Flow polytopes and the graph of reflexive polytopes

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Abstract

We suggest defining the structure of an unoriented graph $\mathcal{R}_d$ on the set of reflexive polytopes of a fixed dimension d. The edges are induced by easy mutations of the polytopes to create the possibility of walks along connected components inside this graph. For this, we consider two types of mutations: Those provided by performing duality via nef-partitions, and those arising from varying the lattice. Then for $d \leq 3$, we identify the flow polytopes among the reflexive polytopes of each single component of the graph $\mathcal{R}_d$. For this, we present for any dimension $d \geq 2$ an explicit finite list of quivers giving all d-dimensional reflexive flow polytopes up to lattice isomorphisms. We deduce as an application that any such polytope has at most $6(d-1)$ facets.

Key words: Quivers, Reflexive polytopes, Toric geometry
MSC: 52B20, 14M25

Introduction

In this paper we report on a project investigating the position of reflexive flow polytopes inside the graph of reflexive polytopes. Reflexive polytopes form an interesting class of lattice polytopes that are being studied for various reasons [4,10,15] and whose number grows very fast with increasing dimension [12,13]. On the other hand, the purely combinatorial notion of quivers furnishes us with one of the very

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rare systematic constructions of reflexive polytopes, called flow polytopes, also in high dimensions [1,2,11]. This provides the motivation to use flow polytopes as good candidates for starting points in determining large sets of reflexive polytopes via an effective procedure of mutating reflexive polytopes, given by lattice modifications and so-called nef partitions. The goal of this paper is to lay the foundations for such an approach by giving an explicit description of the classification of flow polytopes, and to demonstrate these results in dimensions two and three. Throughout this presentation we include several conjectures and ideas for future work.

This article is organized as follows. In the first two sections we recall the definition of reflexive polytopes and how they form a graph via nef-partitions and lattice modifications. We illustrate this concept in dimensions two and sketch the relation to previous notions of connectedness. We then observe in Proposition 5 that it surprisingly suffices to consider mutations of nef-partitions having at most three elements. In section three we recall how quivers yield reflexive polytopes, called flow polytopes, and provide in Theorem 7 explicit instructions on how to determine all flow polytopes in given dimension up to isomorphisms. This is the main theoretical result of our paper. In section four we give detailed descriptions of our computational results in dimension three, which show that most flow polytopes are contained in the 'big' connected component of the graph of reflexive polytopes. Finally, as an application of Theorem 7, we classify in section five the quivers leading to flow polytopes having the maximal possible number of facets.

1 Reflexive polytopes and nef-partitions

In [4], Batyrev introduced the notion of reflexive polytopes. Let $M \cong \mathbb{Z}^d$ denote a finitely generated abelian group sitting as a sublattice inside $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Their duals are denoted by $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Hom}_{\mathbb{R}}(M_{\mathbb{R}}, \mathbb{R})$, respectively.

Definition 1 A lattice polytope $\Delta \subseteq M_{\mathbb{R}}$, i.e. the convex hull of finitely many lattice points, is called *reflexive* if $0 \in \mathrm{int}\Delta$ and, moreover, its dual

$$\Delta^{\vee} := \{a \in N_{\mathbb{R}} \mid \langle a, \Delta \rangle \geq -1\}$$

is a lattice polytope, too. Here the expression $\langle a, \Delta \rangle$ stands for the minimum over the set $\{\langle a, r \rangle \mid r \in \Delta\}$.

While duality between polyhedral cones behaves well, duality among polytopes comes with a problem – it is not translation invariant. However, the previous definition implies that $0 \in \Delta$ is the only interior lattice point of a reflexive polytope. Thus, dualization becomes a canonical operation in this context. It is well known that in each dimension d there is only a finite number of isomorphism classes of reflexive polytopes. For $d = 1,2,3,4$ the precise numbers are $1, 16, 4319, 473800776$ by the classification results due to Kreuzer and Skarke [12,13]. Here, as throughout, two lattice polytopes are called *isomorphic*, if there is a lattice automorphism mapping the vertices onto each other.

