Lattice width directions and Minkowski's 3^d-theorem

Dedicated to Hermann Minkowski (†12.Januar 1909)

Benjamin Nill
(joint work with Jan Draisma & Tyrrell McAllister)

Lattice width

Let K be a full-dimensional convex body in $\mathbb{R}^d$. Then

1. For $u \in (\mathbb{R}^d)^*$:

$$w(K,u) := (\max_{x \in K} \langle u, x \rangle) - (\min_{x \in K} \langle u, x \rangle)$$

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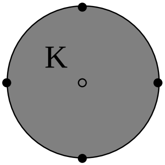
3. $$\mathcal{D}(K) := \{0 \neq u \in (\mathbb{Z}^d)^* \;:\; w(K,u) = w(u)\}$$

   lattice width directions of K.

Example



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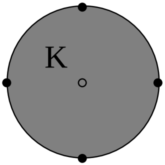

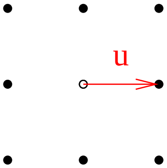

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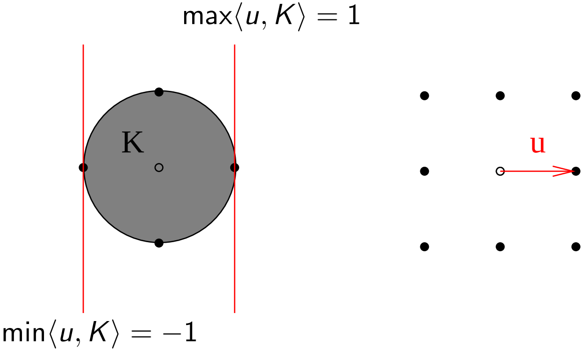
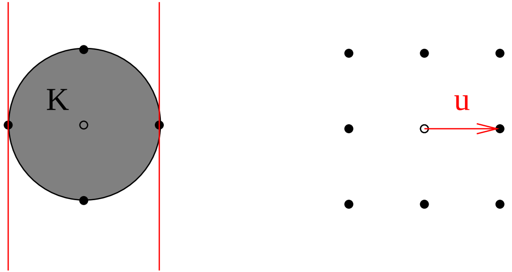


Example

$$w(K,u) = 2$$



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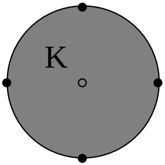

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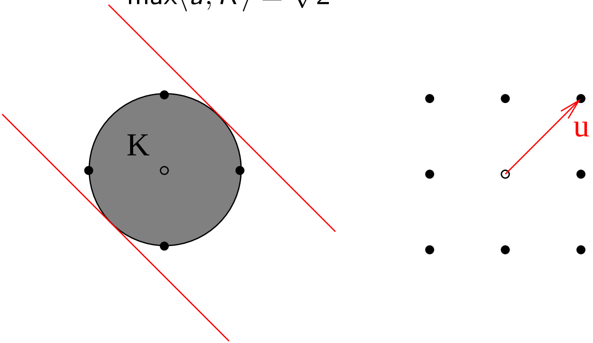
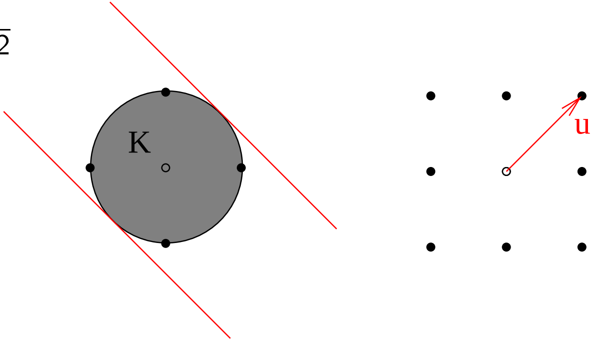
$\max\langle u, K\rangle = \sqrt{2}$



$\min\langle u, K\rangle = -\sqrt{2}$

Example

$w(K, u) = 2\sqrt{2}$



Example

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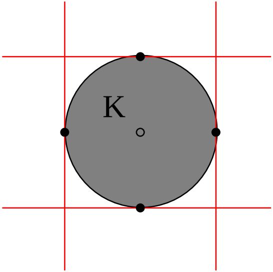
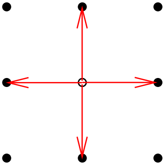


$|\mathcal{D}(K)| = 4$



Question 1

What is the maximal number of lattice width directions?

The convex hull of lattice width directions $\mathcal{D}(K)$

Proposition 1

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$$\exists\, \epsilon > 0 \quad (1 + \epsilon)y \in K'.$$

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Therefore,

$$w(K) \overset{(1)}{\geq} w(K, (1+\epsilon)y) = (1+\epsilon)w(K, y) > w(K, y),$$

a contradiction. □

Minkowski's 3^d-theorem

Theorem I (Minkowski 1910)

Let P be a centrally-symmetric convex set in $\mathbb{R}^d$ with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{0\}$.

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Theorem II

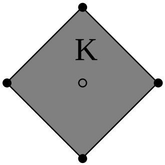
Let K be a d-dimensional convex body in $\mathbb{R}^d$. Then

$$|\mathcal{D}(K)| \leq 3^d - 1.$$

Question 2

What are the equality cases in Theorems I & II?

