so only finitely many choices of such Y for fixed $\dim(Y)$.

THE TABLE of the 'Stringy Landscape'

Hodge numbers of (so far found) Calabi-Yau 3-folds

[Ashmore, He '11, Jurke '13]



More than 90% found as Calabi-Yau complete intersections in Gorenstein toric Fano varieties [Batyrev-Borisov construction].

Publications (with more references)

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- Coates, Corti, Galkin, Golyshev, Kasprzyk: *Fano Search Blog*, http://coates.ma.ic.ac.uk/fanosearch/
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- Øbro: *Classification of smooth Fano polytopes*, thesis, 2007
- Assarf, Joswig, Paffenholz: *Smooth Fano Polytopes With Many Vertices*, arXiv:1209.3186
- Nill, Paffenholz: *On the equality case in Ehrhart's volume conjecture*, arXiv:1205.1270
- Nill, Lorenz: *On smooth Gorenstein polytopes*, arXiv:1303.2138