Polyhedral Adjunction Theory

Benjamin Nill
with Sandra Di Rocco, Christian Haase, and Andreas Paffenholz

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I. Classical Adjunction Theory

Polarized varieties

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(X, L) where

- X is a normal projective variety of dimension n
- L ample line bundle on X

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Adjunction theory $=$ *study of adjoint bundles* $L + c\ K_X$

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Minimal assumption

X is $\mathbb{Q}$-**Gorenstein**, i.e., K_X is $\mathbb{Q}$-Cartier.

Two algebro-geometric invariants

The unnormalized spectral value μ

$$\mu = \sup\{c \in \mathbb{R} \ : \ L + c\ K_X \text{ big}\}^{-1}$$

The nef-value τ

$$\tau = \sup\{c \in \mathbb{R} \ : \ L + c\ K_X \text{ nef}\}^{-1}$$

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$$\mu \leqslant \tau$$

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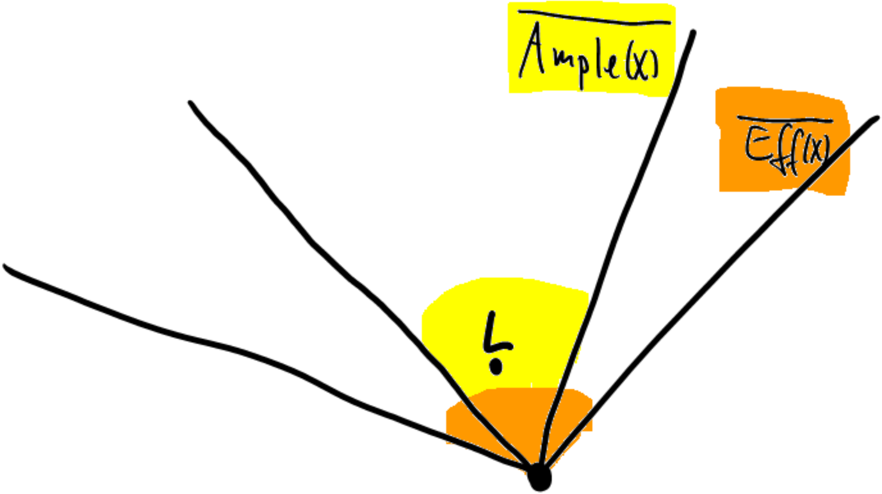

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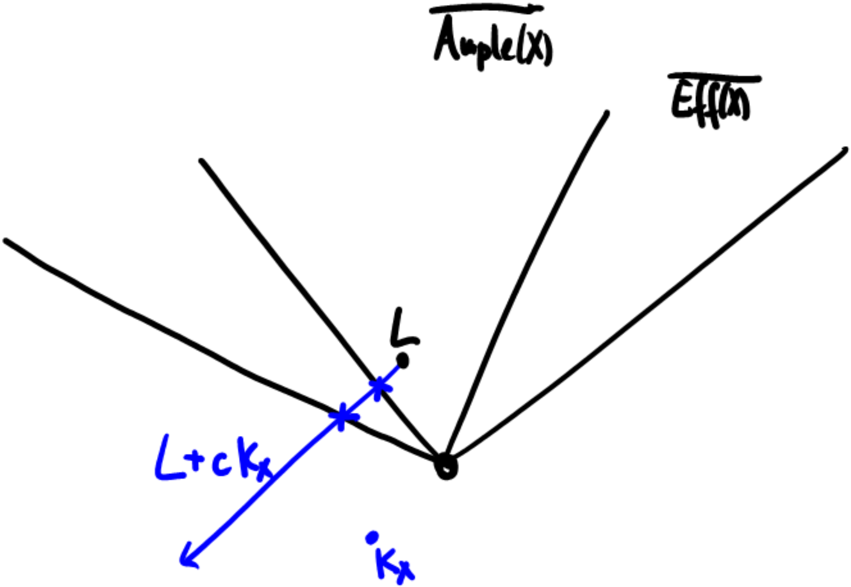

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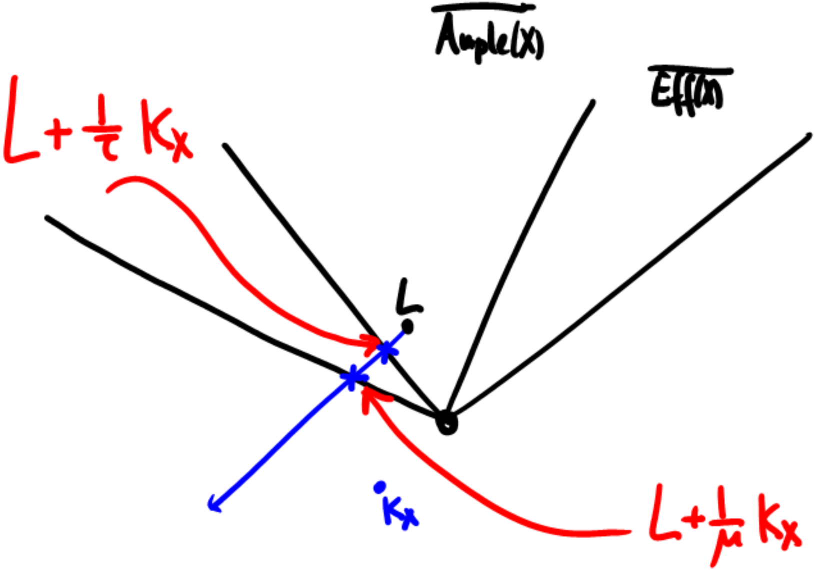

Results and conjectures

Most work on polarized *manifolds*:

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Conjectures on polarized manifolds

- $\mathbb{Q}$**-normality conjecture:**

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Conjectures on polarized manifolds

- **$\mathbb{Q}$-normality conjecture:**

$$\mu > \frac{n+1}{2} \quad \Longrightarrow \quad \mu = \tau$$

- **Spectrum conjecture:**
  For $\varepsilon > 0$, there are only finitely many $\mu > \varepsilon$.

