Poincaré group) $\mathcal{N}$ fermionic spinorial generators Q^A_α, and their complex conjugate ($A = 1, \ldots, \mathcal{N}$), plus **central charges**. The Lorentz scalar **central charges** can emerge only in *extended supersymmetries*, at most $(\mathcal{N}/2) \times (\mathcal{N} - 1)$ of such charges. For instance, the *BPS saturation* of the magnetic **monopoles** in $\mathcal{N} = 2$ gauge theories is related to the existence of the **central charge**. The **supermultiplets** with mass = **central charge** are shortened.

In *supersymmetric theories* with *extended objects*, such as *vortices* and domain walls, tensorial **central charges** can occur in N = 1 theories. For instance, the **central charge** $Z_{\alpha\beta}$ lying in the representation (0,1) of the Lorentz group appears in

$$\{Q_\alpha, Q_\beta\} = 2Z_{\alpha\beta}.$$

It is related to the domain walls. The (1/2,1/2) **central charge** is related to the *vortices*, it can be found in the gauge theories with an *Abelian subgroup* and the *Fayet–Iliopoulos term*. In the *non-Abelian gauge theories* the (0,1) and (1,0) **central charges** are generated [3] as quantum anomalies in the superalgebra. The possibility of existence of the tensorial **central charges** in N = 1 superalgebras was noted in [4]. For a systematic discussion of the **central charges** see [5].

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Mikhail Shifman

Chan–Paton Factor — Group theory factor T_n that can be attached to **open string** *n-point* amplitudes while respecting cyclicity and **planar duality**. In the simplest case it is a trace of a product of matrices in the *fundamental representation* of a unitary group $U(N)$: $T_n = tr(\lambda_1\lambda_1 \ldots \lambda_n)$. Unitarity in the factorization of tree-level (disk) amplitudes combined with the action of the twist operator Ω allow only classical *Lie groups* (unitary, $U(N)$ orthogonal $O(N)$ and symplectic $Sp(2N)$), whose matrices in the *fundamental representation* form an algebra under multiplication. Because of the presence of a massless vector excitation in any *open string theory* the *Chan–Paton group* G_{CP} becomes a local *gauge symmetry* for the theory. For the *type I* open superstring in D = 10 cancellation of chiral gravitational and gauge anomalies through the *Green–Schwarz mechanism* requires $G_{CP} = SO(32)$. In more general backgrounds with open and **unoriented strings**, consistency of the theory is compatible with product group structure with singlets, bi-fundamental, (anti)-symmetric and adjoint representations of the *Chan–Paton group*. Cancellation of chiral gravitational and gauge anomalies through a generalization of the *Green–Schwarz mechanism* is equivalent to the cancellation of the potential **tadpoles** of massless unoriented closed string states belonging to *Ramond–Ramond sectors* $\mathcal{H}_k$ of the spectrum with nonvanishing worldsheet *Witten index* $\mathcal{I}_k = tr_{\mathcal{H}_k}(-)^F \neq 0$.

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Massimo Bianchi

Character Formula — A formula that generalizes *Weyl formula* for the **character** of irreducible finite dimensional module over any simple finite dimensional **Lie algebra** to not necessarily finite dimensional nor *simple Lie algebras* and **Lie superalgebras**.

Let $\mathfrak{h}$ be **Cartan subalgebra** of a **Lie superalgebra** $\mathfrak{g}$. For $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-modules M such that the *weight* subspaces M^χ of *weight* χ are finite dimensional define the **character** $\mathrm{ch}(M)$ of M as an element of $S[[\mathfrak{h}^*]]$ by the formula

$$\mathrm{ch}_M = \sum_{\xi \in \mathfrak{h}^*_{\mathbb{Z}}} \dim M(\xi) e^\xi.$$

By setting $\xi = 0$ one has $\dim M$. Since

$$\mathrm{ch}(M \oplus N) = \mathrm{ch}(M)\mathrm{ch}(N),$$

we see that for a 1-dimensional space M of *weight* λ the **characters** of the symmetric and the **exterior algebra** are as follows

$$\mathrm{ch}(S(M)) = 1 + \lambda + \lambda^2 + \cdots = \frac{1}{1-\lambda},$$

$$\mathrm{ch}(E(M)) = 1 + \varepsilon\lambda,$$

where ε is the *superdimension* of 1-dimensional purely odd space, i.e., $\varepsilon^2 = 1$. Thanks to the Poincaré-Birkhof-Witt theorem these formulas enable one to compute **character** formulas of the *induced modules* or (for superalgebras) the "*typical*" *modules* [1]. The *irreducible modules* are often not induced ones and the corresponding **character** formulas can be very complicated and are sometimes unknown even for Lie (super)algebras of huge interest, cf. [2–8].

