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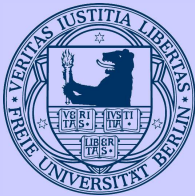
*Proof of Theorem I*

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# The degree of lattice polytopes

Benjamin Nill - FU Berlin

University of Minnesota, October 2, 2007



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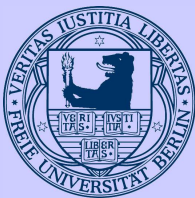
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# 1. Introduction



# The $h^*$ -polynomial

Throughout:

- $M \cong \mathbb{Z}^r$  is a lattice,  $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$ .
- $P \subseteq M_{\mathbb{R}}$  is  $n$ -dimensional lattice polytope.

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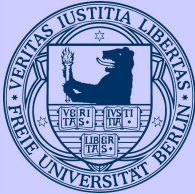
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## Theorem. [Stanley 1980]

There is a polynomial  $h^*(t) \in \mathbb{Z}[t]$  of degree  $\leq n$  with nonnegative integer coefficients such that

$$\sum_{k \geq 0} |kP \cap M| t^k = \frac{h^*(t)}{(1-t)^{n+1}}.$$



## Definition:

- $h_P^*(t) := h^*(t)$  is called  $h^*$ -**polynomial** of  $P$ .
- $h_P^* := (h_0^*, \dots, h_n^*)$  is called  $h^*$ -**vector** of  $P$ .

## Observations:

- $h_0^* = 1$ .
- $h_1^* = |P \cap M| - (n + 1)$ .
- $h_n^* = |\text{int}(P) \cap M|$ .
- $h_0^* + \dots + h_n^* = \text{Vol}(P) = n! \text{ vol}(P)$ .  
(from now on: volume = normalized volume)

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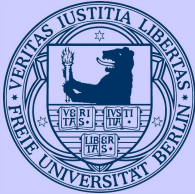
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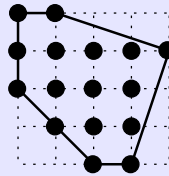
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**Example:**  $n = 2$



$$1 + 13t + 8t^2$$

$$h_P^*(t) = 1 + (|P \cap M| - 3) t + |\text{int}(P) \cap M| t^2.$$

**Pick's formula:**

$$\text{Vol}(P) = h_P^*(1) = |\partial(P) \cap M| + 2|\text{int}(P) \cap M| - 2.$$

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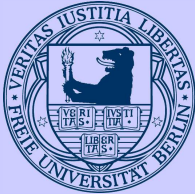
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# The degree

**Definition:** The **degree** of  $P$  is defined as the degree of the  $h^*$ -polynomial.

$$\deg(P) := \max(i : h_i^* \neq 0).$$

Hence

$$0 \leq \deg(P) \leq n = \dim(P).$$

**Philosophy:** The degree of a lattice polytope may be regarded as its "true" dimension.

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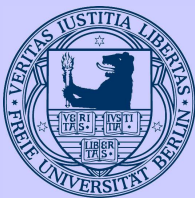
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# The codegree

**Definition:** The **codegree** of  $P$  is defined as:

$$\text{codeg}(P) := \min\{k \in \mathbb{Z}_{\geq 0} : \text{int}(kP) \cap M \neq \emptyset\}$$

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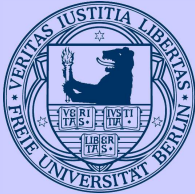
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Reciprocity-Theorem  $\Rightarrow$

$$1 \leq \boxed{\text{codeg}(P) = n + 1 - \deg(P)} \leq n + 1,$$

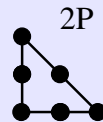
$$h_{\deg(P)}^* = |\text{int}(\text{codeg}(P)P) \cap M|.$$



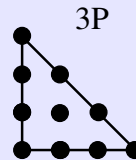
## Example: $n = 2$



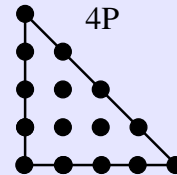
codeg 3  
1



codeg 2  
 $1+3t$



codeg 1  
 $1+7t+t^2$



codeg 1  
 $1+12t+3t^2$

## Observations:

- $\text{codeg}(P) = 1 \Leftrightarrow \deg(P) = n \Leftrightarrow \text{int}(P) \cap M \neq \emptyset$
- $\text{codeg}(P) = n + 1 \Leftrightarrow \deg(P) = 0 \Leftrightarrow P$  is unimodular simplex.

