Adjunction-theoretic invariants of polarized toric varieties

Benjamin Nill
with Sandra Di Rocco, Christian Haase, and Andreas Paffenholz

AMS Statesboro 2011

I. Classical Adjunction Theory

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- L ample line bundle on X

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Minimal assumption

X is $\mathbb{Q}$-**Gorenstein**, i.e., K_X is $\mathbb{Q}$-Cartier.

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$$\mu \leqslant \tau$$

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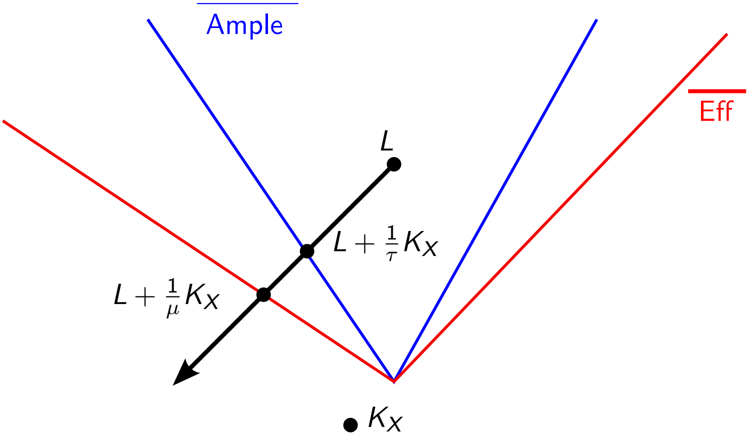

Results and conjectures

Most work on polarized *manifolds*:

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$\mathbb{Q}$-normality conjecture on polarized manifolds

$$\mu > \frac{n+1}{2} \quad \Longrightarrow \quad \mu = \tau$$

II. Polyhedral Adjunction Theory

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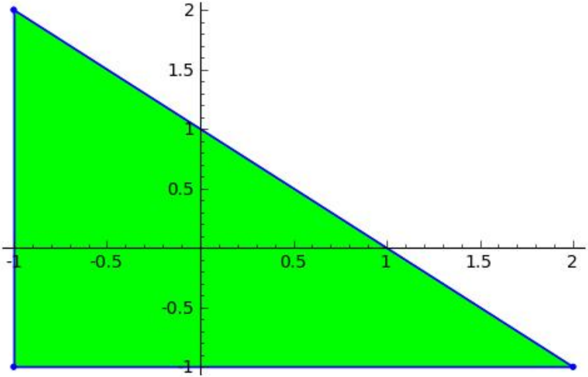
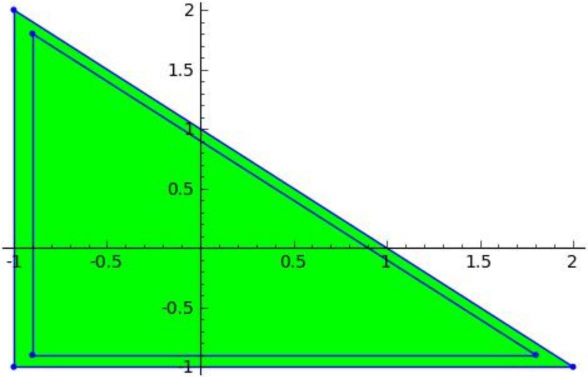
Polyhedral adjunction: "Move facets simultaneously inwards"