Constructing the toric variety $\mathbb{P}(\Delta)$ of the normal fan of a reflexive polytope Δ provides a (possibly singular) Fano variety, and its general hyperplane sections are Calabi-Yau varieties with the same type of singularities. Their Newton polytope is Δ, and the duality notion among reflexive polytopes translates then into the mirror symmetry among these hypersurfaces, cf. [4]. It was Borisov's idea [7] to generalize this setting to complete intersections (see [6] for a survey of these combinatorial mirror constructions):

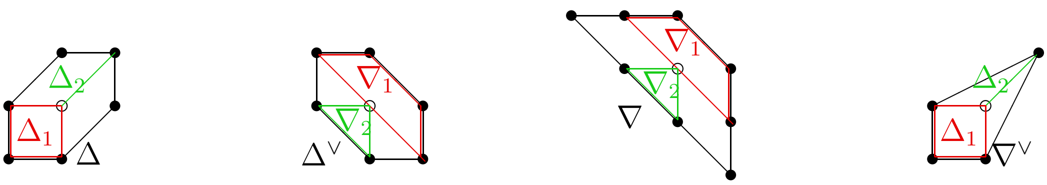
Definition 2 A set $\{\Delta_1, \ldots, \Delta_k\}$ of lattice polytopes $\Delta_i \subseteq M_\mathbb{R}$ containing 0 is called a *nef-partition* of a reflexive polytope Δ if $\sum_{i=1}^k \Delta_i = \Delta$, where this sum denotes the Minkowski sum of polytopes. Its *dual nef-partition* is defined as the set $\{\nabla_1, \ldots, \nabla_k\}$ with

$$\nabla_i := \operatorname{conv}(\{0\} \cup \{a \in \mathcal{V}(\Delta^\vee) \mid \langle a, \Delta_i \rangle = -1\}) \subseteq \Delta^\vee,$$

where $\mathcal{V}(\Delta^\vee)$ denotes the set of vertices.

In particular, this yields $\Delta^\vee = \operatorname{conv}(\nabla_1, \ldots, \nabla_k)$.

Theorem 3 ([7]) *The set $\{\nabla_1, \ldots, \nabla_k\}$ is a nef-partition of the polytope $\nabla := \sum_{i=1}^k \nabla_i$. In particular, ∇ is reflexive, too. Moreover, $\{\Delta_1, \ldots, \Delta_k\}$ is again the dual nef-partition of $\{\nabla_1, \ldots, \nabla_k\}$.*



If nef-partitions $\{\Delta_1, \ldots, \Delta_k\}$ arise as Newton polytopes of hypersurfaces in $\mathbb{P}(\Delta)$, then their corresponding complete intersection is Calabi-Yau, and the duality notion among nef-partitions leads again to mirror symmetric pairs, see [5].

2 The graph of reflexive polytopes

While reflexivity of polytopes is easily defined and for a given polytope can immediately be checked, this notion is not at all easy to handle. For instance, it is rather tricky to provide bunches of examples right away. Despite the existence of explicit lists in low dimensions it seems that reflexivity is not sufficiently understood yet.

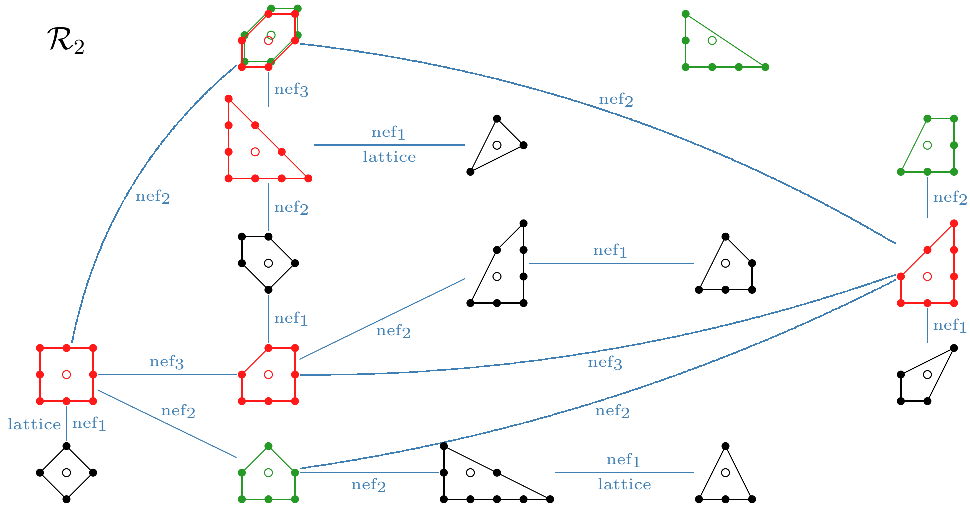
The approach of the present paper is to study the set of reflexive polyhedra in a fixed dimension via introducing the structure of a graph on this set. This creates the method of constructing reflexive polytopes via walking along its edges.