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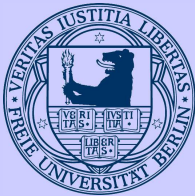
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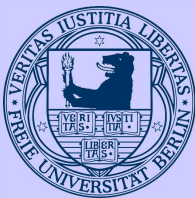
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## 2. Classification of lattice polytopes of degree one



## Dimension two

Let  $P \subseteq M_{\mathbb{R}}$  be a lattice polygon.

$\deg(P) \leq 1 \Leftrightarrow P$  has no interior lattice points.

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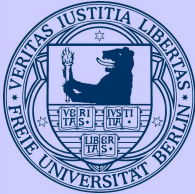
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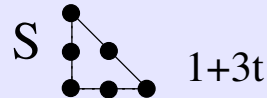
**Theorem. [Arkinstall 1980; Koelman 1991; Khovanskii 1997; Schicho 2003]**

There are precisely *two* cases, in which  $P$  has no interior lattice points:



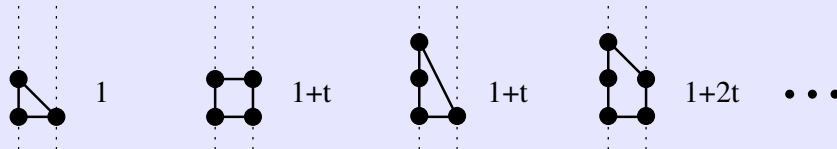
either

$P$  is the **exceptional triangle**  $S$ :



or

$P$  lies between two parallel hyperplanes of integral distance one:



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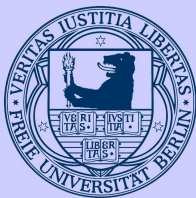
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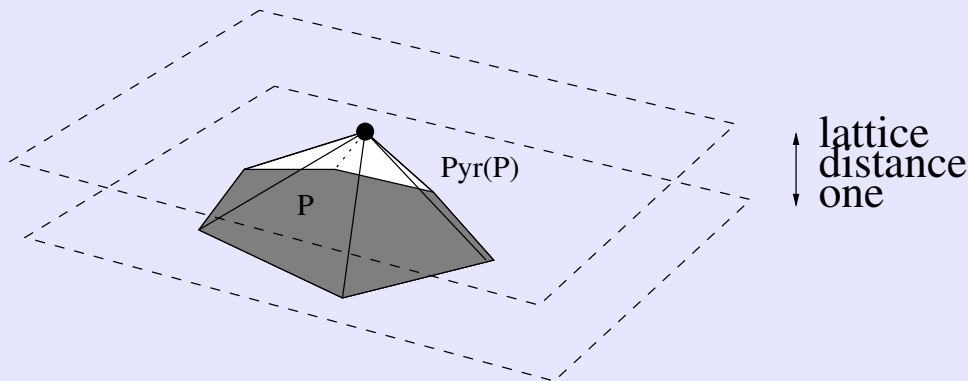
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# Lattice pyramid construction

**Definition:** The **lattice pyramid** over  $P$  is defined as  $\text{Pyr}(P) := \text{conv}(\{0\} \times \{0\}, P \times \{1\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$ .



Recursively:  **$k$ -fold pyramid** over  $P$ :

$$\text{Pyr}^k(P) := \text{Pyr}(\text{Pyr}^{k-1}(P)).$$

**Proposition:**  $P$  and  $\text{Pyr}(P)$  have same  $h^*$ -polynomial.  
 $\Rightarrow$  Degree and volume is unchanged!