Example for equality in Theorem II



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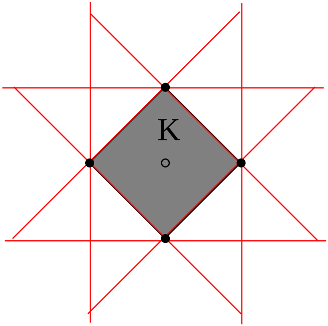
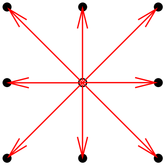
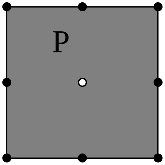




$$|\mathcal{D}(K)| = 3^2 - 1$$

Example for equality in Theorem I

$$P = [-1, 1]^2$$



$$|P \cap \mathbb{Z}^2| = 3^2$$

Our main contribution

Theorem I

Let P be a centrally-symmetric convex set in $\mathbb{R}^d$ with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{0\}$. Then

$$|P \cap \mathbb{Z}^d| \;\leq\; 3^d,$$

Our main contribution

Theorem I

Let P be a centrally-symmetric convex set in $\mathbb{R}^d$ with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{0\}$. Then

$$|P \cap \mathbb{Z}^d| \;\leq\; 3^d,$$

where equality holds if and only if

$$P \cong [-1, 1]^d$$

up to unimodular equivalence (action of $\mathrm{GL}_d(\mathbb{Z})$).

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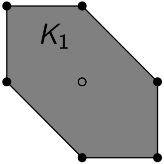
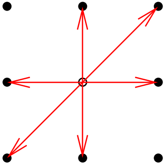
$$K = \mathrm{conv}(\pm e_1, \ldots, \pm e_d) \quad \text{for a lattice basis } e_1, \ldots, e_d$$

up to translations and scalar multiples.

Why the implication (Theorem I $\Rightarrow$ Theorem II) is non-trivial



$\mathcal{D}(K_1)$



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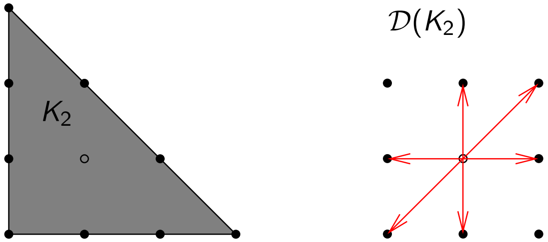

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Let P be a centrally-symmetric convex set in $\mathbb{R}^d$ with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{0\}$. Then

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Why Theorem I is non-trivial

A strengthening of Theorem I ?

Let P be a centrally-symmetric convex set in $\mathbb{R}^d$ with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{0\}$. Then

$$P \overset{?}{\hookrightarrow} [-1, 1]^d$$

Why Theorem I is non-trivial

A strengthening of Theorem I is wrong!

$\exists$ P a centrally-symmetric convex set in $\mathbb{R}^d$ with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{0\}$ ($d \geq 4$) with

$$P \not\to [-1,1]^d$$

Sketch of a Metric Geometry proof of Theorem I

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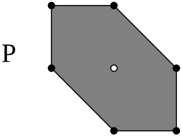

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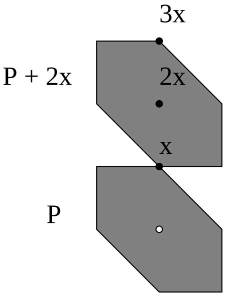

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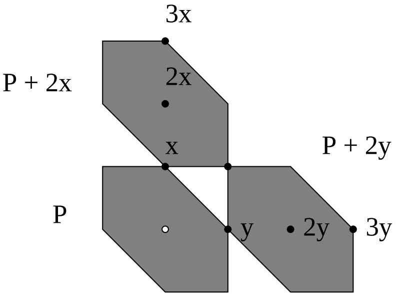

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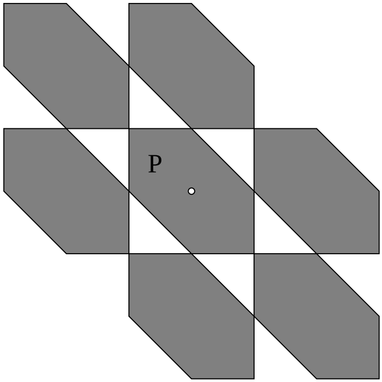

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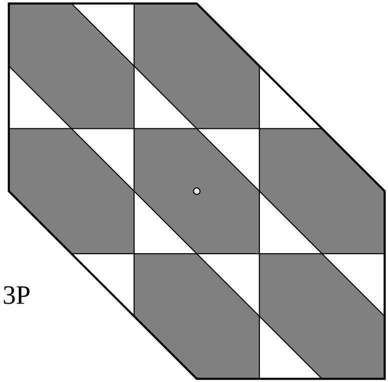

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3. $$|P \cap \mathbb{Z}^d| \operatorname{vol}(P) \leq \operatorname{vol}(3P) = 3^d \operatorname{vol}(P).$$

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Let P be a centrally-symmetric convex set in $\mathbb{R}^d$ with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{0\}$ and $|P \cap \mathbb{Z}^d| = 3^d$.