II. Polyhedral Adjunction Theory

The adjoint polytope

Study initiated by [Dickenstein, Di Rocco, Piene '09].

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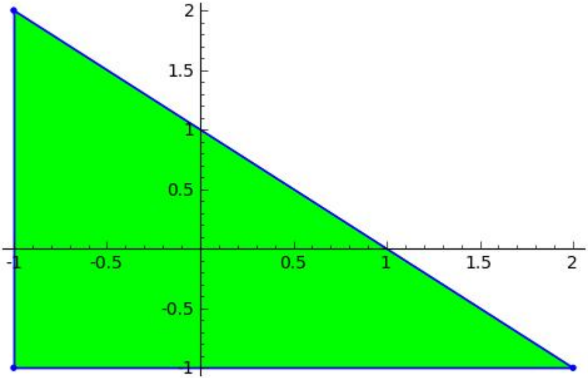
Polyhedral adjunction: "Move facets simultaneously inwards"

The adjoint polytope

$$P = P^{(0)}$$



The adjoint polytope

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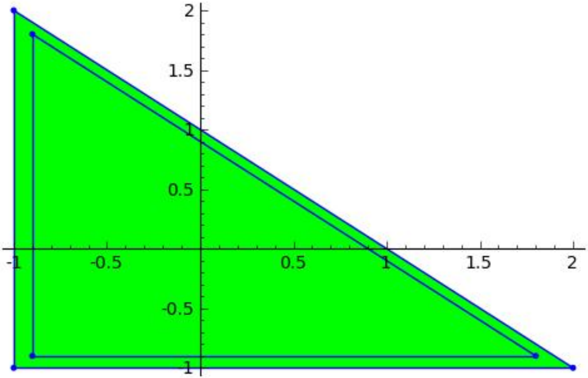
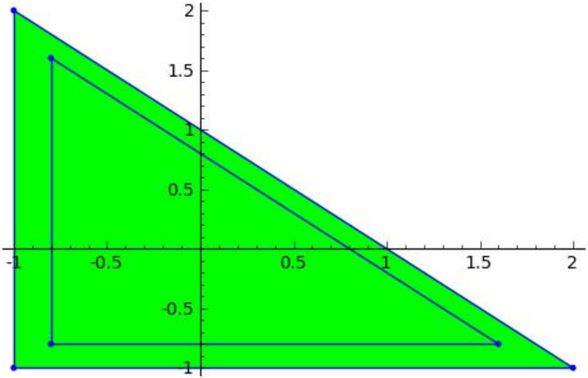


The adjoint polytope

$P^{(0.2)}$



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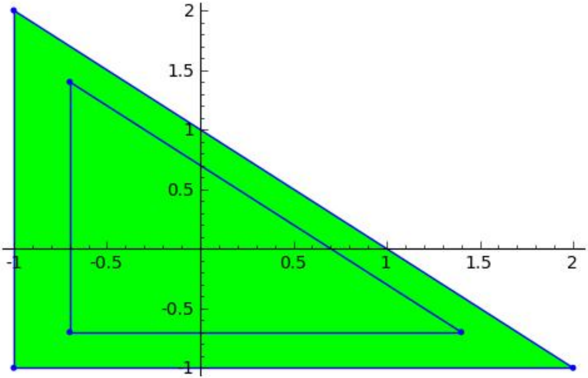

The adjoint polytope

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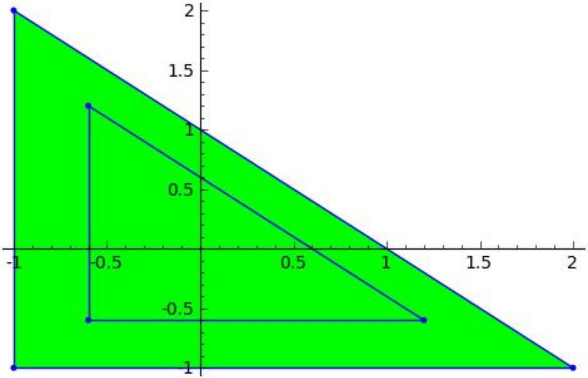

The adjoint polytope

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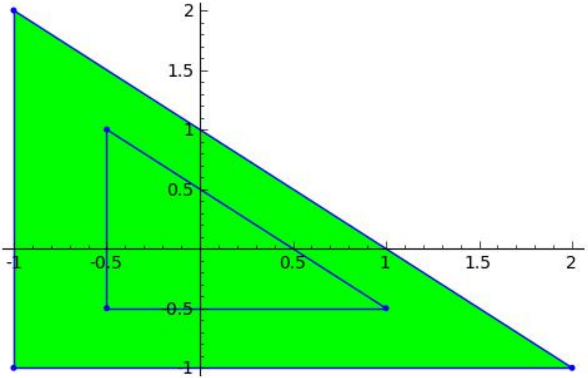