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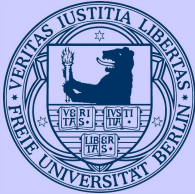
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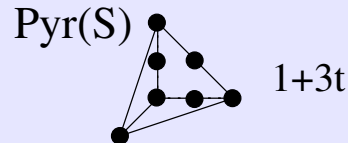
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# Higher dimensions

**Definition:** An  $n$ -dimensional **exceptional simplex** is an  $(n - 2)$ -fold lattice pyramid over  $S$ .

**Example:**  $n = 3$



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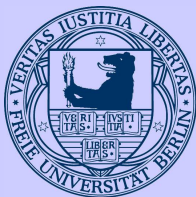
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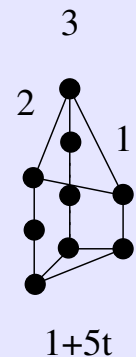
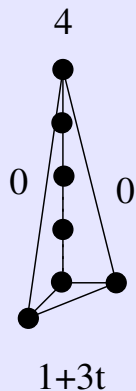
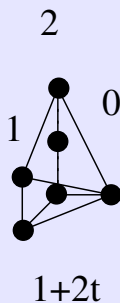
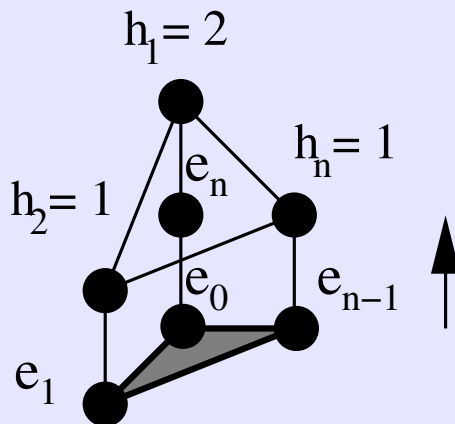
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**Definition:** An  $n$ -dimensional **Lawrence prism** with heights  $h_1, \dots, h_n$ :



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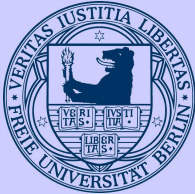
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# Classification result

## Theorem. [Batyrev, N. 2006]

A lattice polytope has degree  $\leq 1$  if and only if it is an exceptional simplex or a Lawrence prism.

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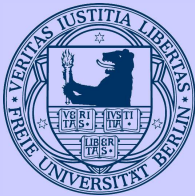
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Proof by induction on the number of lattice points, using the monotonicity property of the degree:

## Theorem. [Stanley 1993]

$Q \subseteq P$  lattice polytopes  $\Rightarrow h_Q^* \leq h_P^*$  coefficientwise.



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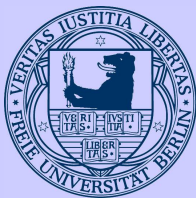
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### 3. How many lattice polytopes with given $h^*$ -polynomial?

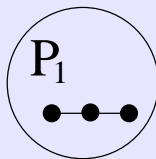




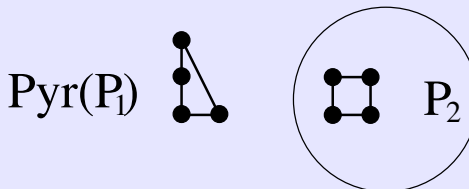
# Case of degree one

**Example:**  $h^* = 1 + t$  (deg = 1 and Vol = 2):

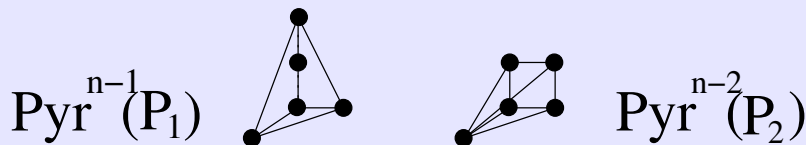
- $n = 1$ :



- $n = 2$ :



- $n \geq 3$ :



**Proposition:** An  $n$ -dimensional lattice polytope  $P$  of degree one is a lattice pyramid, if  $n > \text{Vol}(P)$ .

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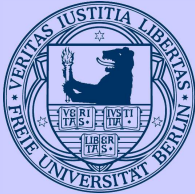
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# The general case

## Theorem. [Batyrev 2006]

A lattice polytope of dimension  $n$ , volume  $V$ , and degree  $d$  is a **lattice pyramid**, if

$$n \geq 4d \binom{2d + V - 1}{2d}.$$

Proof relies heavily on commutative algebra.