1. Exists non-overlapping partition into 3^d translations.
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3. $$|P \cap \mathbb{Z}^d| \operatorname{vol}(P) = \operatorname{vol}(3P) = 3^d \operatorname{vol}(P).$$
4. Theorem (Groemer 1961) $\Rightarrow$ P is a parallelepiped.

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4. Theorem (Groemer 1961) $\Rightarrow$ P is a parallelepiped.
5. $\rightsquigarrow P \cong [-1,1]^d$ □

Sketch of a Geometry of Numbers proof of Theorem I

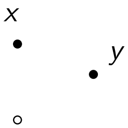
Let P be a centrally-symmetric convex set in $\mathbb{R}^d$ with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{0\}$.

Proposition 2 (Minkowski 1910)

$P \cap \mathbb{Z}^d \ \to \ (\mathbb{Z}/3\mathbb{Z})^d$ via taking congruences mod 3 is *injective*.

Proof of Proposition 2

Let $x, y \in P \cap \mathbb{Z}^d$ such that $x - y$ is divisible by 3.



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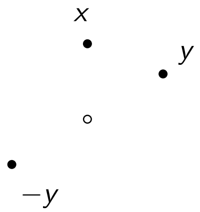

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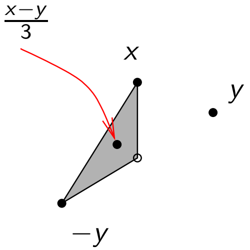

a contradiction. □

Sketch of a Geometry of Numbers proof of Theorem I

Let P be a centrally-symmetric convex set in $\mathbb{R}^d$ with $\mathrm{int}(P) \cap \mathbb{Z}^d = \{0\}$ and $|P \cap \mathbb{Z}^d| = 3^d$.

Corollary 1

$P \cap \mathbb{Z}^d \;\to\; (\mathbb{Z}/3\mathbb{Z})^d$ via taking congruences mod 3 is *bijective*.

Sketch of a Geometry of Numbers proof of Theorem I

The Key-Observation

Given different x and y in $P \cap \mathbb{Z}^d$

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Proof:
By Corollary 1 take z as unique point in $P \cap \mathbb{Z}^d$ such that images of x, y, z lie on a line in $(\mathbb{Z}/3\mathbb{Z})^d$. □

Sketch of a Geometry of Numbers proof of Theorem I

Successive application of Key-Observation yields:

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Proposition 3

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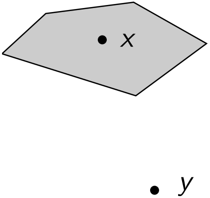
Proposition 4

Given any $y \in P \cap \mathbb{Z}^d$, $y \notin F$, then

$$x + y \in P \cap \mathbb{Z}^d.$$

Proof of Proposition 4



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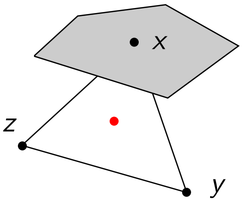

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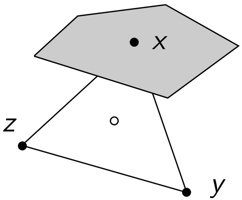

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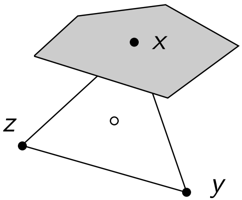
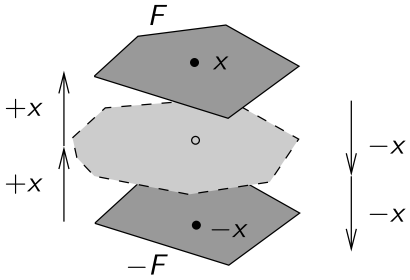


$$\Rightarrow x + y = -z \in P \quad \square$$

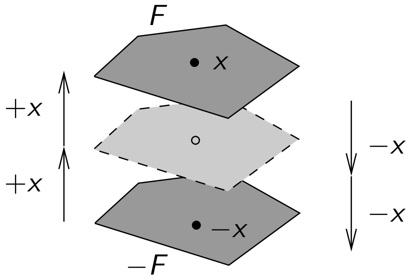
Sketch of a Geometry of Numbers proof of Theorem I

Proposition 4 yields bijection of lattice points.



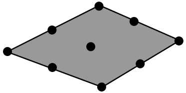
Sketch of a Geometry of Numbers proof of Theorem I

P is a prism over facet F.



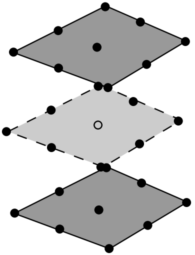
Sketch of a Geometry of Numbers proof of Theorem I

Induction hypothesis for F yields $F \cong [-1,1]^{d-1}$.



Sketch of a Geometry of Numbers proof of Theorem I

$\rightsquigarrow$ $P \cong [-1,1]^d$.



$\square$