From Ehrhart theory to almost-neighborly polytopes

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Almost-neighborly polytopes - Definition

Let P be a d-polytope.

Definition

P is k-**neighborly**, if any subset of k vertices is vertex set of a face.

Much studied class of polytopes (includes cyclic polytopes)

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Definition (see exercise in Grünbaum's book)

P is k-**almost neighborly**, if any subset of k vertices is contained in a proper face.

Not much known!

Almost-neighborly polytopes - Question

Recall:

If P is k-neighborly with $k > \lfloor \frac{d}{2} \rfloor$,
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Main question of this talk

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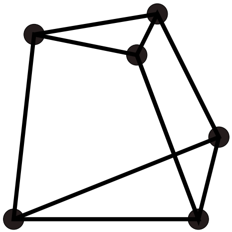
What is the analogue statement for k-almost neighborly polytopes?

Example: P 2-almost neighborly 3-polytope, $2 > \lfloor \frac{3}{2} \rfloor$



Degree of lattice polytopes - Definition

Let $P \subset \mathbb{R}^d$ be a d-dimensional **lattice polytope** (i.e., the vertex set $\mathcal{V}(P) \subset \mathbb{Z}^d$).

Main object of study: *Ehrhart polynomials*

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Theorem [Stanley '80]

$$\sum_{k \geq 0} |\, kP \cap \mathbb{Z}^d \,|\, t^k = \frac{h^*_P(t)}{(1-t)^{d+1}},$$

where h^*_P is a **polynomial of degree $\leq$ d** with coefficients in $\mathbb{N}$.

Definition: The degree of $h^*_P(t)$ is called the **degree** of P.

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Problem: What can you say about lattice polytopes of given degree?

Degree of lattice polytopes - Properties

Isomorphisms of lattice polytopes =
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Basic properties of the degree

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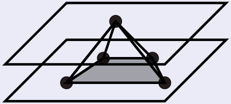

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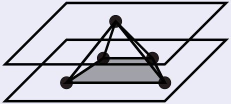

- degree is monotone with respect to inclusion.

Degree of lattice polytopes - Codegree

Definition: The **codegree** of P:

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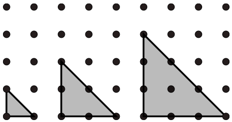

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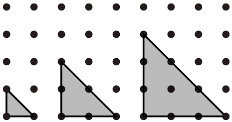

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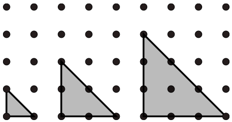


Reciprocity: $1 \leq \boxed{\operatorname{codeg}(P) = d + 1 - \deg(P)} \leq d + 1$

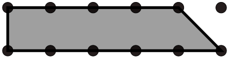
$$\operatorname{codeg}(P) = 1 \quad \Leftrightarrow \quad \deg(P) = d \quad \Leftrightarrow \quad \operatorname{int}(P) \cap \mathbb{Z}^d \neq \emptyset$$

$$\operatorname{codeg}(P) = d + 1 \quad \Leftrightarrow \quad \deg(P) = 0 \quad \Leftrightarrow \quad P \cong \Delta_d$$

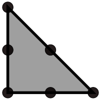
Degree of lattice polytopes - Degree 1

Example: $d = 2$, $\deg(P) \leq 1$ (no interior lattice points)

Infinitely many of this type



and one **exceptional triangle** S



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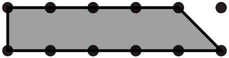

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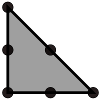


Examples of degree 1 in high dimensions?

Degree of lattice polytopes - Degree 1

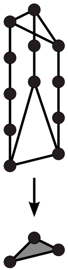
Definition/Lemma (Batyrev, N. '07)

P is a **Lawrence prism**, if exists

$$\phi : \mathbb{Z}^d \twoheadrightarrow \mathbb{Z}^{d-1}, \quad \phi(P) = \Delta_{d-1}.$$

Then

$$\deg(P) \leq 1.$$



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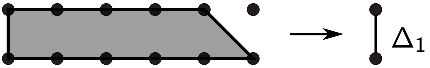

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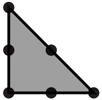

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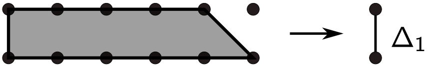

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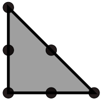

Theorem [Batyrev, N. '07]

P d-dimensional **lattice polytope** which is not a lattice pyramid.
Then $\deg(P) \leq 1$ if and only if P is

- Lawrence prism or
- the exceptional triangle S.

Degree of lattice polytopes - Cayley polytopes

Examples of high dimension but small degree?