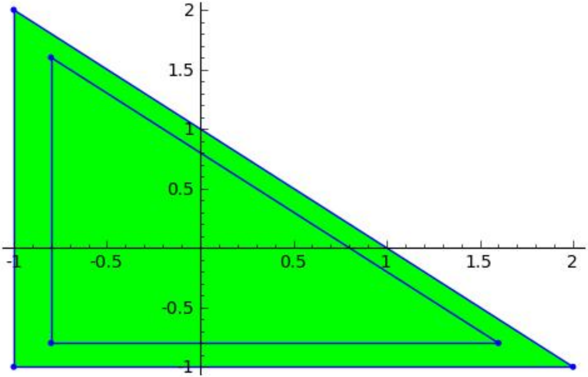
Example - $P^{(0)}$



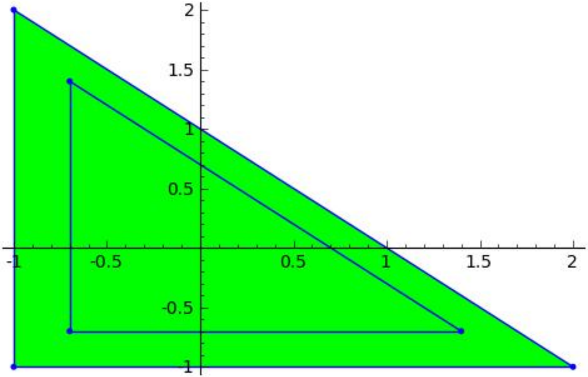
Example - $P^{(0.1)}$



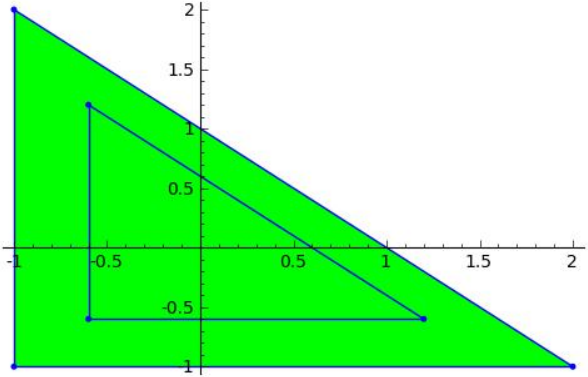
Example - $P^{(0.2)}$



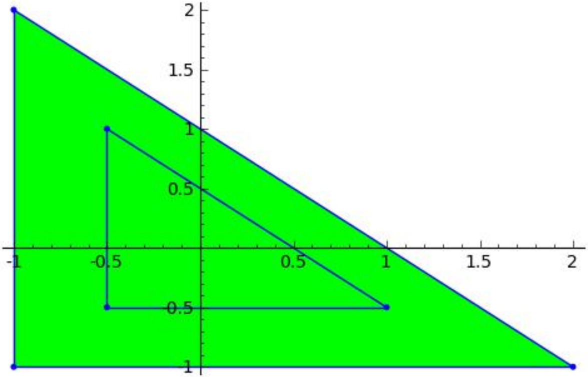
Example - $P^{(0.3)}$



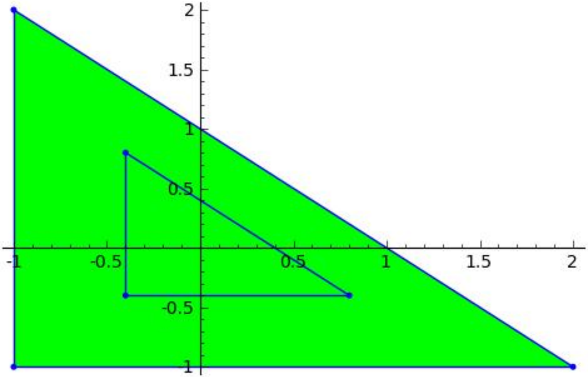
Example - $P^{(0.4)}$



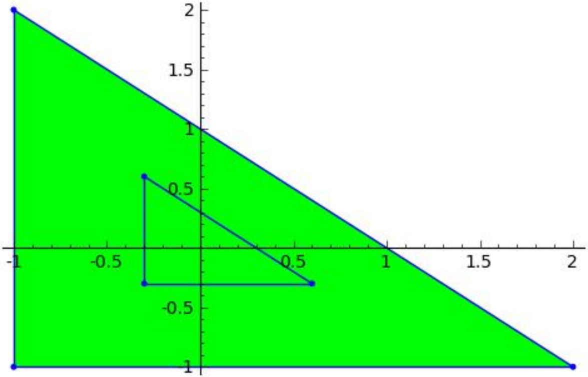
Example - $P^{(0.5)}$



Example - $P^{(0.6)}$



Example - $P^{(0.7)}$



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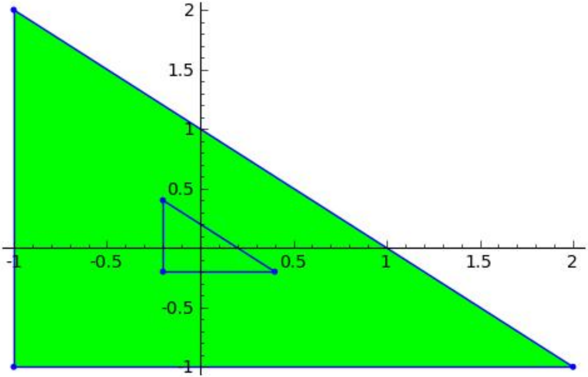
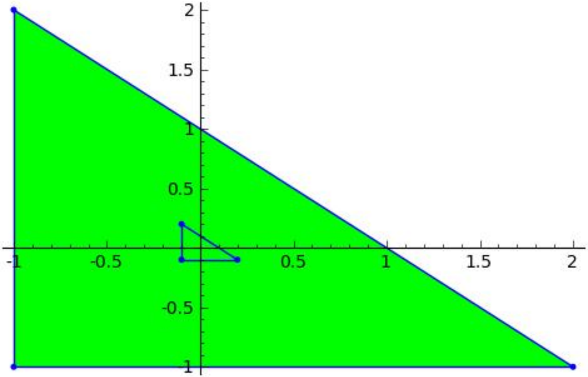
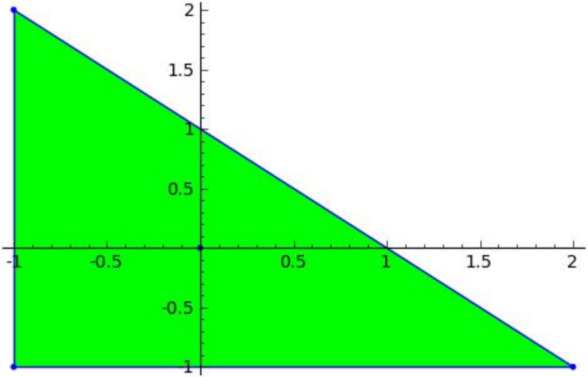


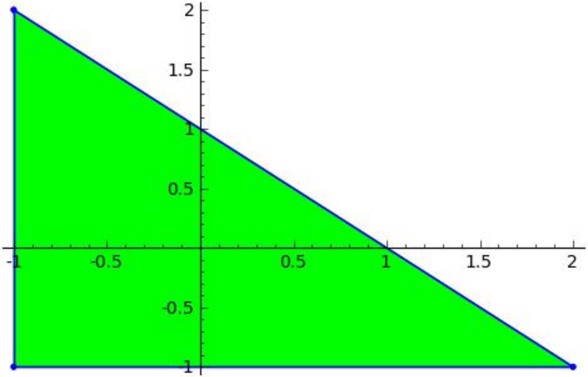
Example - $P^{(0.9)}$



Example - $P^{(1)}$ point



Example - $P^{(c)} = \emptyset$ for $c > 2$



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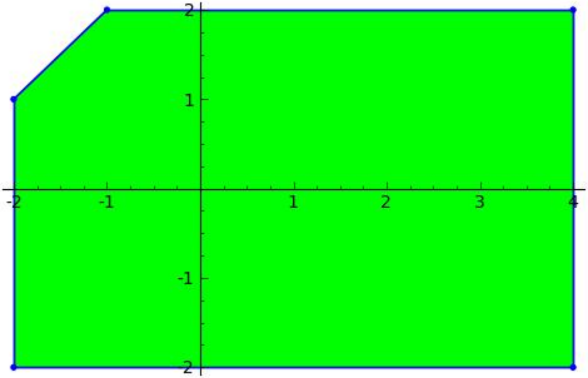

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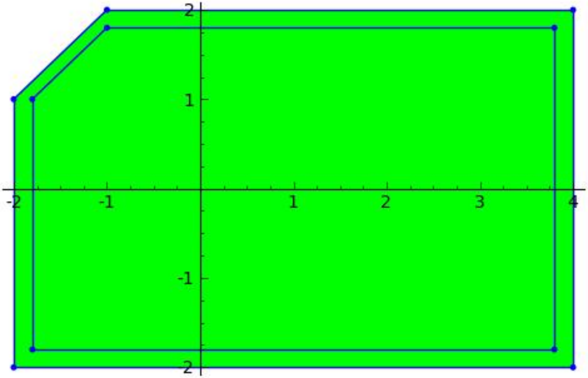

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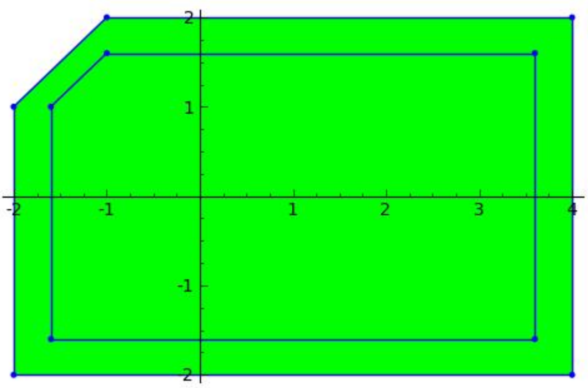