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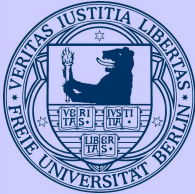
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**Definition:**  $C(V, d, n) :=$  number of isomorphism classes of lattice polytopes of volume  $V$ , degree  $d$ , dimension  $n$ .

- *finite* (theorem of Lagarias and Ziegler),
- *upper monotone in  $n$*  (pyramid construction),
- *eventually become constant* (Batyrev's result).

**Corollary:** There are only **finitely** many lattice polytopes with the same  $h^*$ -polynomial up to isomorphisms and pyramid constructions.

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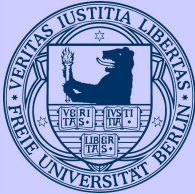
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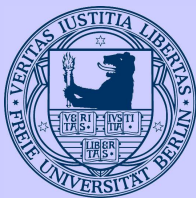
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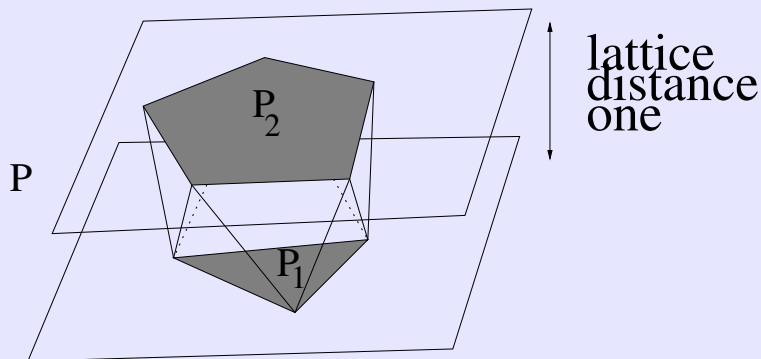
## 4. What about lattice polytopes of given degree?



# The Cayley polytope conjecture

**Definition:** A lattice polytope  $P$  is a **Cayley polytope**

$$P = P_1 * P_2, \quad \text{if}$$



On the other hand, let  $P_1, \dots, P_s \subseteq M_{\mathbb{R}}$  be given.

**Definition:**  $P_1 * \dots * P_s$  is defined as

$$\text{conv}(P_1 \times \{e_1\}, \dots, P_s \times \{e_s\}) \subseteq M_{\mathbb{R}} \oplus \mathbb{Z}^s,$$

where  $e_1, \dots, e_s$  is a lattice basis of  $\mathbb{Z}^s$ .

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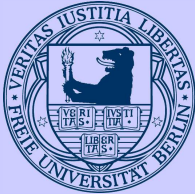
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**Conjecture I [Batyrev, N.]:** Let  $d$  be fixed.

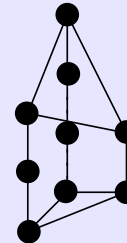
There exists  $N$  such that any lattice polytope of degree  $d$  and dimension  $n$  is a **Cayley polytope**, if

$$n \geq N.$$

**Example:** Conjecture I holds for  $d = 1$  with  $N = 3$ .

- Pyramids are Cayley polytopes.
- Lawrence prisms are Cayley polytopes of segments.

$$[0,2] * [0,3] * [0,1]$$



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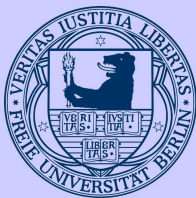
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# A surprise ...

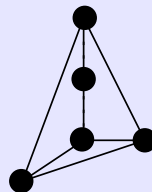
## Theorem 1. [N. 2007]

A lattice simplex of degree  $d$  and dimension  $n$  is a **lattice pyramid**, if

$$n \geq 4d - 1.$$

**Example:** A lattice simplex of degree  $d = 1$  is a lattice pyramid, if  $n \geq 3$ .

- Exceptional simplices are lattice pyramids for  $n \geq 3$ .
- Simplices that are Lawrence prisms are lattice pyramids for  $n \geq 2$ .



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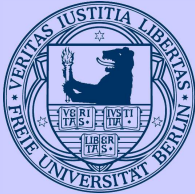
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## ... and its generalization

### Theorem II. [N. 2007]

A lattice polytope of degree  $d$  and dimension  $n$  is a **lattice pyramid**, if

$$n \geq (f_0 - n - 1)(2d + 1) + 4d - 1,$$

where  $f_0$  equals the number of vertices of  $P$ .