The adjoint polytope

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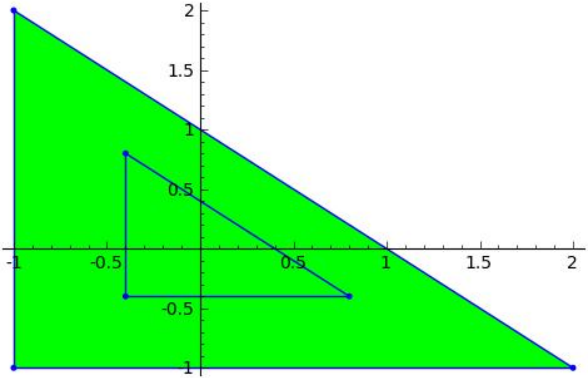

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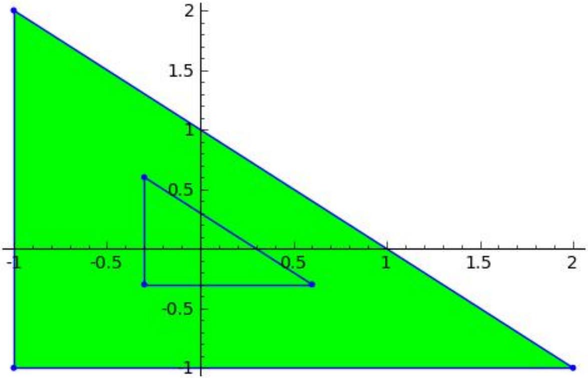
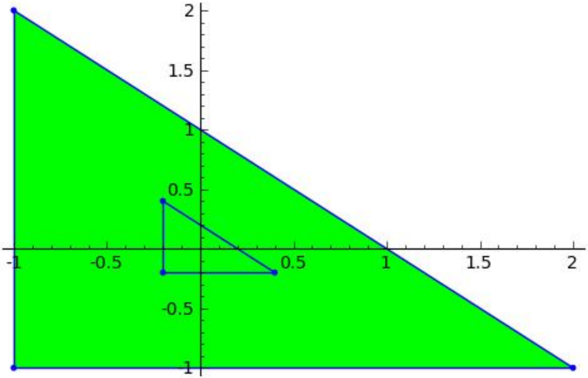


The adjoint polytope

$P^{(0.8)}$



The adjoint polytope

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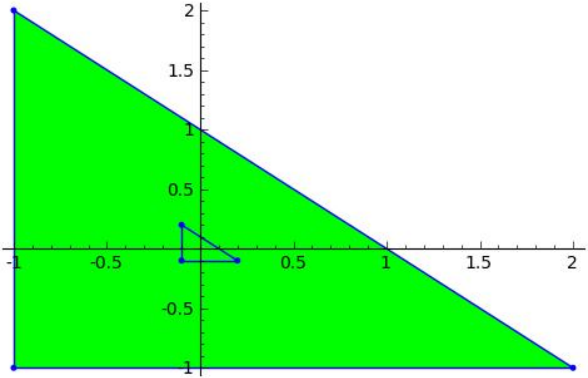
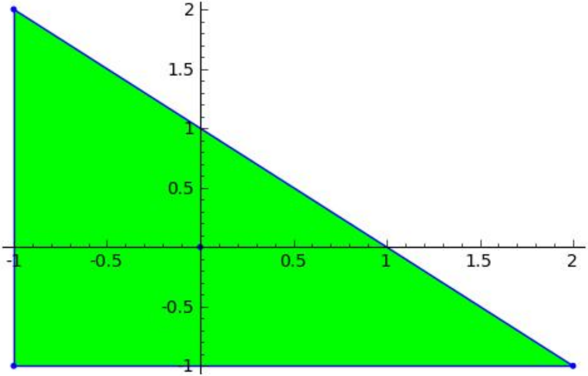


The adjoint polytope

$P^{(1)}$ point



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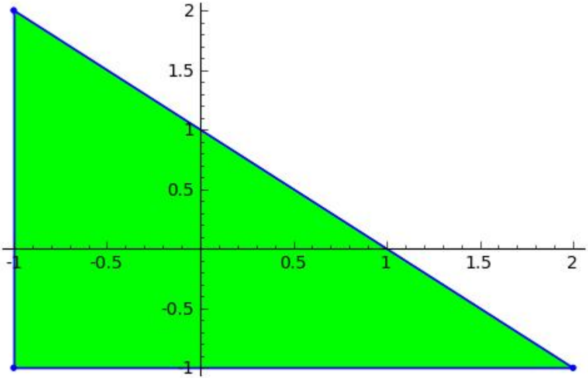
$$P^{(c)} = \varnothing \text{ for } c > 1$$



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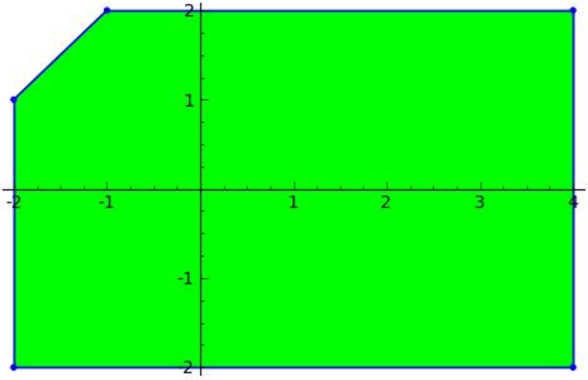

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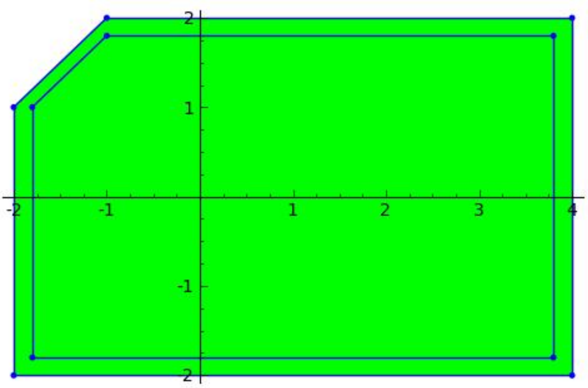