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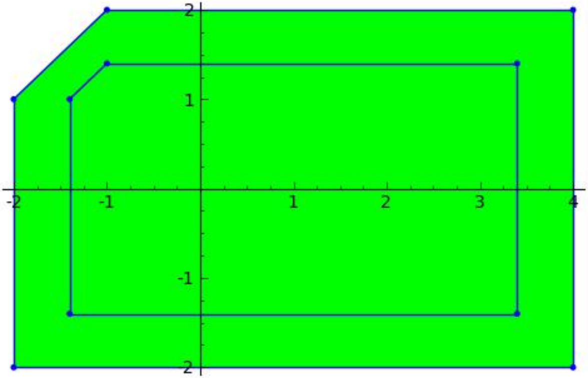

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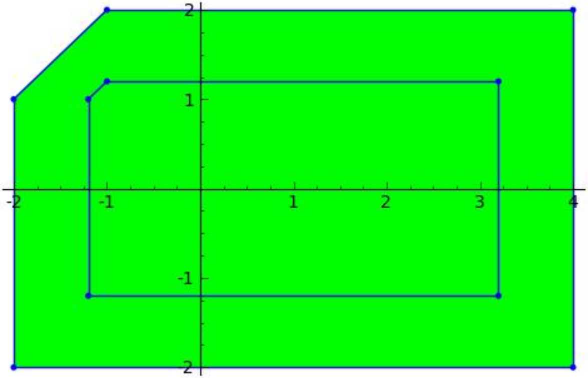
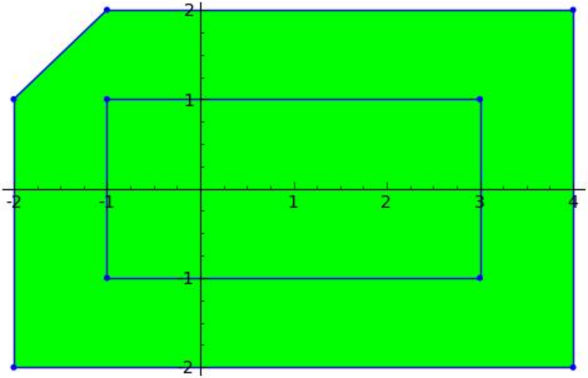


Example - $P^{(1)}$ combinatorics changes



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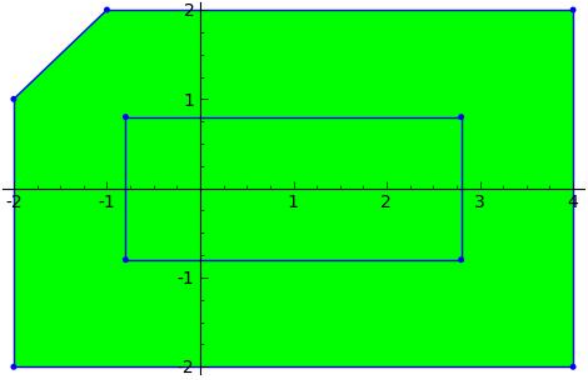

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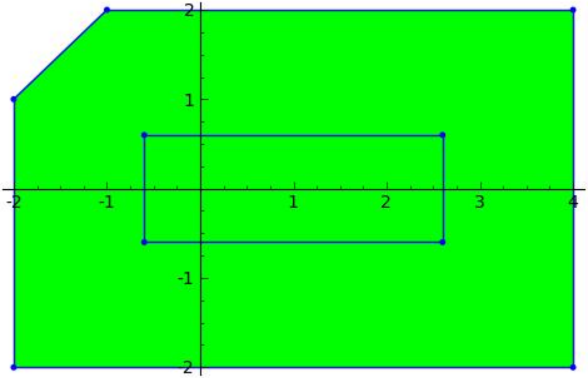

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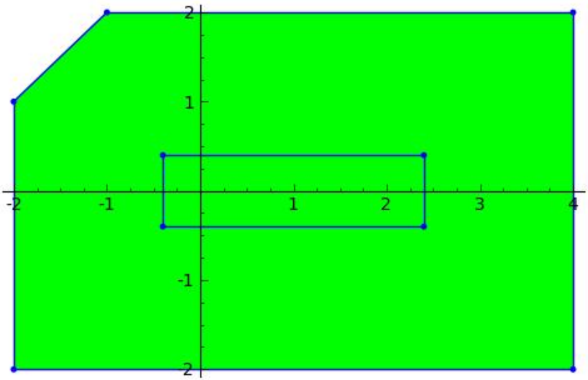

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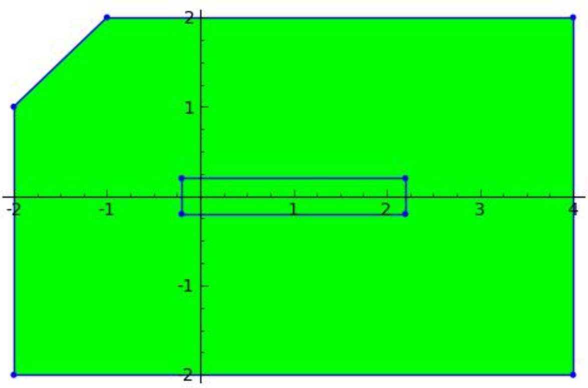
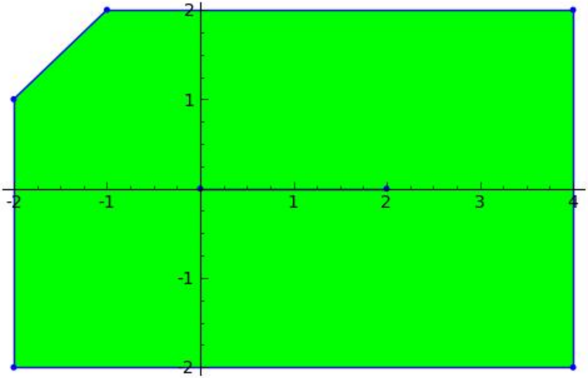


Example - $P^{(2)}$ interval



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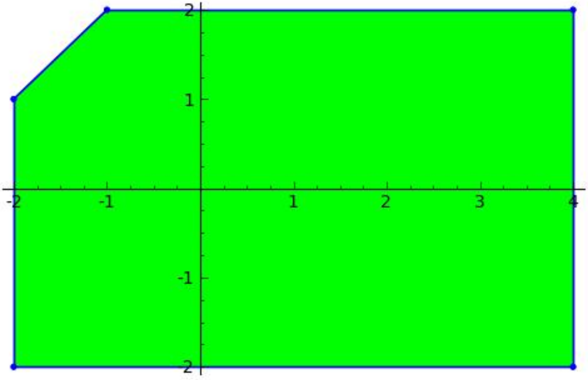

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Definition makes sense for **general** lattice polytopes!