Proofs of Theorems I and II are purely combinatorial.

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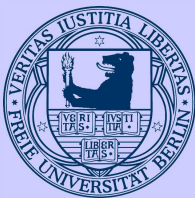
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# Back to $h^*$ -polynomial

$$f_0 - n - 1 \leq |P \cap M| - n - 1 = h_1^* \Rightarrow$$

**Corollary:** A lattice polytope of degree  $d$  and dimension  $n$  is a **lattice pyramid**, if

$$n \geq h_1^*(2d + 1) + 4d - 1.$$

Fine tuning of Theorem II  $\rightsquigarrow$

**Corollary:** A lattice polytope of dimension  $n$ , volume  $V$ , and degree  $d$  is a **lattice pyramid**, if

$$n \geq (V - 1)(2d + 1) \approx O(V d).$$

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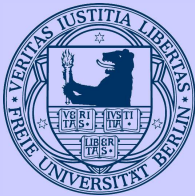
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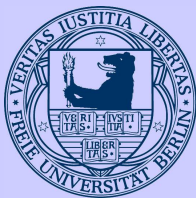
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## 5. Proof of Theorem I



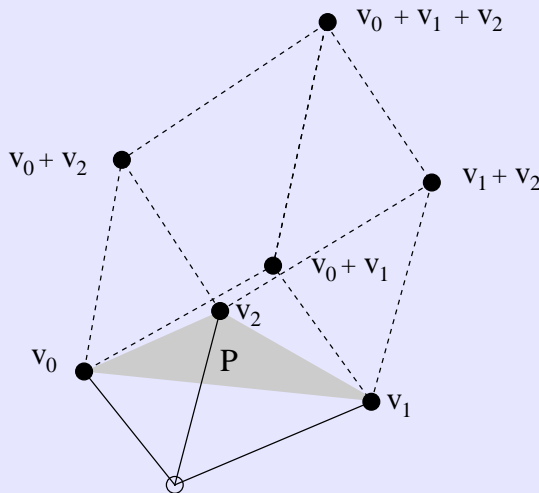
# The setup

Let  $P$  be an  $n$ -dimensional lattice simplex of degree  $d$ .

$$P := \text{conv}(v_0, \dots, v_n) \subseteq M_{\mathbb{R}} \times \{1\}, \quad \overline{M} := M \oplus \mathbb{Z}.$$

**Definition:** (half-open) **paralelepiped**

$$\Pi(P) := \left\{ \sum_{i=0}^n \lambda_i v_i : 0 \leq \lambda_i < 1 \right\}.$$



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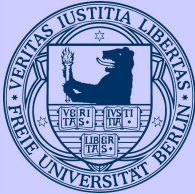
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# The support

1. Let  $m \in M_{\mathbb{R}} \oplus \mathbb{R}$ . Then

$$m = \sum_{i=0}^n \lambda_i v_i \quad \text{for } \lambda_i \in \mathbb{R},$$

$$\text{supp}(m) := \{i \in \{0, \dots, n\} : \lambda_i \neq 0\}.$$

2. 
$$\text{supp}(P) := \bigcup_{m \in \Pi(P) \cap \overline{M}} \text{supp}(m).$$

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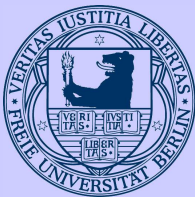
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# A simple reformulation

**Lemma:**

$P$  lattice pyramid  $\Leftrightarrow \text{supp}(P) \subsetneq \{0, \dots, n\}$

**Proof:** Follows from

$$\text{Vol}(P) = |\Pi(P) \cap \overline{M}|.$$

T.f.a.e. for  $P' := \text{conv}(v_0, \dots, v_{n-1})$ :

- $P$  is a lattice pyramid over  $P'$  (with apex  $v_n$ )
- $\text{Vol}(P) = \text{Vol}(P')$
- $\Pi(P) \cap \overline{M} = \Pi(P') \cap \overline{M}$ .
- $\text{supp}(P) \subseteq \{0, \dots, n-1\}$