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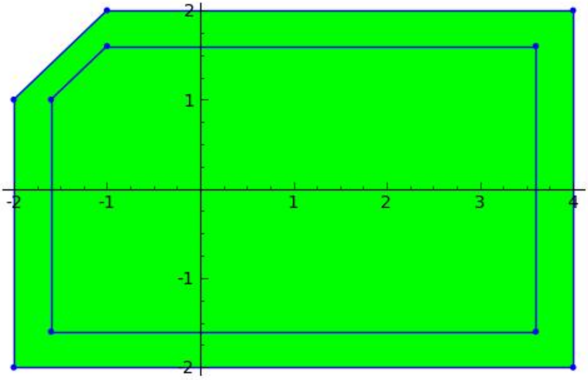

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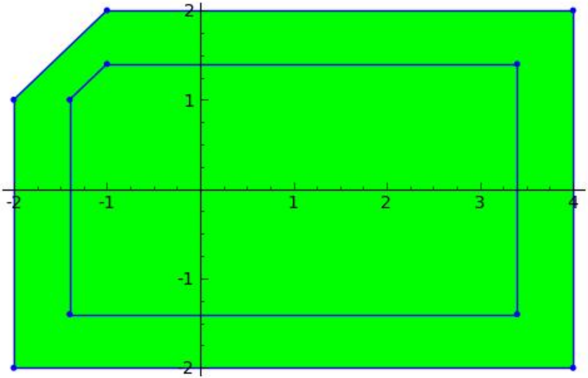

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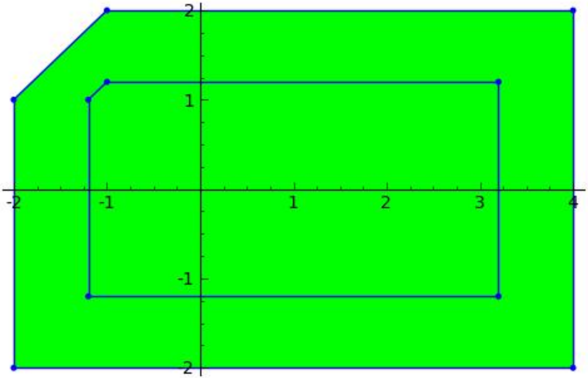

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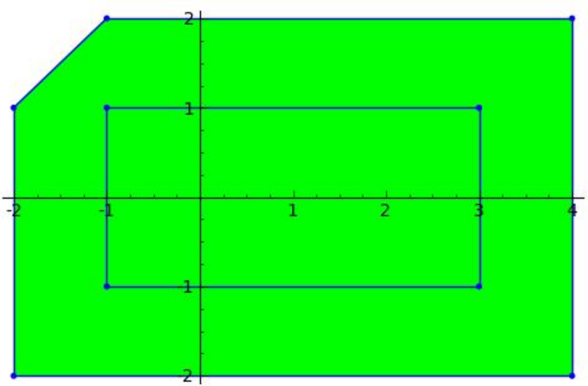
$P^{(1)}$ combinatorics changes!



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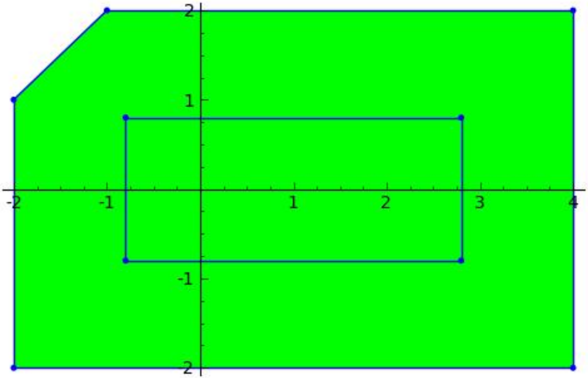

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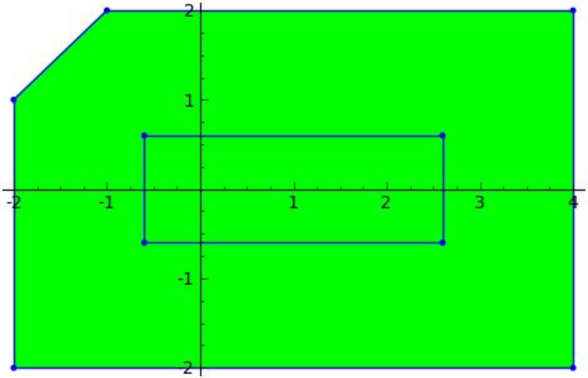

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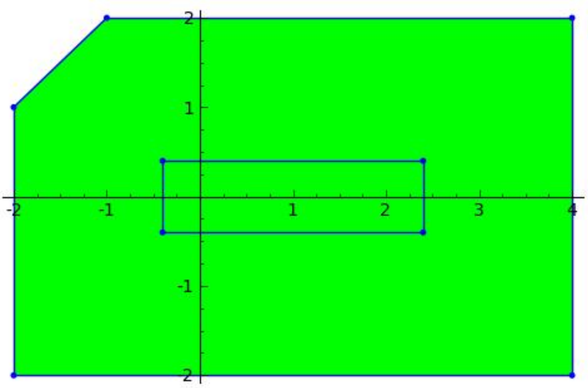