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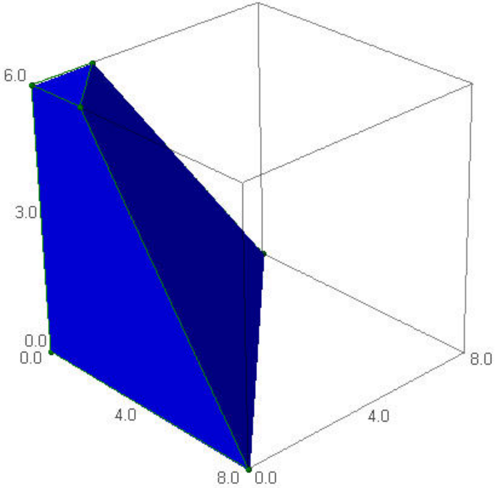

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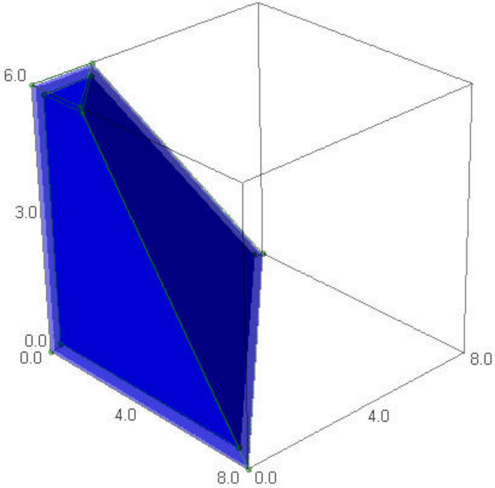

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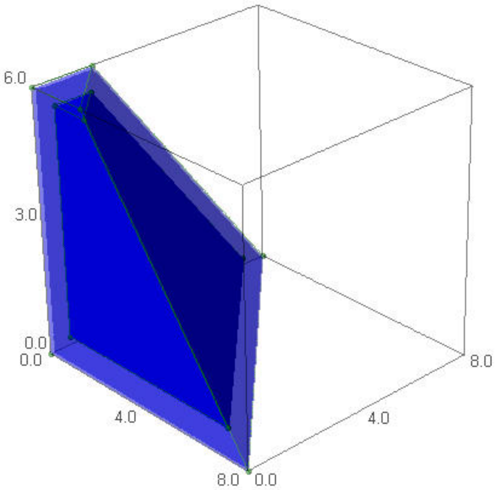

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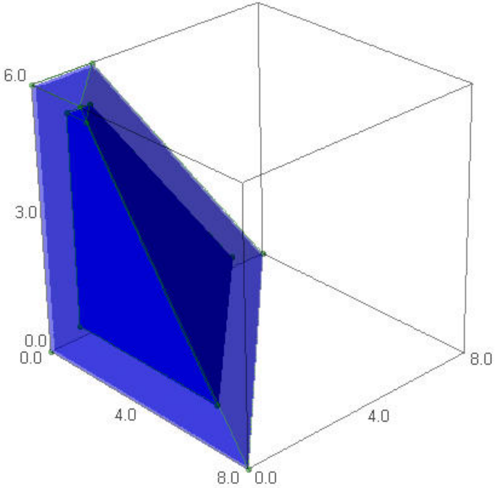

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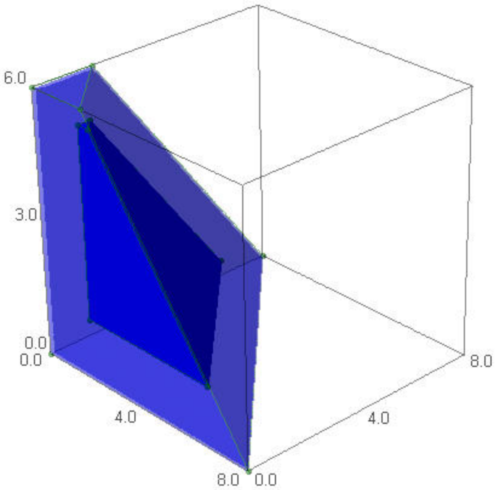
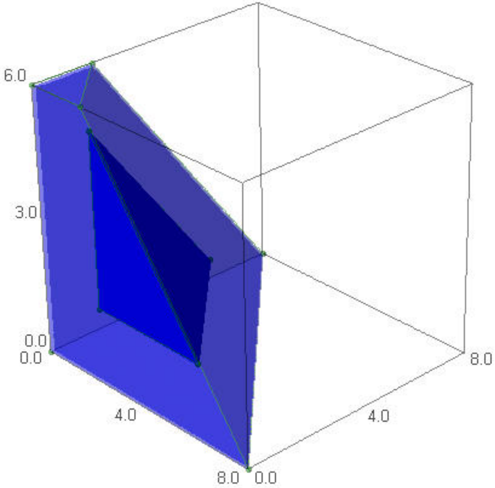


Example - $P^{(1)} \implies \tau_P = 1^{-1} = 1$



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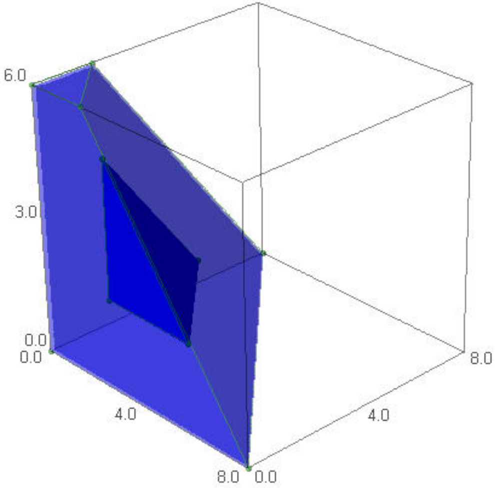