Definition 4 The graph $\mathcal{R}_d$ of reflexive polytopes in dimension d has this set as its *vertices*, and two reflexive polytopes are connected by an *edge* if they either

(1) result from the duality $\Delta \rightsquigarrow \nabla$ provided by a nef-partition of Δ, or

(2) if both polyhedra become isomorphic when considered with the sublattices generated by their vertices, respectively.

Note that, for a reflexive polytope Δ, the sublattice generated by its vertices keeps the polyhedron reflexive – the dual lattice becomes larger and the dual polytope remains a lattice polytope. This sublattice is called the *minimal lattice* $L(\Delta)$ for Δ.



The above figure shows the graph $\mathcal{R}_2$; the green color indicates selfdual polytopes, while red denotes flow polytopes, which will be discussed later. Moreover, we have labeled the edges by the type of mutation they represent. The label nef_k means that both reflexive polytopes host mutually dual nef-partitions with k elements each. In particular, nef_1 stands for ordinary duality.
The striking observation in dimension two is that the 16 reflexive polytopes of dimension two split into a major component and a single isolated polygon. The latter corresponds to the toric variety $\mathbb{P}(1,2,3)$, i.e., to the weighted projective plane.

In general, it is known (see [6, Prop. 6.16]) that any nef-partition of a d-dimensional reflexive polytope has at most $2d$ elements. Indeed, in $\mathcal{R}_2$ the square with 9 lattice points is adjacent to itself via a nef_4-edge, which we omitted in the figure. On the other hand, we observe that removing all nef_3-edges splits the big component into two parts. This leads to the question, which is of enormous computational importance, whether it is actually necessary to compute all nef-partitions in order to determine the connected components of $\mathcal{R}_d$. As the following proposition shows, surprisingly it suffices for any dimension d to consider only nef-partitions of length three.

Proposition 5 *If two reflexive polytopes in $\mathcal{R}_d$ are adjacent via duality of some nef-partition, then there is a connecting path using only nef-partitions having at most three elements.*

Proof. Let Δ and ∇ be adjacent via a nef-partition $\{\Delta_1, \ldots, \Delta_k\}$ of length $k \geq 4$. We show that there is an edge path $\Delta \overset{\text{nef}_{k-1}}{\sim} \nabla' \overset{\text{nef}_2}{\sim} \Delta' \overset{\text{nef}_3}{\sim} \nabla$, hence using only nef-partitions of length at most $k-1$:
By [6, Prop. 6.2], $\{\Delta_1, \ldots, \Delta_{k-2}, \Delta_{k-1} + \Delta_k\}$ is a nef-partition of Δ of length $k-1$ with dual nef-partition $\{\nabla_1, \ldots, \nabla_{k-2}, \text{conv}(\nabla_{k-1}, \nabla_k)\}$ of some reflexive polytope ∇'. In the same way, $\{\nabla_1 + \cdots + \nabla_{k-2}, \text{conv}(\nabla_{k-1}, \nabla_k)\}$ is a nef-partition of length two of ∇' with dual nef-partition $\{\text{conv}(\nabla_1, \ldots, \nabla_{k-2}), \Delta_{k-1} + \Delta_k\}$ of some reflexive polytope Δ'. Eventually, $\{\text{conv}(\nabla_1, \ldots, \nabla_{k-2}), \Delta_{k-1}, \Delta_k\}$ is a nef-partition of length three of Δ' with dual nef-partition $\{\nabla_1 + \cdots + \nabla_{k-2}, \nabla_{k-1}, \nabla_k\}$ of ∇. □

