A combinatorial generalization of the degree of lattice polytopes

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joint work with Arnau Padrol

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I. The degree of lattice polytopes - Ehrhart theory

Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytope
(i.e., the vertex set $\mathcal{V}(P) \subset \mathbb{Z}^d$).

Main problem: Classify *Ehrhart polynomials* of lattice polytopes:

$$k \mapsto \left|kP \cap \mathbb{Z}^d\right|$$

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Theorem [Stanley '80]

$$\sum_{k \geq 0} |\, kP \cap \mathbb{Z}^d \,|\; t^k = \frac{h^*_P(t)}{(1-t)^{d+1}},$$

where h^*_P is a polynomial of degree $\leq d$ with coefficients in $\mathbb{N}$.

I. The degree of lattice polytopes - Definition

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$$\sum_{k \geq 0} |\, kP \cap \mathbb{Z}^d \,|\; t^k = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

where h_P^* is a polynomial of degree $\leq d$ with coefficients in $\mathbb{N}$.

Definition: The degree of h_P^* is called the **degree** of P.

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Main problem (dimension-free version):
Classify *h^*-polynomials of lattice polytopes of given degree*.

$\rightsquigarrow$ Degree fixed, dimension arbitrary!

I. The degree of lattice polytopes - Properties

Basic properties of the degree

- $\deg(P) = 0 \Leftrightarrow h^*_P = 1 \Leftrightarrow P \cong \Delta_d = \mathrm{conv}(0, e_1, \ldots, e_d)$

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- $\deg(P)$ does not change by taking **lattice pyramids**.
- degree is monotone with respect to inclusion.

I. The degree of lattice polytopes - Meaning

Definition: The **codegree** of P:

$$\operatorname{codeg}(P) := \min\{k \in \mathbb{Z}_{>0} \,:\, \operatorname{int}(kP) \cap \mathbb{Z}^d \neq \emptyset\}$$

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$$\operatorname{codeg}(P) = 1 \quad \Leftrightarrow \quad \deg(P) = d \quad \Leftrightarrow \quad \operatorname{int}(P) \cap \mathbb{Z}^d \neq \emptyset$$

$$\operatorname{codeg}(P) = d + 1 \quad \Leftrightarrow \quad \deg(P) = 0 \quad \Leftrightarrow \quad P \cong \Delta_d$$

I. The degree of lattice polytopes - Degree 1

Examples of high dimension but degree 1 ?

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Examples of high dimension but degree 1 ?

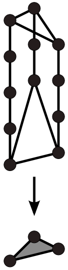
Definition/Lemma (Batyrev, N. '07)

P is a **Lawrence prism**, if exists

$$\phi : \mathbb{Z}^d \twoheadrightarrow \mathbb{Z}^{d-1}, \quad \phi(P) = \Delta_{d-1}.$$

Then

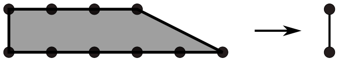
$$\deg(P) \leq 1 \;\Leftrightarrow\; \mathrm{codeg}(P) \geq d.$$



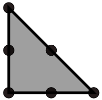
I. The degree of lattice polytopes - Degree 1

Example: Dimension $d = 2$ (no interior lattice points)

Lawrence prisms



and one **exceptional triangle** S



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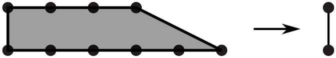

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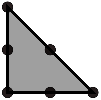

Theorem [Batyrev, N. '07]

$\deg(P) \leq 1$ if and only if P is

- Lawrence prism or
- $(d-2)$-fold lattice pyramid over S.

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Definition/Lemma [Batyrev, N. '07]

P is a **Cayley polytope** of lattice polytopes in $\mathbb{R}^s$, if exists

$$\phi : \mathbb{Z}^d \twoheadrightarrow \mathbb{Z}^{d-s}, \quad \phi(P) = \Delta_{d-s}.$$

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$$\deg(P) \leq s \;\Leftrightarrow\; \operatorname{codeg}(P) \geq d + 1 - s.$$

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Observe: Let $\deg(P) \leq 1$. Then $d \geq 3 \;\Rightarrow\; P$ Cayley polytope

I. The degree of lattice polytopes - Structure result

Theorem [Haase, N., Payne '09]

If $d > 20(\deg(P))^2$, then

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Theorem [Haase, N., Payne '09]

There exists a function f such that

$$\mathrm{Vol}_{\mathbb{Z}}(P) = 1 + h_1^* + \cdots + h^*_{\deg(P)} \leq f(\deg(P), h^*_{\deg(P)}).$$

I. The degree of lattice polytopes - Cayley conjecture

Cayley conjecture

If

$$d > 2\mathrm{deg}(P) \quad \Leftrightarrow \quad \mathrm{codeg}(P) > \frac{d+2}{2},$$

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If $\mathrm{codeg}(P) > \frac{d+r}{2}$, then P is a Cayley polytope of r lattice polytopes.