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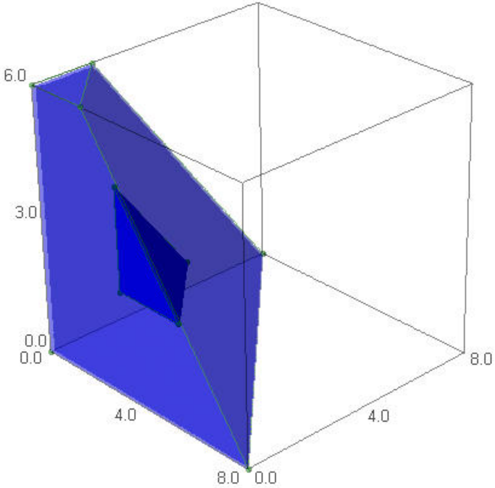

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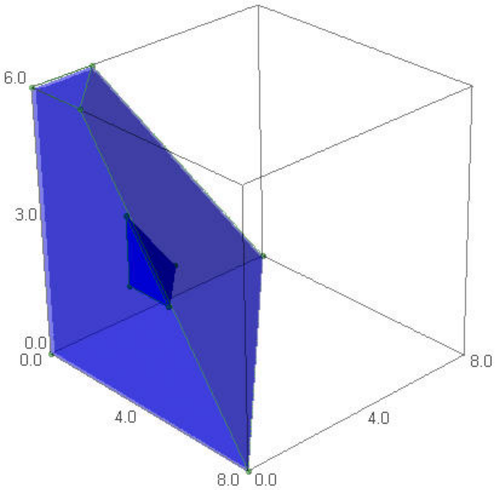

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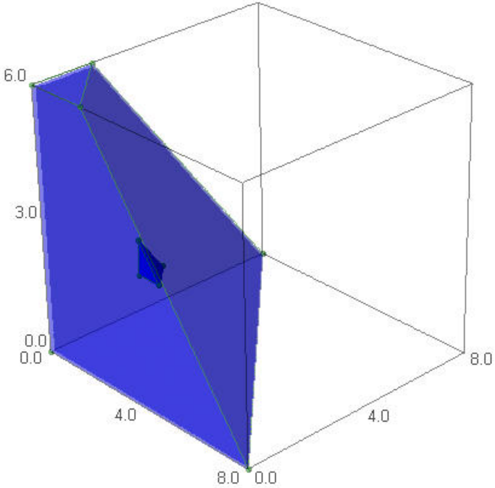
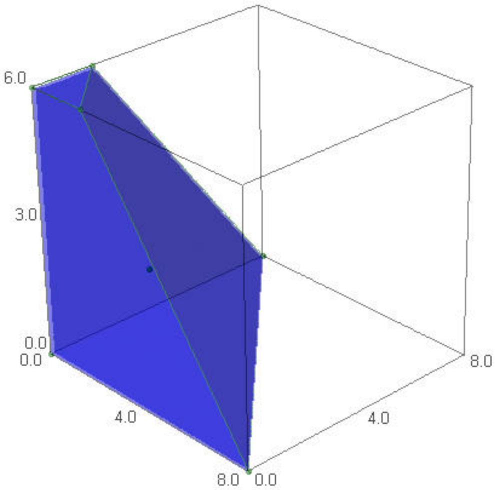


Example - $P^{(2)} \implies \mu_P = 2^{-1} = \frac{1}{2}$



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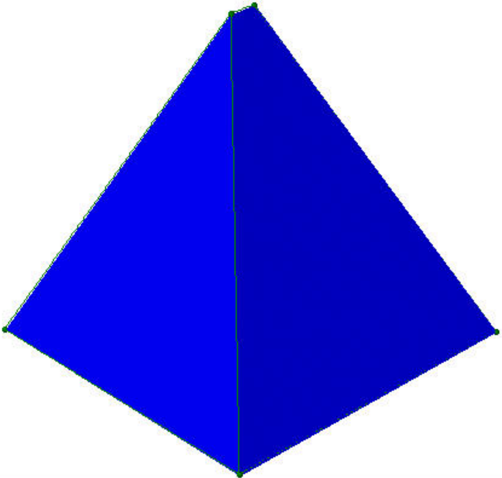

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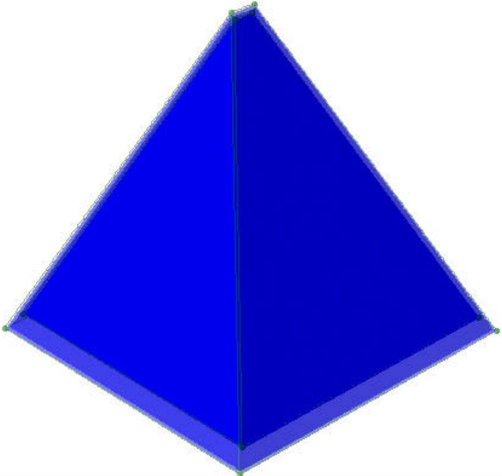

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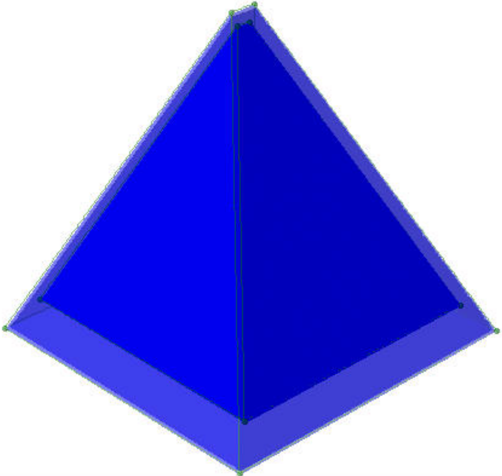

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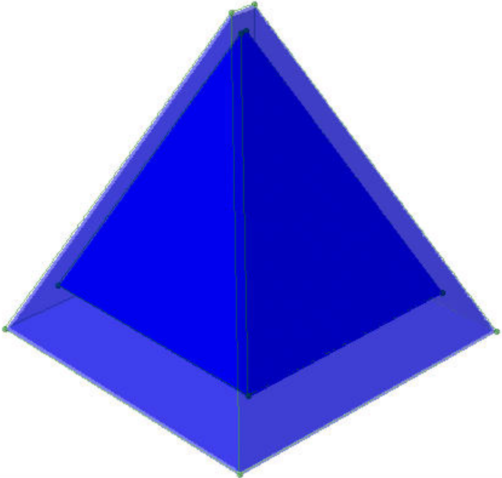