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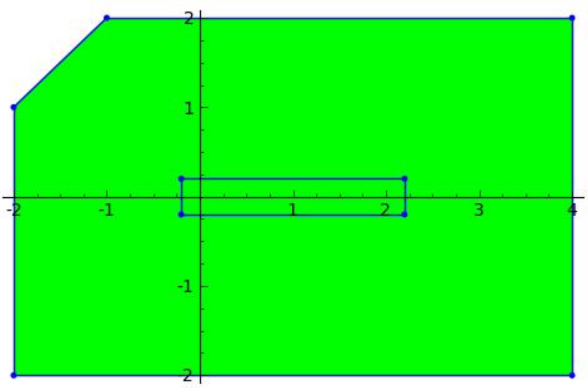

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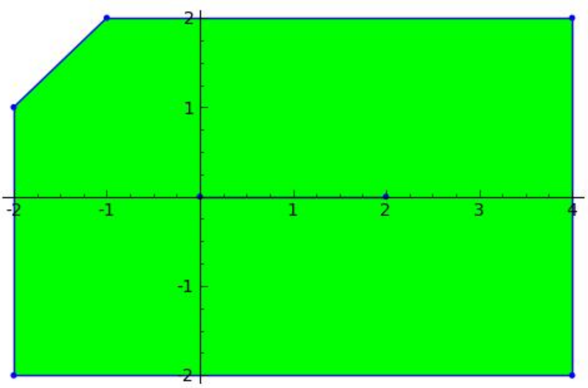
$P^{(2)}$ interval



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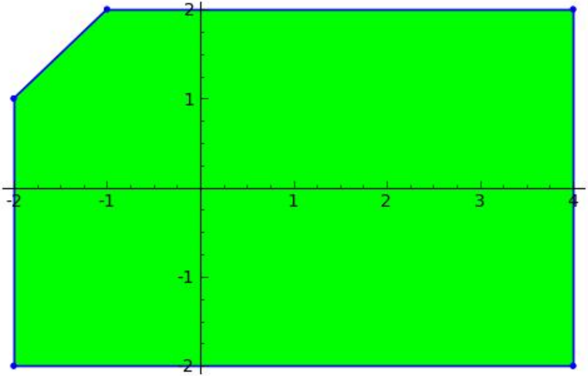

μ, τ for polarized toric varieties

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Definition makes sense for **general** lattice polytopes!

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- $\tau_P := \left(\sup\{c > 0 \;:\; P^{(c)} \text{ combinatorially equal to } P\}\right)^{-1}$, with $(\sup\{\})^{-1} := \infty$.

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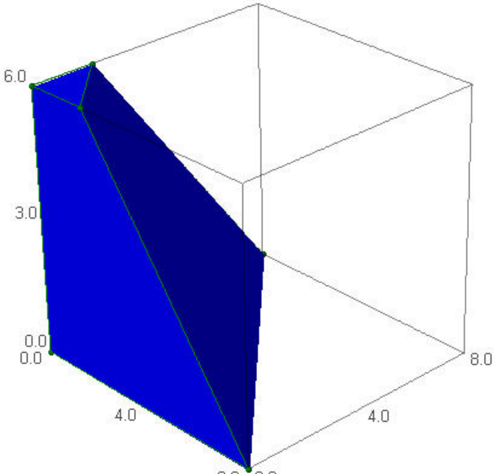
$$P = P^{(0)}$$



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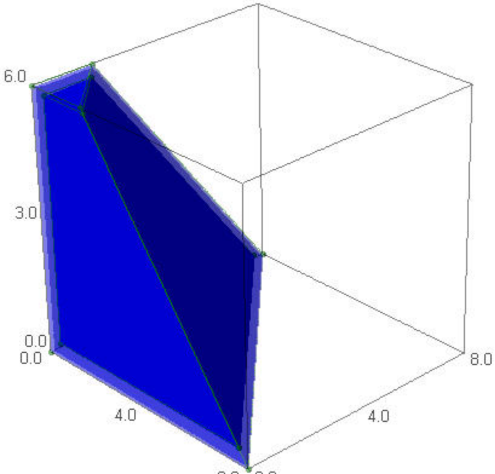
$P^{(0.2)}$



Two polyhedral invariants

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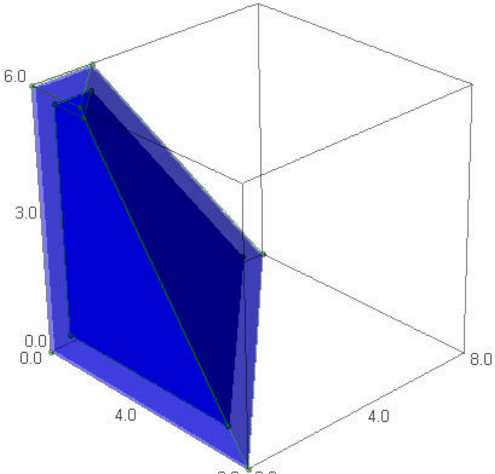
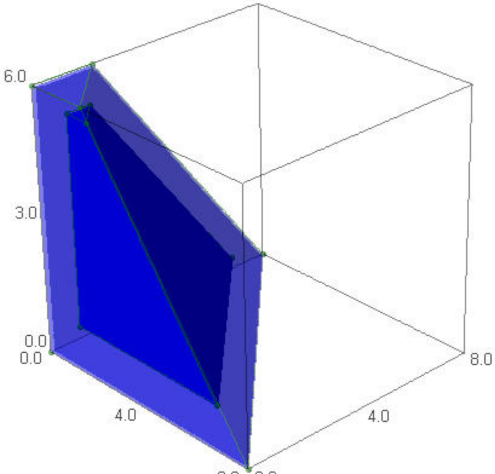