The *Shapovalov determinant* is a useful criterion for determination whether the induced module is induced [9,10,11].

For the **character** formula of irreducible finite dimensional modules over simple finite dimensional **Lie algebras** and its generalization to infinite dimensional, not necessarily simple, **Lie algebras** and **Lie superalgebras** see [2–8].

Character formula is a source of various combinatorial identities, cf. [7,10].

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Dimitry Leites

CHARACTER, in CFT — A map from the upper half plane **H** into the complex numbers defined via the dimensions of certain sub-modules. **Rational conformal field theories** are defined by vertex algebras V which have finitely many *irreducible modules* R_j, where j is element of some index set. Associated to these are (normalized) **characters** as maps $\mathrm{ch}_{R_j} : \mathbf{H} \longrightarrow \mathbf{C}^*$ defined by

$$\mathrm{ch}_j(\tau) = \mathrm{tr}_{R_j} q^{L_0 - \frac{c}{24}},$$

where c is the rank of the vertex algebra and $q = e^{2\pi\tau}$, for τ in the upper half plane **H**. The vertex algebra V is a vector space with a grading induced by the eigenspaces of the operator L_0 and the **character** ch_{R_j} as defined reflects this L_0 grading of V.

The importance of the **characters** derives from the fact that they provide a representation of the *modular group* SL(2,**Z**) over the integers **Z** of the **torus**. On the half plane the action of this group is given by

$$SL(2, Z) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a\tau + b}{c\tau + d}.$$

The representation of SL(2,**Z**) on the **characters** is usually described in terms of the generators $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which induce the matrices S and T respectively via $\mathrm{ch}_{R_j}(-1/\tau) = \sum_k S_{jk}\mathrm{ch}_k(\tau)$ and $\mathrm{ch}_j(\tau+1) = \sum_k T_{jk}\mathrm{ch}_k(\tau)$.

The modular behavior illuminates the physical behavior of the theory since it enters in the *torus partition function* of the theory, defined by the trace over the total *Hilbert space H*

$$Z(\tau) = tr_H q^{L_0 - \frac{c}{24}} \bar{q}^{L_0 - \frac{c}{24}}.$$

H admits the decomposition $H = \bigoplus_{i,j} N_{ij} R_i \otimes R_j$ into the R_j modules with multiplicities N_{ij} and therefore the *partition function* becomes

$$Z(\tau) = \sum_{ij} N_{ij} ch_i(\tau) \overline{ch_j(\tau)},$$

an extremely useful form for many applications.

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Rolf Schimmrigk

CHERN–SIMONS FORM — We consider a group G with generators T^I, the gauge potential 1-form $A = A^I_m T^I dx^m$ and the curvature 2-form $F = dA + A^2$, defined on a manifold M^{2N+2} of even dimension $2N+2$. The covariant derivative is $D = d + [A,]$ and it follows that F satisfy the **Bianchi identity** $DF = 0$. An invariant polynomial $P(F)$ is defined as the formal sum

$$P(F) = \sum_{n=0}^{N} \alpha_n STr\big(F^{n+1}\big),$$

where $STr\big(T^{I_1} \dots T^{I_{n+1}}\big) = g^{I_1 \cdots I_{n+1}}$ stands for an invariant symmetric trace on the algebra of G and the wedge product of forms is understood. Under *gauge transformations* we have

$$A^g = g^{-1}(A + d)g, \quad F^g = g^{-1}Fg$$

where g is an element of the group. Then $P(F^g) = P(g^{-1}Fg) = P(F)$. An important property is $d\,STr(\ ) = STr(D\ )$ which together with the **Bianchi identity** implies that $P(F)$ is closed, $dP(F) = 0$ and then that $P(F)$ is locally exact. The **Chern–Simons form** [1,2,3] is defined as

$$I^0_{2n+1} = k\,STr[F_t^{n+1}] = (n+1)\int_0^1 dt\,STr[AF_t^n]$$

where k is the **Cartan homotopy operator** in the case that $A_1 = A$ and $A_0 = 0$, and then $F_t = dA_t + A_t^2 = tF + t(t-1)A^2$. The Cartan *homotopy formula* in that case is

$$[kd + dk]\mathcal{P}(F_t, A_t) = \mathcal{P}(F, A)$$

where $\mathcal{P}(F, A)$ is an arbitrary polynomial in A and F. Then locally

$$dI^0_{2n+1} = STr[F^{n+1}]$$