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Definition/Lemma [Batyrev, N. '07]

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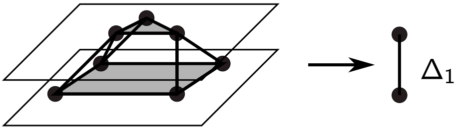

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$$d > 2\deg(P) \implies P \text{ is Cayley polytope}$$

Bound is **tight** (for d even):
$d = 2\deg(2\Delta_d)$, but $2\Delta_d$ is not Cayley polytope.

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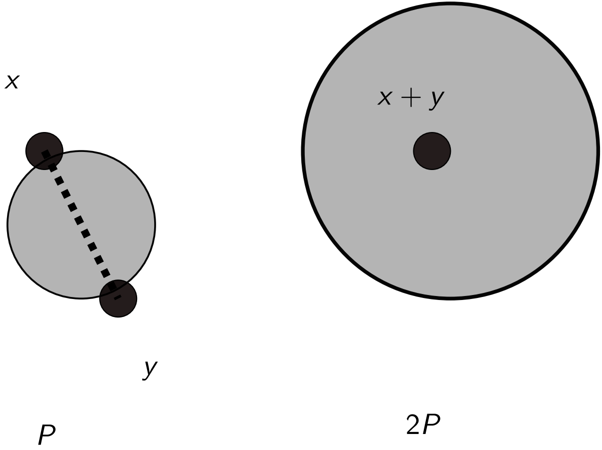
- P is smooth $\implies$ Cayley conjecture holds [Dickenstein, N. '10]
- $d > 20(\deg(P))^2 \implies P$ Cayley polytope [Haase, N., Payne '09]

Combinatorial degree of polytopes - Motivation

What about the combinatorics?



Combinatorial degree of polytopes - Motivation

Observation

Let P be d-dimensional lattice polytope.

$$\mathrm{codeg}(P) > k \quad \Longrightarrow$$

P is k-**almost neighborly**

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The **combinatorial codegree** of d-dimensional polytope P is

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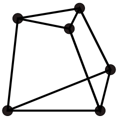

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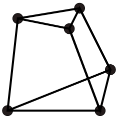

(Co-)degree also be defined for point configurations.

Combinatorial degree of polytopes - Properties

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Combinatorial degree of polytopes - $\deg_c(P) = 1$

Theorem [Batyrev, N. '07]

P d-dimensional **lattice polytope** which is not a lattice pyramid.
Then $\deg(P) \leq 1$ if and only if P is

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Theorem [Joswig, Herrmann '10; N., Padrol '12]

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- polygon.

One can extend this result to point configurations.

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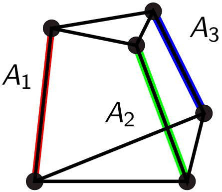

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is equivalent to

$$P \text{ is } k\text{-almost neighborly with } k > \lfloor \tfrac{d}{2} \rfloor$$

This would solve question at beginning!

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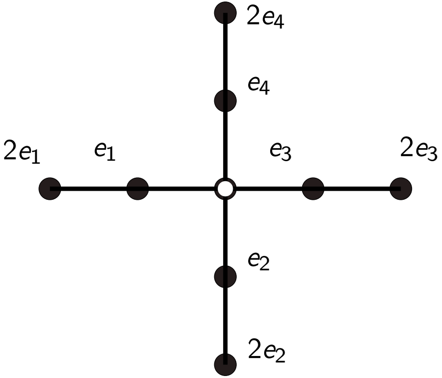
Example: P is 3-neighborly 5-polytope ($3 > \lfloor \frac{5}{2} \rfloor$),
P **not** combinatorial Cayley



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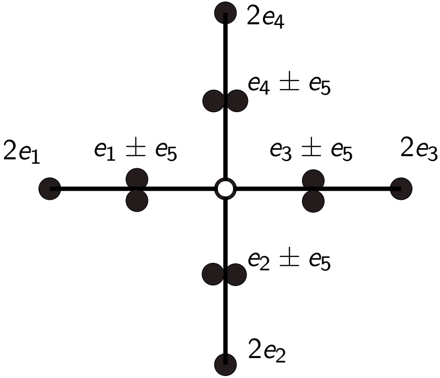

II. The combinatorial degree of polytopes - Conjecture

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P is a **weak combinatorial Cayley polytope** of r polytopes, if there exists *cover* $\mathcal{V}(P) = A_1 \cup \cdots \cup A_r$ such that

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Theorem [N., Padrol '12]

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Observation

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Generalized lower bound conjecture [Murai, Nevo '12]

If P is simplicial, then

$$\deg(g_P) = \min\{\deg(h_\tau) \ : \ \tau \text{ triangulation with vertex set } \mathcal{V}(P)\}$$

where g_P is g-polynomial of face poset ∂P.