We also observe in $\mathcal{R}_2$ that there is no pair of reflexive polygons that is connected by a lattice transformation (condition (2) in Definition 4) but not by some nef-partition (condition (1)). However, this peculiar behavior is restricted to dimension two; it does not hold in dimension three anymore. For instance, the standard simplex $P_1 := \text{conv}(e_1, e_2, e_3, -e_1 - e_2 - e_3)$ maps via an integer linear map $\left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right]$ onto a reflexive simplex P_2. Here, P_1 corresponds to $\mathbb{P}^3$, thus P_2 to a quotient of $\mathbb{P}^3$ by the action of a finite group of order 4. However, P_2 and $P_2^\vee$ both admit no non-trivial nef-partitions.

Remark 6 There is another notion of a graph of reflexive polytopes, called *nesting* of reflexive polytopes [3]: Here we say two reflexive polytopes P, P' are connected by an edge, if $P \subseteq P'$ or $P' \subseteq P$. Relying on heavy computations, it was shown by Kreuzer and Skarke [13] that in dimensions up to four this graph is connected. The motivation to define such a graph comes from mirror symmetry: passing from one reflexive polytope to another one such that both have a common reflexive subpolytope induces a so-called extremal or singular transition of the corresponding Calabi-Yau varieties. The idea goes back to Reid who conjectured that the web of all Calabi-Yau manifolds should be connected via so-called conifold transitions, which are even more special operations. The reader is refered to [3,8]. The nesting-graph is related to our graph $\mathcal{R}_d$: If Δ is connected to ∇ via a nef-partition $\{\Delta_1, \ldots, \Delta_k\}$, then $\Delta^\vee = \text{conv}(\nabla_1, \ldots, \nabla_k) \subseteq \nabla$, where the non-zero vertices of the ∇_i are also vertices of $\Delta^\vee$. In the same way, we get $\nabla^\vee \subseteq \Delta$. Comparing the computational complexity of a walk in this graph or in $\mathcal{R}_d$ along the edges, note that already in dimension two the nesting-graph has *many* more edges than $\mathcal{R}_2$. And this discrepancy is expected to get worse if the dimension increases. One reason for this is that by the duality of nef-partitions, in order to determine all nef-partitions of Δ it suffices to look at subsets of the vertices of $\Delta^\vee$. On the other hand, in the nesting-graph one has to determine all lattice subpolytopes of a reflexive polytope. Note that for instance four-dimensional reflexive polytopes have at most 36 vertices but 680 lattice points [13].

3 Classification of d-dimensional flow polytopes

3.1 Definition

In [1], it was shown that every *quiver*, i.e., connected oriented graph Q without oriented cycles automatically leads to a reflexive polytope. Assume that the quiver $Q = (Q_0, Q_1)$ is given by its incidence $(\#Q_0 \times \#Q_1)$-matrix $\mathcal{I}$, i.e. each column corresponds to an arrow, and its tail and head are indicated by the entries ± 1, respectively (1 for tail, -1 for head). This leads to an exact sequence

$$\mathbb{Z}^{Q_1} \xrightarrow{\mathcal{I}} \mathbb{Z}^{Q_0} \xrightarrow{\underline{1}} \mathbb{Z} \to 0, \tag{1}$$

and we may consider flow polytopes $\Delta(\theta) := \mathbb{R}^{Q_1}_{\geq 0} \cap \mathcal{I}^{-1}_{\mathbb{R}}(\theta)$ for every weight $\theta \in \ker \underline{1} \subseteq \mathbb{Z}^{Q_0}$. These polytopes are always lattice polytopes, and for the canonical weight $\theta^c := \mathcal{I}(\underline{1})$ one even obtains a reflexive polytope

$$\Delta(Q) := \Delta(\theta^c).$$

Note: When referring to a *flow polytope* in this paper we always mean such a reflexive polytope $\Delta(Q)$ (for flow polytopes with general weights, see [11]).

3.2 Classifying flow polytopes

Given the dimension d of $\Delta(Q)$, the question is how to determine a finite list of quivers defining all d-dimensional flow polytopes up to isomorphisms.