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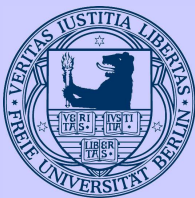
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# Line of argument

Goal:

$$|\text{supp}(P)| \leq 4d - 1.$$

*Idea: Cover the support of  $P$  in a greedy manner.*

Choose recursively lattice points

$$m_0, m_1, m_2, \dots \in \Pi(P) \cap \overline{M}$$

such that

$$|\text{supp}(m_0)| \text{ maximal,}$$

$$|\text{supp}(m_0) \cup \text{supp}(m_1)| \text{ maximal,}$$

$$|\text{supp}(m_0) \cup \text{supp}(m_1) \cup \text{supp}(m_2)| \text{ maximal,}$$

...

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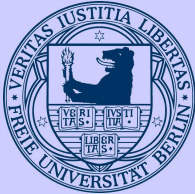
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**Claim:**

$$|\text{supp}(m_0)| \leq 1 \cdot 2d,$$

$$|\text{supp}(m_0) \cup \text{supp}(m_1)| \leq \left(1 + \frac{1}{2}\right) \cdot 2d,$$

$$|\text{supp}(m_0) \cup \text{supp}(m_1) \cup \text{supp}(m_2)| \leq \left(1 + \frac{1}{2} + \frac{1}{4}\right) \cdot 2d,$$

...

Finiteness yields

$$|\text{supp}(P)| = \left| \bigcup_{j=0}^{\infty} \text{supp}(m_j) \right| < 2 \cdot 2d = 4d.$$

Therefore

$$|\text{supp}(P)| \leq 4d - 1.$$

□

Proof of claim by induction.

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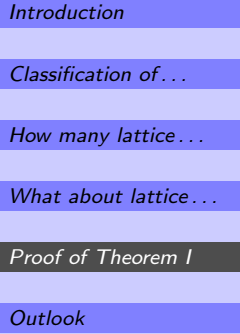
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In the case of a *simplex* the following equation holds:

where  $\text{ht}(m)$  equals its last coordinate.

ht=3

ht=2

ht=1

ht=0

$v_0 + v_1 + v_2$

$v_0 + v_2$

$v_1 + v_2$

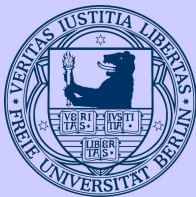
$v_0 + v_1$

$v_0$

$v_1$

$v_2$





# Proof of claim for $m_0$

We have to show

$$m \in \Pi(P) \cap \overline{M} \Rightarrow |\text{supp}(m)| \leq 2d.$$

The previous slide yields

$$d = \deg(P) \geq \text{ht}(m).$$

Let

$$\text{supp}(m) = \{0, \dots, s\},$$

$$P' := \text{conv}(v_0, \dots, v_s)$$

$$\Rightarrow m \in \text{int}(\Pi(P')) \cap \overline{M}.$$

This yields

$$d \geq \text{ht}(m) \geq \text{codeg}(P') = s + 1 - \deg(P')$$

$$\geq s + 1 - \deg(P) = s + 1 - d.$$

Hence

$$|\text{supp}(m)| = s + 1 \leq 2d.$$



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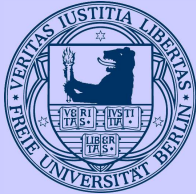
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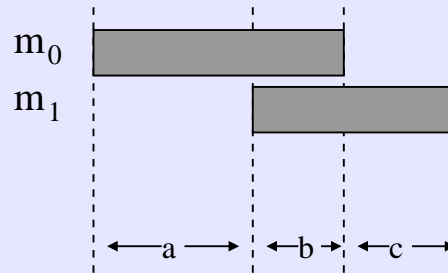
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# Proof of induction step $m_0 \rightarrow m_1$

Schematic figure of  $\text{supp}(m_0) \cup \text{supp}(m_1) \subseteq \{0, \dots, n\}$ :



$$b + c = |\text{supp}(m_1)| \leq |\text{supp}(m_0)| = a + b. \quad (1)$$

We are going to show

$$c \leq d = \deg(P).$$

Since then

$$|\text{supp}(m_0) \cup \text{supp}(m_1)| = a + b + c \leq 2d + d = (1 + \frac{1}{2})2d.$$