Two polyhedral invariants

$P^{(0.6)}$



Two polyhedral invariants

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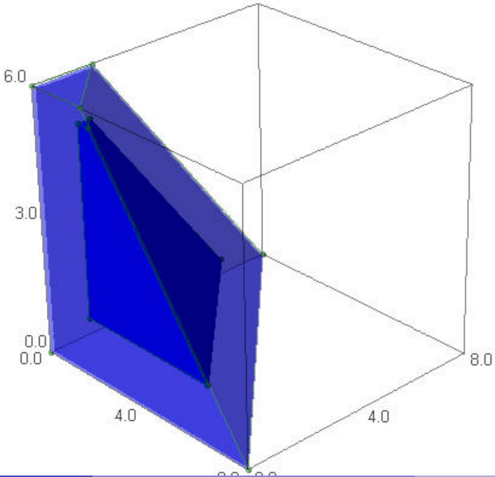
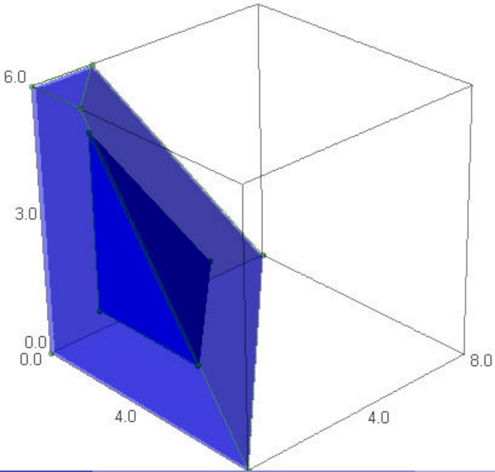


Two polyhedral invariants

$P^{(1)}$ combinatorics changes $\implies \tau_P = 1^{-1} = 1$



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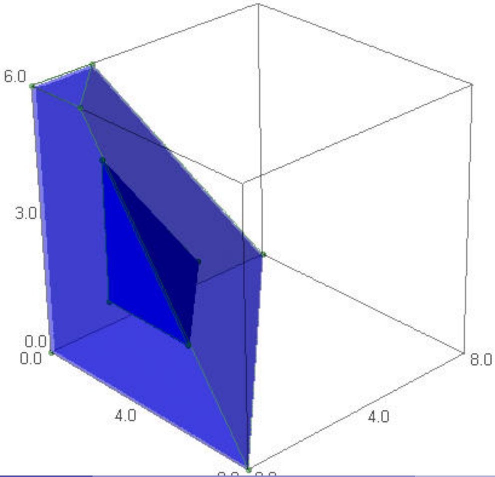

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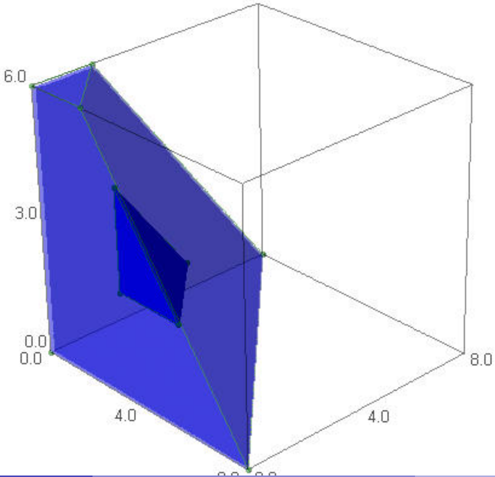

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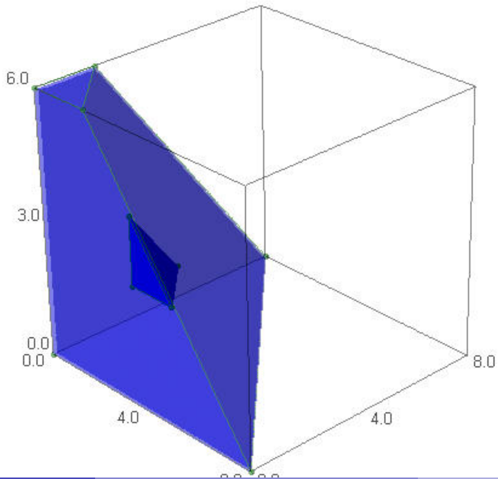

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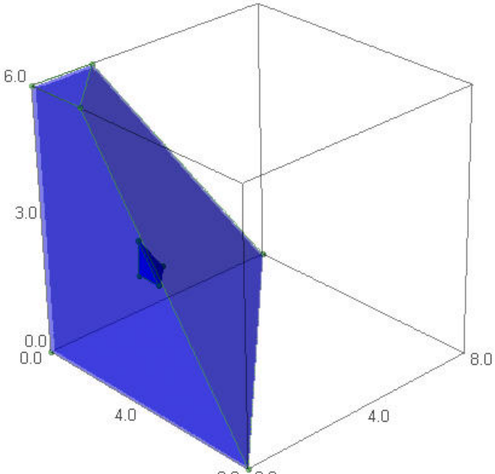
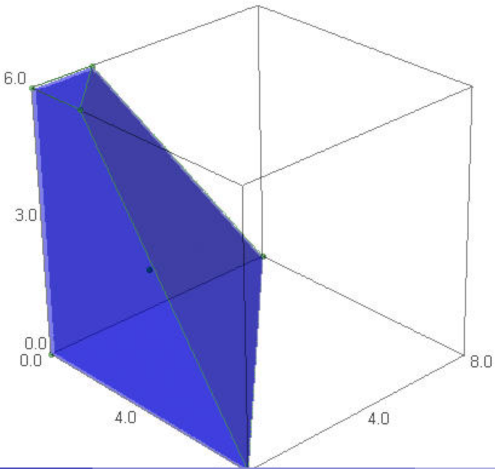