For this, let us introduce some notation. The *valence* of the vertex of an (undirected) graph (possibly with loops) is the number of incident edges. A *loop* is an edge connecting a vertex with itself. When e is an edge of a graph G (incident to vertices v_1, v_2), let us say *we put the vertex v on the edge e* to denote a new graph that is obtained by removing e but adding a vertex of valence two, namely v, that is adjacent to v_1 and v_2 by new edges e_1, e_2. This notion is illustrated in Figure 1.

Fig. 1. Putting a vertex onto an edge

Now, we can give our main result.

Theorem 7 *Let $d \geq 2$ be an integer.*

(1) Let G' be a connected graph (possibly with loops) satisfying the following properties:
   (a) $|G'_0| \leq 2(d-1)$,
   (b) $|G'_1| = |G'_0| + d - 1$,
   (c) $\mathrm{val}(v) \geq 3$ for all $v \in G'_0$,
   (d) Any edge of G' is contained in a cycle.

(2) Let G be a loop-free graph obtained by putting vertices on some of the edges of G', at most one per each edge. In particular, we have to put a vertex on any loop of G'.

(3) Let Q be a quiver (i.e., connected oriented graph without oriented cycles) having G as its underlying graph such that for any vertex v with $\mathrm{val}(v) = 2$ the two arrows incident to v are both pointing towards v.

If G', G, Q have been chosen in this way, then $\Delta(Q)$ is a d-dimensional flow polytope, and, up to isomorphisms, any flow polytope of dimension d is obtained in this way.

Let us deduce a simple consequence, which stems from the fact that G' has at most $2(d-1)$ vertices and $3(d-1)$ edges.

Corollary 8 *For $d \geq 2$ any d-dimensional flow polytope is defined up to isomorphisms by a quiver Q with at most $5(d-1)$ vertices and at most $6(d-1)$ arrows.*

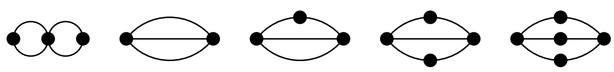
Theorem 7 shows that in order to classify d-dimensional flow polytopes it suffices to enumerate all such possible G', G, Q. We remark that the associated flow polytopes $\Delta(Q)$ may still be mutually isomorphic. In particular, isomorphisms of quivers lead to isomorphisms of flow polytopes.

3.3 Dimension two

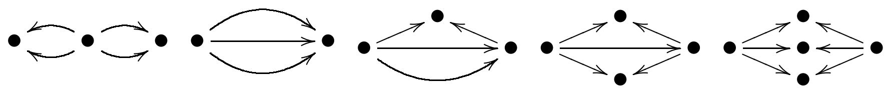
Before giving the proof of Theorem 7 we illustrate the classification procedure in dimension two.

First we determine all graphs G' satisfying the conditions in Theorem 7(1), there are precisely two of these:

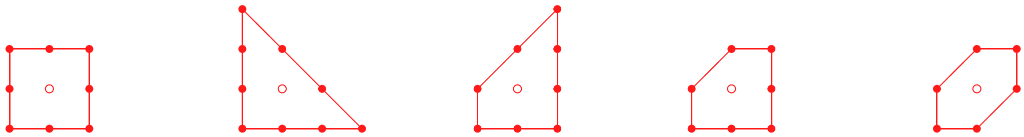
Second, we choose up to symmetry all essentially different possibilities to put vertices on the edges according to Theorem 7(2):



Three, we choose orientations for the arrows as described in Theorem 7(3). By our assumptions on Q there is (up to symmetry) for any G only one choice for Q:



Now, computing the flow polytope for these five quivers we see that they are pairwise non-isomorphic, so we have shown that there are up to isomorphisms precisely 5 flow polytopes in dimension two.



3.4 Proof of Theorem 7

For this we need a notion defined in [2].

Definition 9 Let Q be an oriented graph.

(1) Q is called *θ-stable* for a weight $\theta \in \mathbb{Z}^{Q_0}$, if for all non-empty proper subsets S of vertices ($\emptyset \neq S \subsetneq Q_0$) that are closed under arrows (no arrow of Q has tail in S but head in $Q_0 \backslash S$) the total weight is negative, i.e.,

$$\theta(S) := \sum_{v \in S} \theta(v) < 0.$$

(2) Q is called *tight* if removing any arrow yields an oriented graph that is θ^c-stable, where θ^c is the canonical weight of Q.