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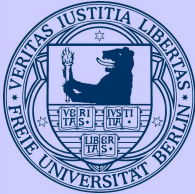
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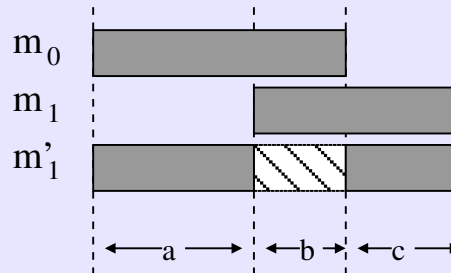


$$m_0 = \sum_{i=0}^n \lambda_i v_i, \quad m_1 = \sum_{i=0}^n \mu_i v_i.$$

Then

$$m'_1 := \sum_{i=0}^n \{\lambda_i + \mu_i\} v_i \in \Pi(P) \cap \overline{M},$$

where  $\{\gamma\} \in [0, 1[$  is the fractional part of a real number  $\gamma$ .



$$a + c \leq \text{supp}(m'_1) \leq \text{supp}(m_0) = a + b,$$

so

$$c \leq b. \quad (2)$$

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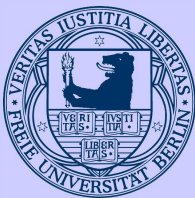
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Combining

$$b + c \leq a + b \quad (1)$$

and

$$c \leq b \quad (2)$$

yields

$$c \leq b \leq a + b - c,$$

so

$$2c \leq a + b = \text{supp}(m_0) \leq 2d,$$

$$\Rightarrow c \leq d.$$



The general induction step works similarly.

This finishes the proof of Theorem I.

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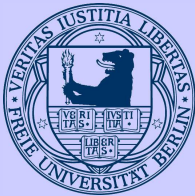
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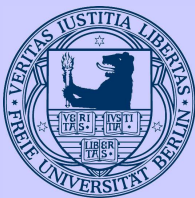
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## 6. Outlook



# The leading term conjecture

**Conjecture:** Let  $i \in \{1, \dots, d\}$ .

There exists a constant  $N$  such that any  $n$ -dimensional lattice polytope of degree  $d$  and with coefficient  $h_i^*$  is a **lattice pyramid**, if

$$n > N.$$

For  $i = d$  the conjecture is equivalent to:

**Conjecture II [Batyrev]:**

There exists a **uniform upper bound** on the volume of lattice polytopes of degree  $d$  and leading coefficient  $h_d^*$ .

Holds for  $d \leq 2$  due to Treutlein (2007).

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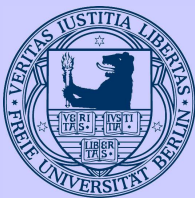
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# The Cayley conjecture - refined

**Conjecture I'**: Let  $d$  be fixed.

There exists  $N$  such that any lattice polytope  $P$  of degree  $d$  and dimension  $n \geq N$  is a **Cayley polytope**  $P_1 * \dots * P_s$  of non-empty lattice polytopes for  $s = n + 2 - N$ .

Holds for  $d = 1$  with  $N = 3$  by the classification.

**Theorem. [Haase, N., Payne 2007]**

Conjecture I' implies Conjecture II.

Conjecture I' also implies the qualitative statements of Batyrev's result and its generalization Theorem II.

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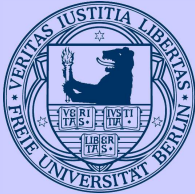
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# Gorenstein polytopes

**Definition:**  $P$  is **Gorenstein**, if  $h_P^*$  is symmetric.  
In particular,  $h_d^* = h_0^* = 1$ .

**Proposition [N.]:**

Conjecture I' holds for Gorenstein polytopes.

**Corollary:**

There exists a uniform bound on the volume of Gorenstein polytopes of degree  $d$ .

Proposition follows from combining

- [Batyrev, Borisov 1997] criterion for Gorenstein polytopes in terms of **Gorenstein cones**,
- [Batyrev, N. 2007] characterization of Cayley polytopes by **special simplices**.

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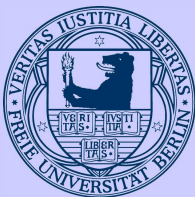
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