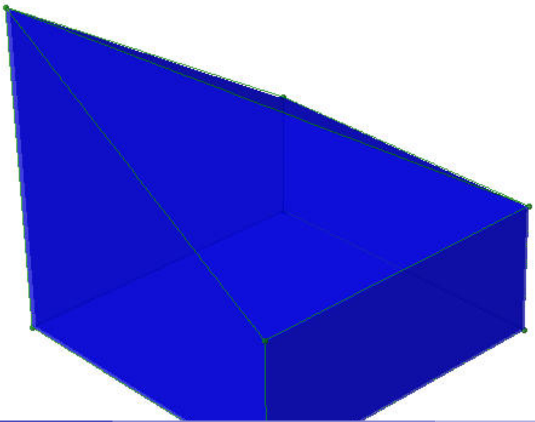
Two polyhedral invariants

$$P^{(2)} \text{ point } \implies \mu_P = 2^{-1} = \frac{1}{2}$$



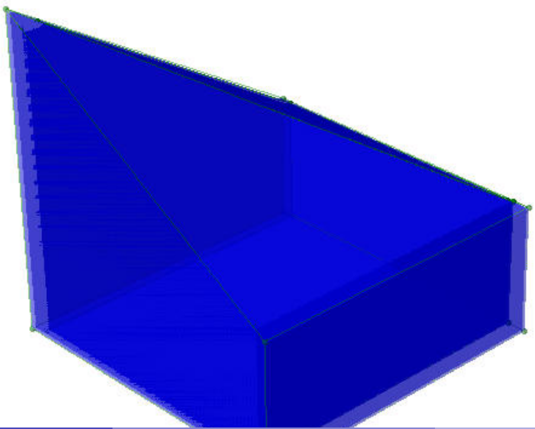
Two polyhedral invariants

$P = P^{(0)}$ combinatorics changes immediately $\implies \tau = \infty$



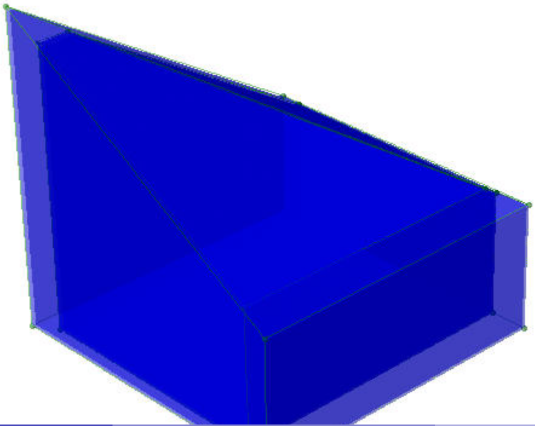
Two polyhedral invariants

$P^{(0.05)}$



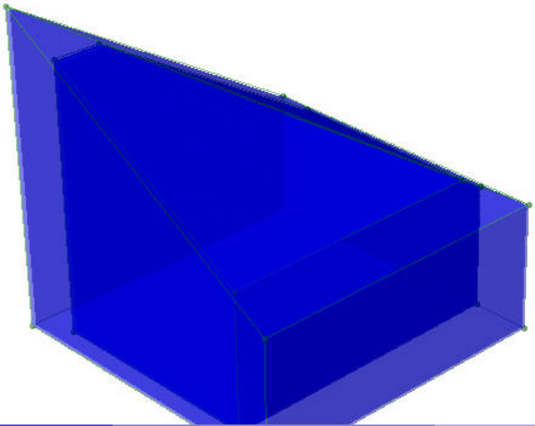
Two polyhedral invariants

$P^{(0.1)}$



Two polyhedral invariants

$P^{(0.15)}$



Two polyhedral invariants

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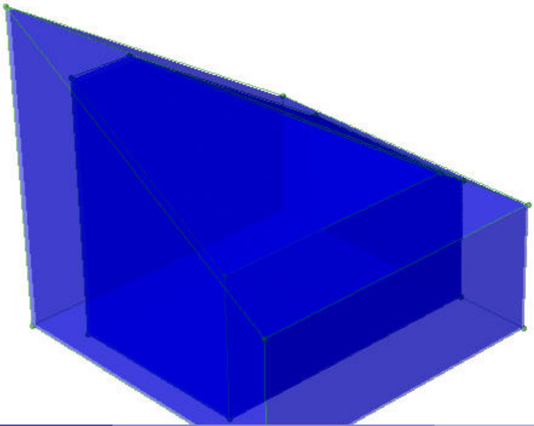
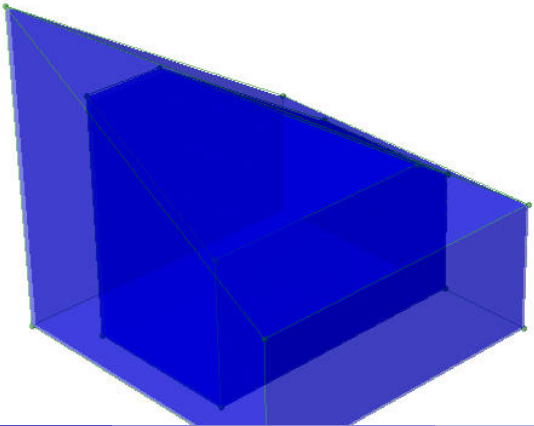