Note that an oriented graph Q is θ^c-stable if and only if Q is connected. This follows easily from $\theta^c(Q) = 0$, cf. [2, Proposition 7].

The following crucial observation is contained in [2, Proposition 13].

Lemma 10 *For any quiver there exists a tight quiver such that the associated flow polytopes are isomorphic.*

The precise statement is that, if removing an arrow yields an oriented graph that is not θ^c-stable, then we may contract this arrow without changing the isomorphism type of the associated flow polytopes.

For tight quivers we may deduce the following statements:

Lemma 11 *Let Q be a quiver. Let $v \in Q_0$ be a vertex with* $\mathrm{val}(v) = 2$.

(1) If Q is tight, then the two arrows incident to v are either both pointing towards v or both pointing away from v.

(2) *Inverting the orientation of both arrows incident to v yields a quiver defining a flow polytope isomorphic to $\Delta(Q)$.*
(3) *Let e, e' be the arrows incident to v, both pointing towards v. Then putting a vertex v' on the edge e, and orientating both new edges incident to v' as pointing away from v' yields a quiver defining a flow polytope isomorphic to $\Delta(Q)$. In Figure 2 the statement is illustrated in the only two possible cases.*

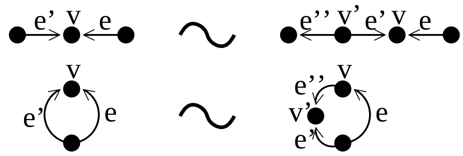


Fig. 2. The two cases of Lemma 11(3)

Proof. (1): Assume one arrow e incident to v was pointing inwards to v, the other arrow outwards. Then removing e yields an oriented graph that is not stable with respect to the canonical weight of Q, a contradiction to the tightness of Q. This is seen by regarding the set $S := Q_0 \backslash \{v\}$. (2): This is left to reader. (3): Let Q' denote the new quiver, having one vertex and one arrow more than Q. We have to compare the kernels of the two incidence matrices $\mathcal{I}, \mathcal{I}'$ associated to Q and Q'. There are two cases illustrated in Figure 2. As the second case uses a similar argument, let us only deal with the first case. Here, the relevant rows of the respective incidence matrices look as follows:

$$\begin{pmatrix} \cdots A \cdots & 1 & 0 & \cdots B \cdots \\ \cdots 0 \cdots & -1 & -1 & \cdots 0 \cdots \\ \cdots C \cdots & 0 & 1 & \cdots D \cdots \end{pmatrix} \quad , \quad \begin{pmatrix} \cdots A \cdots & -1 & 0 & 0 & \cdots B \cdots \\ \cdots 0 \cdots & 1 & 1 & 0 & \cdots 0 \cdots \\ \cdots 0 \cdots & 0 & -1 & -1 & \cdots 0 \cdots \\ \cdots C \cdots & 0 & 0 & 1 & \cdots D \cdots \end{pmatrix}$$

From this one deduces that the projection $\mathbb{Z}^{Q'_1} \to \mathbb{Z}^{Q_1}$, given by forgetting about the entry corresponding to the new edge e'' of Q'_1, induces a lattice isomorphism between $\ker \mathcal{I}$ and $\ker \mathcal{I}'$. This yields the statement. □

Now we are in the position to prove Theorem 7. By Lemma 10, we may assume we have a tight quiver Q whose flow polytope $\Delta(Q)$ has dimension $d \geq 2$. From the exact sequence (1) we can read off the dimension from the quiver:

$$d = \dim(\Delta(Q)) = |Q_1| - |Q_0| + 1.$$

In particular, we see that putting a vertex of an edge (and then orientating the edge) does not change the dimension of the associated flow polytope.

Let G be the graph underlying the quiver Q. Double counting yields

$$\sum_{v \in G_0} \mathrm{val}(v) = 2|G_1|,$$

or equivalently

$$\sum_{v \in G_0} (\mathrm{val}(v) - 2) = 2(d-1). \tag{2}$$

Since $d \geq 2$, there is at least one vertex of valence larger than two. Now, by Lemma 11, we may assume that there are no two adjacent vertices of valence two, and, moreover, that all vertices of Q of valence two may be supposed to have their incident arrows pointing inwards. Hence, G is obtained by putting vertices on different edges of a unique graph G', where all vertices of G' have valence at least three. In particular, by equation (2), G' satisfies conditions (a),(b),(c).