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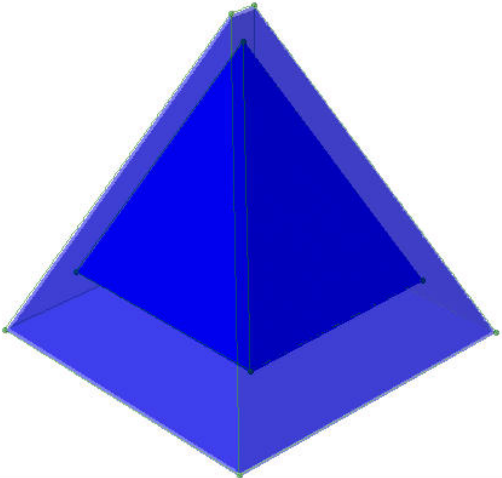

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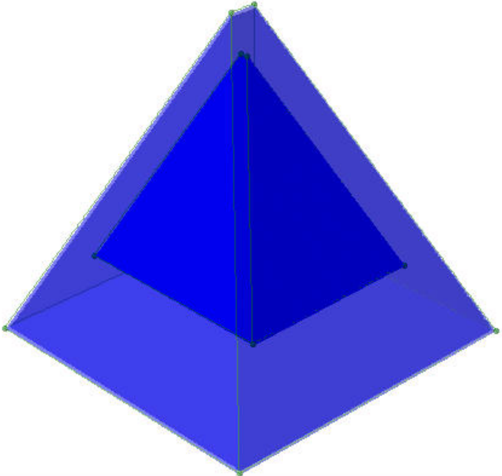
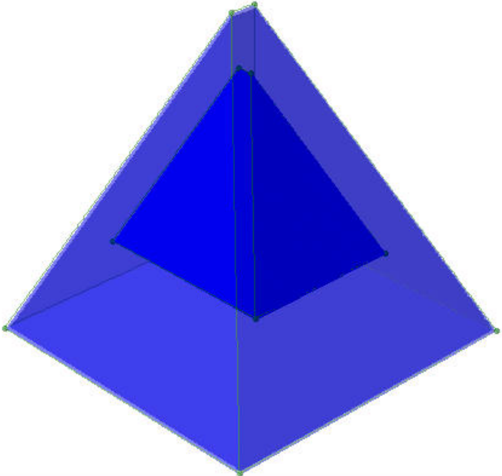
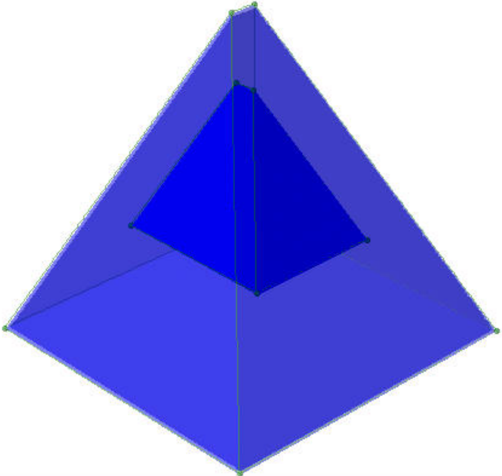


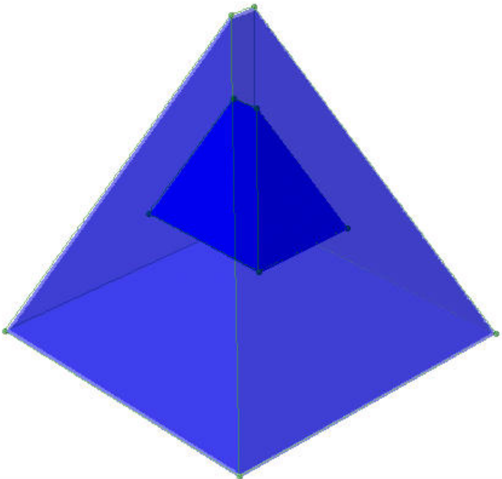
Example - $P^{(3)}$



Example - $P^{(3.5)}$



Example - $P^{(4)}$



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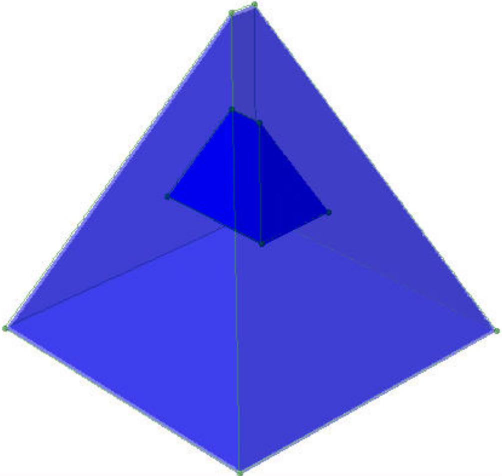
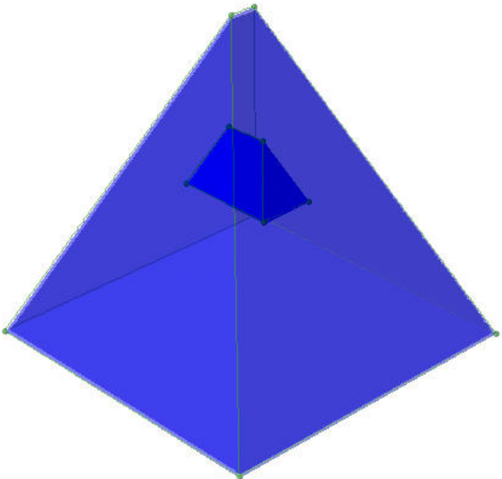
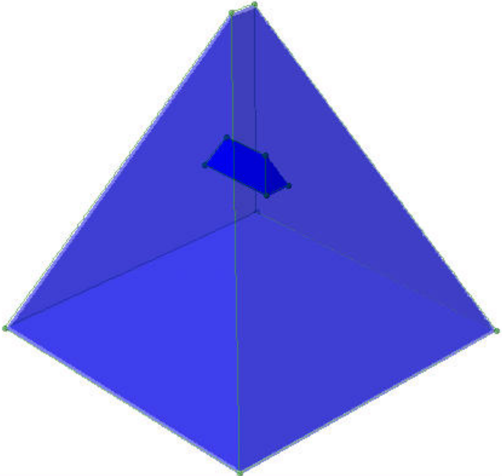