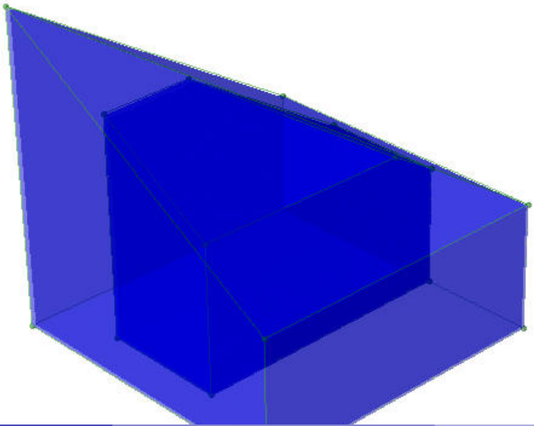
Two polyhedral invariants

$P^{(0.25)}$



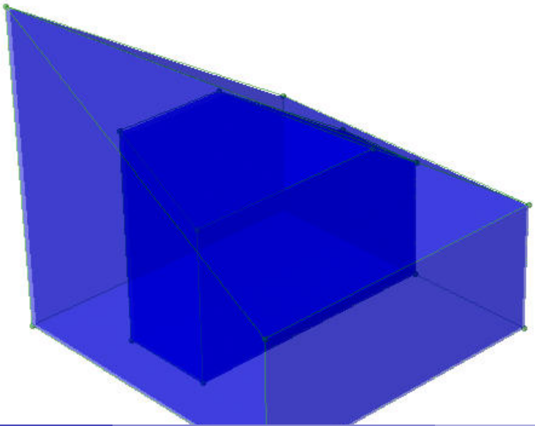
Two polyhedral invariants

$P^{(0.3)}$



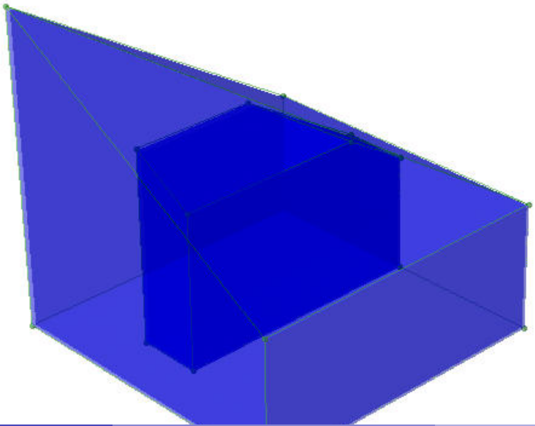
Two polyhedral invariants

$P^{(0.35)}$



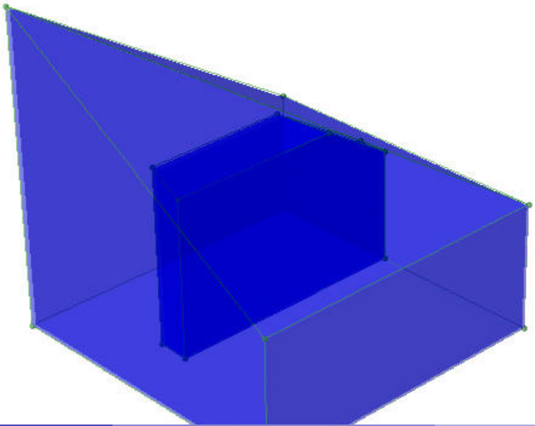
Two polyhedral invariants

$P^{(0.4)}$



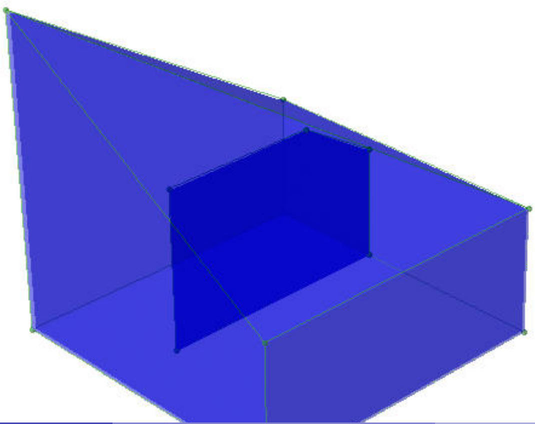
Two polyhedral invariants

$P^{(0.45)}$



Two polyhedral invariants

$$P^{(0.5)} \text{ polygon } \implies \mu_P = 0.5^{-1} = 2$$



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Polyhedral adjunction theory

$\supsetneq$

Adjunction theory of polarized toric varieties

III. The Main Theorem

Large μ_P implies P flat

Theorem [Di Rocco, Haase, N., Paffenholz '11]

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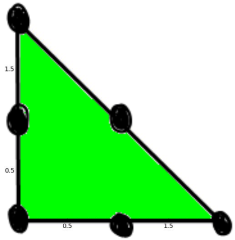

Relation to Ehrhart theory

[Dickenstein, Di Rocco, Piene '09]: μ_P is called $\mathbb{Q}$-**codegree** of P.

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Proof follows from

$$\mathrm{int}(kP) \cap \mathbb{Z}^n \subset (kP)^{(1)} = kP^{(\frac{1}{k})}.$$

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Philosophy: $\mathbb{Q}$-codegree is more tractable than codegree!

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$$\mu_P > \frac{n+2}{2} \implies \mu_P = \tau_P.$$

This is (nearly) the $\mathbb{Q}$-normality conjecture!

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Do methods also help to attack the Spectrum Conjecture ?