Recent developments in the geometry of numbers of lattice polytopes

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Outline of talk

Lattice polytope $=$ d-polytope in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$
Isomorphisms $=$ affine lattice automorphisms

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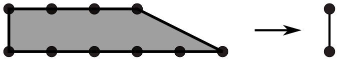
Isomorphisms = affine lattice automorphisms

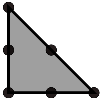
I. Lattice polytopes with no interior lattice points
II. Lattice polytopes with $k \geq 1$ interior lattice points
III. Unifying approach via the notion of (co)degree

I. NO interior lattice points - Dimension two

One **infinite** class



and one **exceptional** triangle S



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All but finitely many hollow lattice d-polytopes admit a lattice projection onto a hollow lattice polytope of smaller dimension.

Any hollow lattice polytope

- **either** projects onto hollow lattice polytope
- **or** is contained in one of finitely many inclusion-maximal hollow lattice polytopes

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Proposition [N., Ziegler '11]

Any inclusion-maximal hollow lattice polytope of dimension $d = 3$ has volume $\leq 4,106$.

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Expected maximal volume: 6.

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Theorem [Andersen, Averkov, Wagner, Weismantel '09, '10]

There are **12** inclusion-maximal hollow lattice polytope of dimension $d = 3$ where any facet has an interior lattice point.

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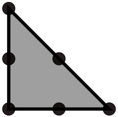
$d = 2$: Yes.



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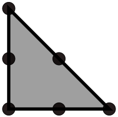

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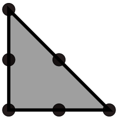

$d \geq 4$: No! [N., Ziegler '11]

$d = 3$: Open.

II. Lattice d-polytopes P with $k \geq 1$ interior lattice points

Theorem [Hensley '83; Lagarias&Ziegler '91; Pikhurko '01]

$$\mathrm{vol}(P) \leq (8d)^d 15^{d2^{2d+1}} k$$

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Proposition [Zaks, Perles, Wills '82]

$$\max_P \mathrm{vol}(P) \geq \mathrm{vol}(S_{d,k}) \geq 2^{2^{d-1}} (k+1)/d!$$

where

$$S_{d,k} := \mathrm{conv}(0, s_1 e_1, \ldots, s_{d-1} e_{d-1}, (k+1)(s_d - 1) e_d)$$

and

$$s_1 = 2, s_2 = 3, s_3 = 7, s_4 = 43, \ldots, \quad s_k := s_1 \cdots s_{k-1} + 1$$

II. Lattice d-polytopes P with $k \geq 1$ interior lattice points

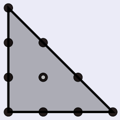
[Pikhurko '01]:

"... we have the correct type of dependence of d,k ... but the gap between the known bounds is huge. The ultimate aim would be to find exact values, which is probably not hopeless, because the above constructions, believed to be extremal, are rather simple."

II. Lattice d-polytopes P with ONE interior lattice point

Maximal volume of lattice d-polytopes P with ONE interior lattice point

- $d = 2$: 4.5,



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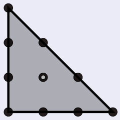

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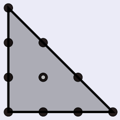

- $d = 3$ [Kasprzyk '08]: 12, attained by two simplices
- $d \geq 4$: Conjecture [Zaks, Perles, Wills '82]: $\mathrm{vol}(S_{d,1}) = 2(s_d - 1)^2/d!$, only attained by $S_{d,1}$.

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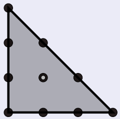

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Theorem [Nill '07]

Conjecture holds for reflexive simplices.

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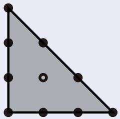

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Theorem [Averkov '11]

Upper bound in conjecture holds for lattice simplices.

Uniqueness of equality case still open!

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Let P be lattice simplex with 0 only interior lattice point.
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Lemma [N.'07]

Let $x_0 \geq \cdots \geq x_d > 0$ such that $x_0 + \cdots + x_d = 1$ and

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$$d! \operatorname{vol}(P) \leq \frac{1}{\beta_0 \cdots \beta_{d-1}} \leq 2(s_d - 1)^2 \quad \square$$

II. Faces of lattice simplices with ONE interior lattice point

Let P be lattice simplex with 0 only interior lattice point.

Theorem [Averkov, N., Krümpelmann '12]

Let $d \geq 4$, F be a face of P dimension ℓ. Then

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Theorem [Averkov, N., Krümpelmann '12]

For $d \geq 4$, if P has edge with maximal number of lattice points $2(s_d - 1) + 1$, then $P \cong S_{d,1}$.

III. Unifying - The degree

$P \subset \mathbb{R}^d$ lattice polytope of dimension d.

The **codegree**:

$$\operatorname{codeg}(P) := \min\{k \in \mathbb{N}_{\geq 1} \;:\; \operatorname{int}(kP) \cap \mathbb{Z}^d \neq \emptyset\}$$

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- $\deg(P) = d \quad \Leftrightarrow \quad \operatorname{codeg}(P) = 1 \quad \Leftrightarrow \quad \operatorname{int}(P) \cap \mathbb{Z}^d \neq \emptyset$
- $\deg(P) = 0 \quad \Leftrightarrow \quad \operatorname{codeg}(P) = d+1 \quad \Leftrightarrow \quad P \cong \operatorname{conv}(0, e_1, \ldots, e_n)$

III. Unifying - Volume bounds (Section II)

Ehrhart theory: $\deg(P) = \max(i \ : \ h_i^* \neq 0)$ for

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Theorem [Haase, N., Payne '09]

Upper bound on $d!\mathrm{vol}(P)$ for lattice d-polytopes P in terms of $\deg(P)$ and $h^*_{\deg(P)}$.

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Corollary to main theorem on hollow lattice polytopes

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Daring conjecture [N., Padrol '12]

There exists **finite list** $\mathcal{P}_s$ of lattice polytopes of degree s and dimension $\leq 2s$ s.t. if a lattice polytope has degree s then

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- True for $s \leq 1$ [Batyrev, N. '07]
- If true: $d > 2s \implies$ lattice width one
  (conjectured; holds for $d > 20s^2$ [Haase, N., Payne '09])