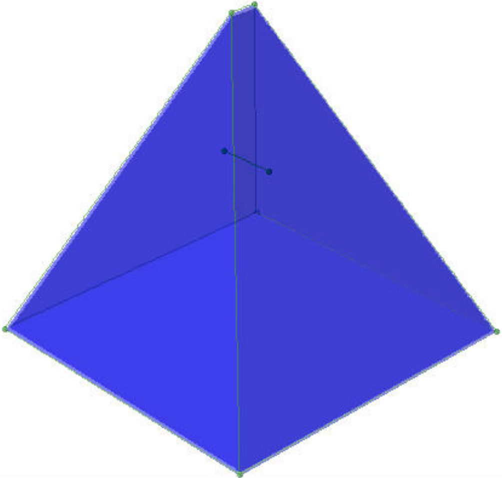
Example - $P^{(5)}$



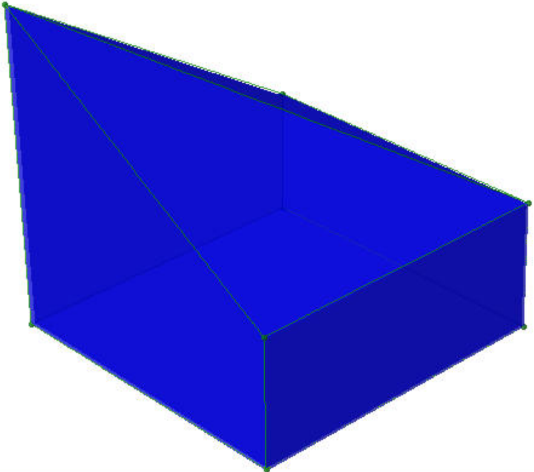
Example - $P^{(5.5)}$



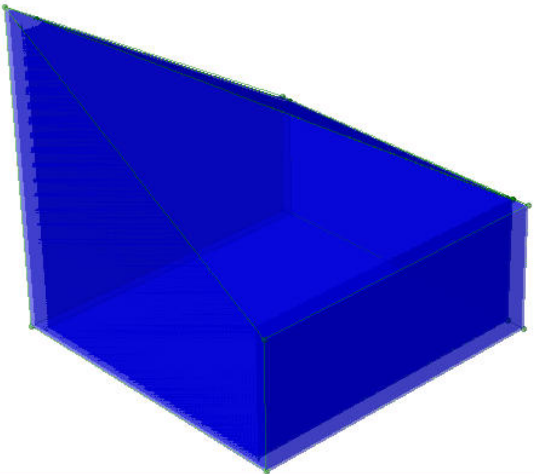
Example - $P^{(6)} \implies \mu_P = 6^{-1} = \frac{1}{6}$



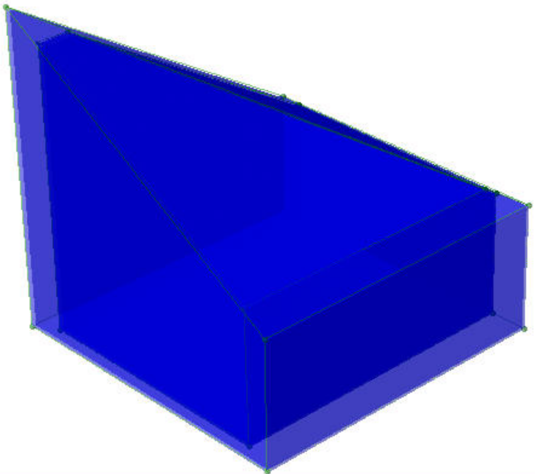
Example - $P^{(0)} \implies \tau = 0^{-1} = \infty$