It remains to show condition (d). So assume that there is an edge of G' that is not contained in a cycle. This implies that there is also an arrow e of Q that is not contained in an oriented cycle of Q. Therefore, the tail of e is contained in a proper subset S of Q_0 such that e is the only arrow of Q whose tail is in S and whose head in $Q_0 \backslash S$. Hence, we see that removing e yields an oriented graph that is not stable with respect to the canonical weight of Q, a contradiction to the tightness of Q.

This finishes the proof of Theorem 7. □

4 Computational results in dimensions two, three, and four

Reflexive polytopes were classified in dimensions $2, 3, 4$ by Kreuzer and Skarke using the computer package `PALP` [14]. Moreover, `PALP` allows the computation of nef-partitions and normal forms of lattice polytopes with respect to isomorphisms. Based on this, a `Perl`-script was implemented to calculate the graphs $\mathcal{R}_2$ and $\mathcal{R}_3$ on a standard computer, see [17]. In the case of flow polytopes of small dimensions, a list of possible quivers is easy to compile given the results of the previous section. Then given a quiver Q the corresponding flow polytope $\Delta(Q)$ can be computed using `Polymake` [9]. This was achieved in [16] via a `Perl`-script. All programs can be downloaded from [18].

The point for considering the set of reflexive polytopes as a graph $\mathcal{R}_d$ is the following: The edges given by the mutations of reflexive polytopes provide an easy way to pass from one reflexive polytope to another one. In particular, as soon as one knows one reflexive polytope from a certain connected component of $\mathcal{R}_d$, one can easily obtain the whole component. On the other hand, the elementary construction of flow polytopes provides a couple of initial vertices in the graph $\mathcal{R}_d$.

Now, the original hope was that every component would contain at least one initial vertex. In the present form this already fails in dimension two. Here, there are 16 reflexive polygons up to lattice isomorphisms. The graph $\mathcal{R}_2$ is shown in the figure of Section 2, the flow polytopes are presented in red. Still, this method covers almost all reflexive polytopes – only $\mathbb{P}(1,2,3)$ is left. For future considerations, further methods of providing initial polytopes as well as mutations are required. The latter would merge different connected components.

In dimension three, we had to check the list of all 4319 reflexive polytopes. While the graph $\mathcal{R}_3$ splits into 1056 connected components, almost every other reflexive polytope is captured by a single one, the "big" component containing 1868 vertices. Here is the complete list of components:

# of components	size
1	1868
1	64
1	14
1	12
1	10

# of components	size
8	8
30	6
68	4
890	2
55	1

There are 55 isolated polytopes, and these are necessarily selfdual, i.e., isomorphic to their dual. They play the same role as $\mathbb{P}(1,2,3)$ does in dimension two. The total number of selfduals is 79, and of the remaining 24 all but two sit in the big component. The two exceptions are contained in the unique component with ten elements. It also contains the $\mathbb{P}(1,2,3)$ "derivatives", i.e., the pyramid, the double pyramid, and the prism over this polygon. Here, given a reflexive polytope $P \subset \mathbb{R}^d$, we define the pyramid, double pyramid, and prism over P as $\text{conv}(P \times \{-1\}, (0,1))$, $\text{conv}(P \times \{0\}, (0,1), (0,-1))$, and $P \times [-1,1]$ respectively. These are again reflexive polytopes, and it is easy to see that they necessarily lie in a common connected component of $\mathcal{R}_{d+1}$. Moreover, prisms over flow polytopes are again flow polytopes.

In dimension three, the classification procedure described in Section 3 yields 55 isomorphism classes of quivers. From this list we get exactly 39 flow polytopes, where none is dual to each other. Sadly all of them, up to one exception, are contained in the big component and do not contribute to a better understanding of reflexive polytopes. The exception stems from the quiver given in Figure 3 below and belongs to the component with 14 elements.