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Refined conjecture holds, if P is **smooth** [Dickenstein, N. '10].

II. The combinatorial degree of polytopes - Observation

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P is k-**neighborly**, if any subset of k vertices is vertex set of a face.

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Recall:
P is k-**neighborly**, if any subset of k vertices is vertex set of a face.

Goal: *Study $(d-s)$-almost neighborly polytopes for s fixed and d large!*

II. The combinatorial degree of polytopes - Definition

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The **combinatorial codegree** of d-dimensional polytope P is

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$$\deg_c(P) := d + 1 - \mathrm{codeg}_c(P)$$

which is the maximal *codimension* of an interior face of P.

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If P is a lattice polytope, then

$$\deg_c(P) \leq \deg(P).$$

II. The combinatorial degree of polytopes - Properties

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- $\deg_c(P)$ does not change by taking **pyramids**.
- Comb. degree is monotone with respect to inclusion (of vertex sets).

II. The combinatorial degree of polytopes - $\deg_c(P) = 1$

Theorem [Batyrev, N. '07]

P d-dimensional **lattice polytope** which is not a lattice pyramid.
Then $\deg(P) \leq 1$ if and only if P is

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Theorem [N., Padrol '12]

P d-dimensional **polytope** which is not a pyramid.
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- prism over $(d-1)$-simplex or
- polygon.

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Def. P is an **affine Cayley polytope** of r polytopes, if exists affine projection mapping $\mathcal{V}(P)$ onto vertex set of $(r-1)$-simplex.

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$$\Leftrightarrow$$

There exists **partition** $\mathcal{V}(P) = A_1 \uplus \cdots \uplus A_r$ such that

$$\forall\, I \subsetneq \{1, \ldots, r\} : \qquad \operatorname{conv}\left(\bigcup_{i \in I} A_i\right) \text{ is face of } P.$$

In this case, $\operatorname{codeg}_c(P) \geq r$.

II. The combinatorial degree of polytopes - Analogy?

Analogue to Cayley conjecture:

If $\operatorname{codeg}_c(P) > \frac{d+2}{2}$,
then P is a combinatorial Cayley polytope.

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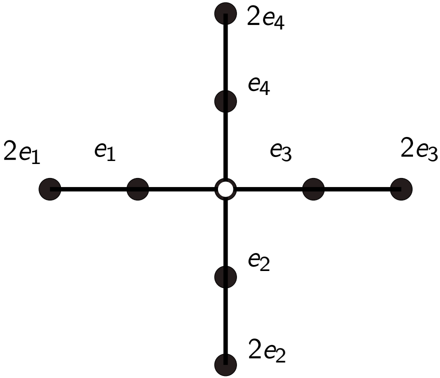
Recall:

If P is k-neighborly with $k > \lfloor \frac{d}{2} \rfloor$,
then P is a simplex.

II. The combinatorial degree of polytopes - ~~Analogy?~~

Example: $d = 5$, $\mathrm{codeg}_c(P) = 4 > \frac{5+2}{2}$, P **not** combinatorial Cayley



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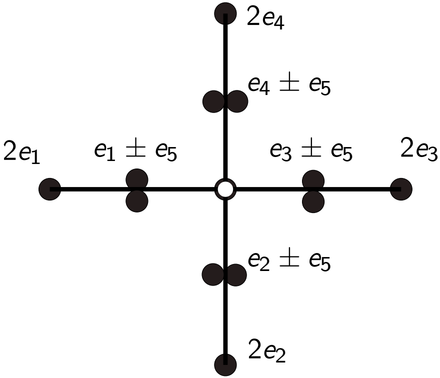

II. The combinatorial degree of polytopes - Conjecture

Definition

P is a **weak Cayley polytope** of r polytopes, if there exists **cover** $\mathcal{V}(P) = A_1 \cup \cdots \cup A_r$ such that

$$\forall\ I \subsetneq \{1, \ldots, r\} : \qquad \operatorname{conv}\left(\bigcup_{i \in I} A_i\right) \text{ is face of } P.$$

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Correct analogue to Refined Cayley Conjecture?

If $\operatorname{codeg}_c(P) > \frac{d+r}{2}$, then P is a **weak Cayley polytope** of r polytopes.

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$\deg_c(P)$ is maximal codimension of interior face of P.

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Question:

$$\deg_c(P) = \max\{\deg(h_\tau) \ : \ \tau \text{ triangulation with vertex set } \mathcal{V}(P)\} \ ?$$

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Open part of Generalized Lower Bound Conjecture

P simplicial, then

$$\deg(g_P) = \min\{\deg(h_\tau) \ : \ \tau \text{ triangulation with vertex set } \mathcal{V}(P)\}$$

where g_P is g-polynomial of face poset ∂P.