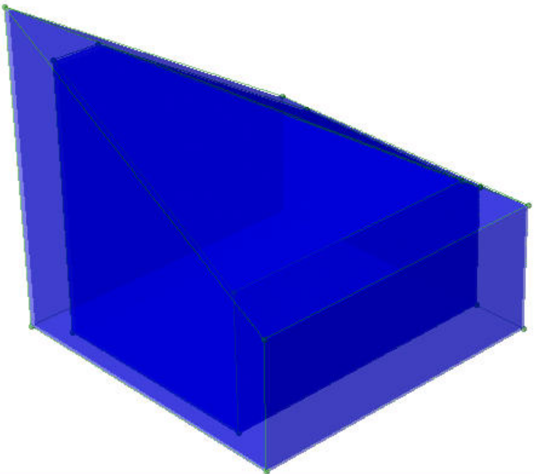
Example - $P^{(0.05)}$



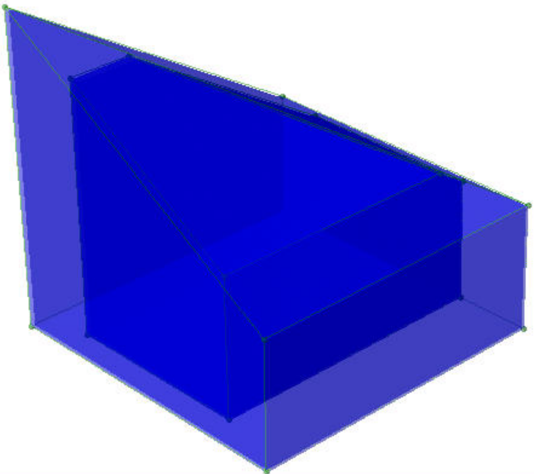
Example - $P^{(0.1)}$



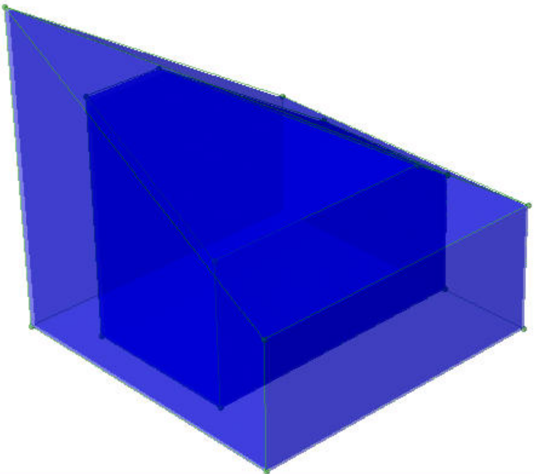
Example - $P^{(0.15)}$



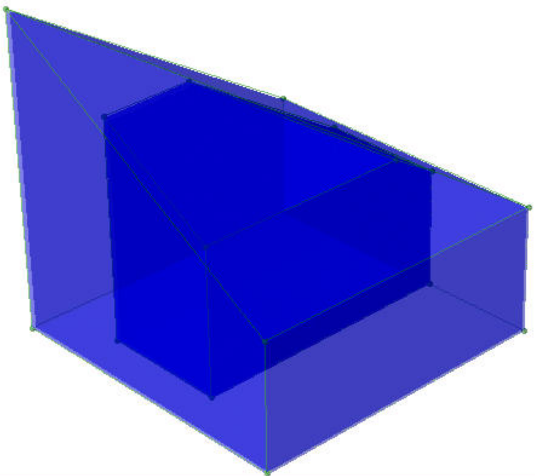
Example - $P^{(0.2)}$



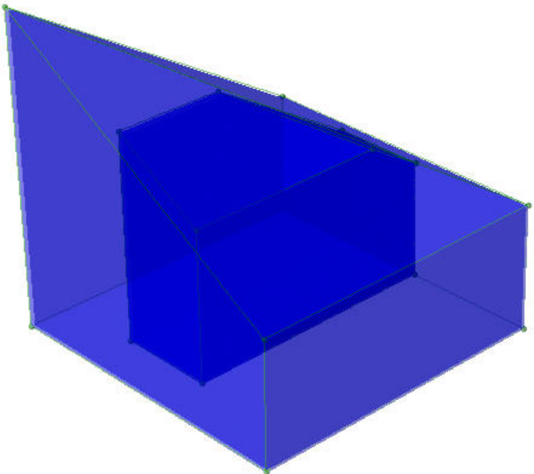
Example - $P^{(0.25)}$



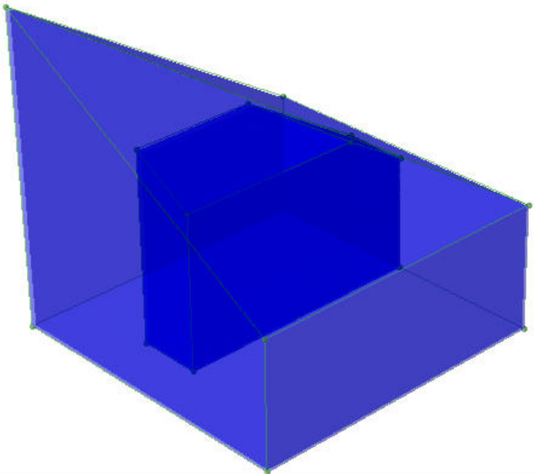
Example - $P^{(0.3)}$



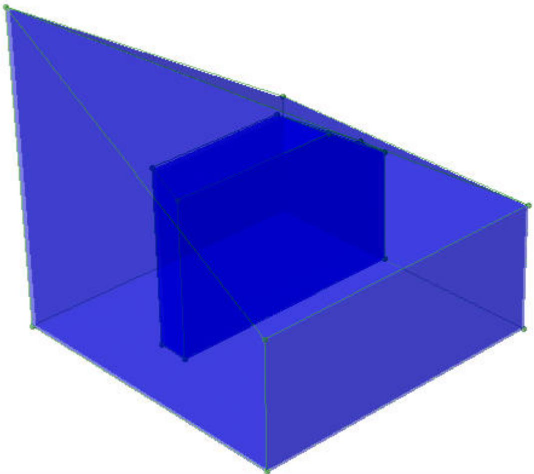
Example - $P^{(0.35)}$



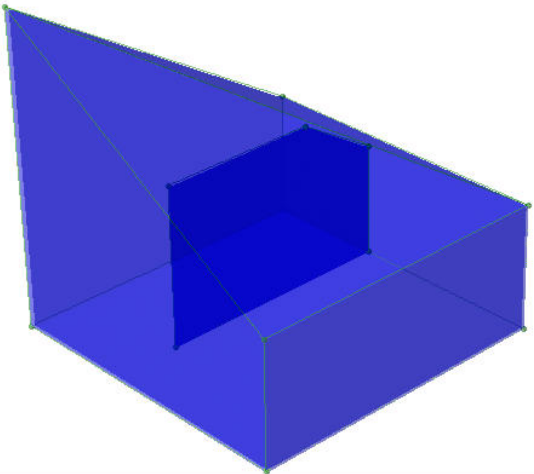
Example - $P^{(0.4)}$



Example - $P^{(0.45)}$



Example - $P^{(0.5)} \implies \mu_P = 0.5^{-1} = 2$



Two polyhedral invariants

Criterion

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Polyhedral adjunction theory
$\supsetneq$
Adjunction theory of polarized toric varieties

III. The Main Theorem

Large μ_P implies P Cayley

Theorem (Di Rocco, Haase, N., Paffenholz '11)

If $\mu_P > \frac{n+2}{2}$

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If $\mu_P > \frac{n+2}{2}$ (equivalently, $n > 2(n+1-\mu_P)$),

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If $\mu_P > \frac{n+2}{2}$ (equivalently, $n > 2(n+1-\mu_P)$),
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i.e., for $m := \lfloor 2(n+1-\mu_P) \rfloor$ there exists

$$\varphi : \mathbb{Z}^n \twoheadrightarrow \mathbb{Z}^{n-m}$$

such that

$$\varphi(P) = \mathrm{conv}(0, e_1, \ldots, e_{n-m}).$$

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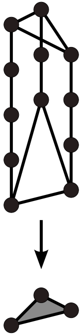
Example: $n = 3$, $m = 1$, $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^{3-1} = \mathbb{Z}^2$



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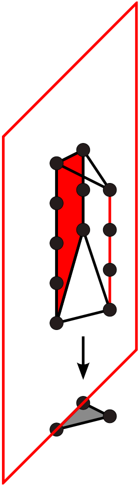

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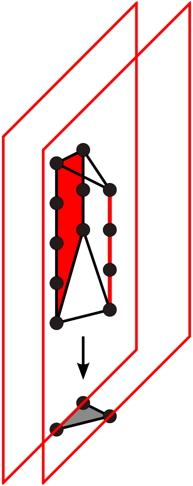

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Corollary

$$\mu_P > \frac{n+2}{2} \implies P \text{ has lattice width one.}$$

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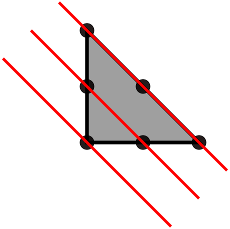
Corollary is sharp: $(\mathbb{P}^n, O(2))$, $\mu = \frac{n+1}{2}$, lattice width > 1



Large μ_P implies P Cayley

Algebro-geometric version

If $\mu_P \geqslant \frac{n+2}{2}$, then there exists proper birational toric morphism

$$\pi : X' = \mathbb{P}(H_0 \oplus H_1 \oplus \cdots \oplus H_m) \to X$$

where H_i are line bundles on a toric variety of dimension $\leqslant 2(n+1-\mu_P)$ and $\pi^* L \cong O(1)$.

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This is (nearly) the $\mathbb{Q}$-normality conjecture!

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Corollary to Theorem

$$\mu_P > \frac{3n+4}{4} \implies X_P \text{ dual defective.}$$