In dimension four, only partial computations could be performed because of limited memory resources. In particular, we focused only on nef-partitions and ignored lattice mutations. It could be calculated that the 'big' component of those reflexive

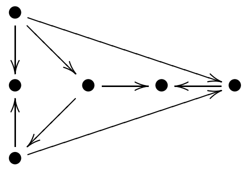


Fig. 3. Quiver giving the only flow 3-polytope outside of the big component

polytopes that can be reached via nef-partitions starting from the reflexive polytope associated to $\mathbb{P}^4$ contains 1.538.185 reflexive polytopes. Compared to the number of nearly half a billion reflexive 4-polytopes this is an astonishingly small amount, and it reflects the diversity of reflexive polytopes as the dimensions grows. The computation of flow polytopes, and the connected components they are contained in, in dimension four, and even more interestingly in dimension five, is part of a future project.

5 Outlook: the number of vertices and facets

In dimension three there is exactly one three-dimensional reflexive polytope attaining the maximal possible number of vertices – 14. As in the two-dimensional case, where it was the hexagon, this is a flow polytope. It is given by the quiver in Figure 4. In general, the maximal number of vertices of a d-dimensional reflexive polytope is unknown. However, even in dimension d it is conjectured that the $d/2$-fold product of the hexagon attains the maximal number of vertices, cf. [15]. Since this reflexive polytope is again a flow polytope, determining the possible number of vertices for flow polytopes would give some evidence or even produce counterexamples to this open question. However, this seems to be a difficult task, cf. [11, Lemma 3.1], which we leave open at the moment.

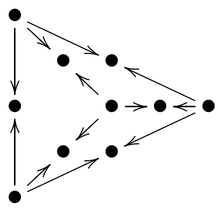


Fig. 4. Quiver giving the unique flow 3-polytope with 12 facets

Here, we determine as a consequence of Theorem 7 the maximal number of *facets* of a d-dimensional flow polytope.

Corollary 12 *For $d \geq 2$ any d-dimensional flow polytope $\Delta(Q)$ has at most $6(d-1)$ facets. Equality is attained if and only if in Theorem 7 the graph G is obtained by putting vertices on any edge of a 3-regular loop-free graph G' with $2(d-1)$ vertices.*

Proof. Let G', G, Q be given as in Theorem 7. Since Q is tight, the number of facets of $\Delta(Q)$ equals the number of arrows of Q, by [2, Cor. 8]. Hence, the upper bound follows from Corollary 8. Now, let G have $6(d-1)$ edges. Then, since $|G'_1| = |G'_0| + d - 1 \leq 3(d-1)$, G is obtained by putting a vertex on any edge of G'. In particular, $|G_0| = |G'_0| = 2(d-1)$. From equation (2) we deduce that any vertex of G' has valence three. Finally, assume there is a loop in G' at a vertex v. Since $\mathrm{val}(v) = 3$, there is a unique edge e incident to v that is not a loop. Hence, e is not contained in a cycle, a contradiction to property (d) of G'. □

The corollary proves once again the observation that the hexagon and the flow polytope given by the quiver in Figure 4 are actually the only flow polytopes in dimension 2 and 3, with the maximal number of facets, namely 6 and 12, respectively. However, in dimension 4 there are precisely two flow polytopes with 18 facets. This follows, since there are precisely two 3-regular loop-free graphs with 6 vertices, and the corresponding quivers Q_1, Q_2 shown in Figure 5 lead to non-isomorphic reflexive polytopes.

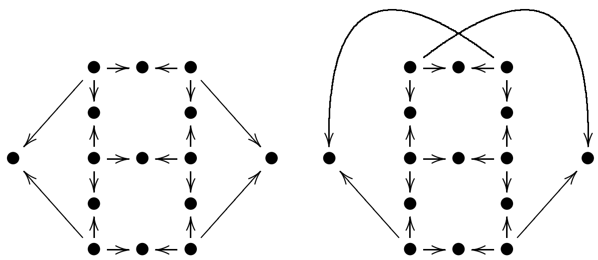


Fig. 5. Quivers giving flow 4-polytopes with 18 facets and 28 and 30 vertices, respectively

In general, one may conjecture that the surjective map between isomorphism classes of 3-regular loop-free graphs and isomorphism classes of d-dimensional flow polytopes with $6(d-1)$ facets is even a bijection.

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