Multiple priors valuation of a liability cash flows subject to capital requirements

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based on recent work with H. Engsner and J. Thøgersen,
and earlier work with H. Engsner and K. Lindensjö

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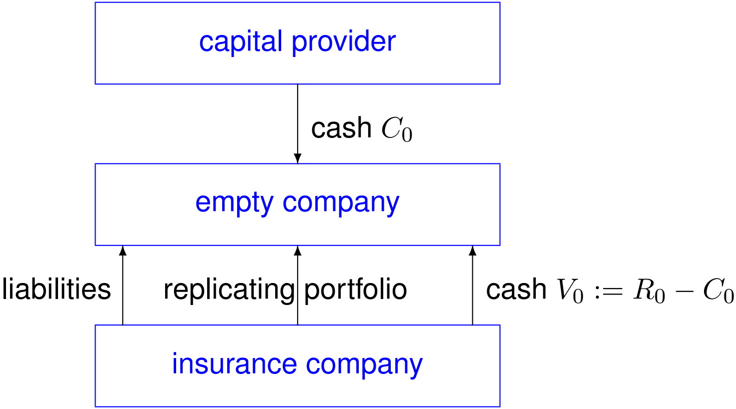
Valuation of aggregate insurance liabilities

- Aim: Present a conceptually sound market consistent approach to valuing an insurance company's aggregate liability to its policyholders
- Insurance companies are regulated, subject to externally imposed capital requirements. Sharpened capital requirements should lead to increased values of non-hedgeable risks
- Owners (share holders) are not liable for losses that exceed the total asset value of the company. Limited liability should enter in the valuation of non-hedgeable risks
- The liability value should depend on the assets held by the insurer for replication/hedging of liabilities

Hypothetical transfer of liabilities



Suppose liabilities and replicating portfolio together, after transfer, imply capital requirement of size R_0 according to some regulatory rule

Silly example

- Liability: payment of a Google share in three years
- Replicating portfolio: A forward contract with forward price F on the delivery of a Google share in three years, a three-year zero-coupon bond with face value F
- Liability is perfectly hedged by the replicating portfolio $\Rightarrow$ no risk $\Rightarrow R_0 = 0 \Rightarrow V_0 = C_0 = 0$
- Real insurance liability cash flow are complex, not perfectly replicable, and there is no obvious "best" replicating portfolio
- Solvency 2/EIOPA says that the replicating portfolio should minimize solvency capital requirements of the liability-receiving entity (Article 38: Reference undertaking)

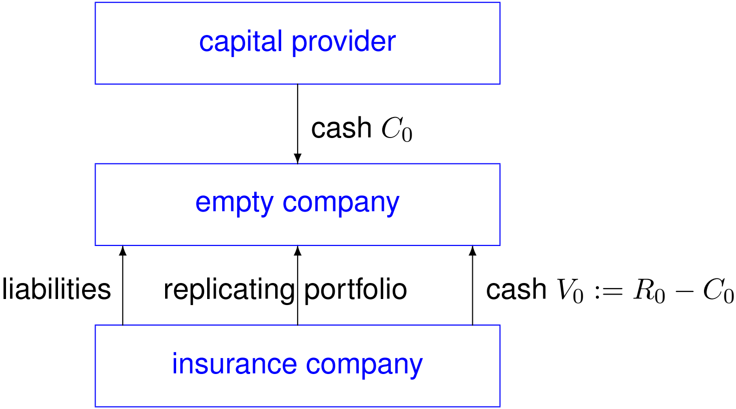
Hypothetical transfer of liabilities

- Hypothetical transfer at time 0 of the liability together with a replicating portfolio to an empty corporate entity whose role is to manage the liability run-off (and then cease to exist)
- All values are discounted by a bank account numeraire
- The following are transferred: original liability with cash flow $(X^o_t)_{t=1}^T$, replicating portfolio with cash flow $(X^r_t)_{t=1}^T$ and a cash amount V_0 invested in the numeraire asset.
- Equivalently: residual liability with cash flow $X = X^o - X^r$ and cash amount V_0 are transferred
- V_0 is value of X; V_0 + [market price of X^r] is the value of X^o
- How should V_0 be determined?

The new liability owner's perspective

- At each time t, ownership of new entity requires meeting externally imposed capital requirements R_t, where $R_T = 0$
- C_t is the value of continued ownership of the entity managing the liability run-off
- Ownership can be terminated at any time, option to default, without costs (but possibly with a net loss)
- Upon termination of ownership all assets of the entity are transferred to the policyholders
- The owner's benefit is the option to collect dividends/surplus throughout the run-off or until terminating ownership

Hypothetical transfer of liabilities



If the option to collect dividends/surplus is worth C_0 to the new owner, then the insurance company should pay $V_0 := R_0 - C_0$ to make the entity managing the liability run-off solvent

The cash flows to owner and policyholders

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$, adapted processes $(X_t^o)_{t=1}^T, (X_t^r)_{t=1}^T, (R_t)_{t=0}^T$
- Upon stopping at time τ the cumulative cash flow (dividends) to the owner in return for providing initial capital C_0 is

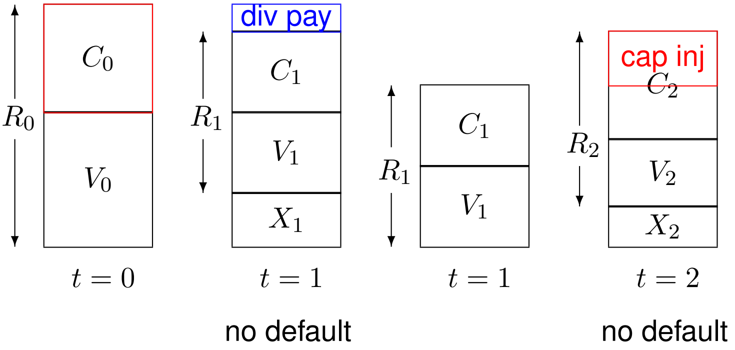
$$\sum_{t=1}^{\tau-1} \big(R_{t-1} - R_t - X_t\big) = R_0 - R_{\tau-1} + \sum_{t=1}^{\tau-1} X_t^r - \sum_{t=1}^{\tau-1} X_t^o,$$

  $\tau \in \{1, \dots, T+1\}$ and $\tau = T+1$ means a complete run-off
- Upon stopping at time τ the cumulative cash flow to the policyholders is

$$\sum_{t=1}^{\tau-1} X_t^o + R_{\tau-1} + \Big[\text{time-}\tau \text{ market price of } \sum_{t=\tau}^{T} X_t^r\Big]$$

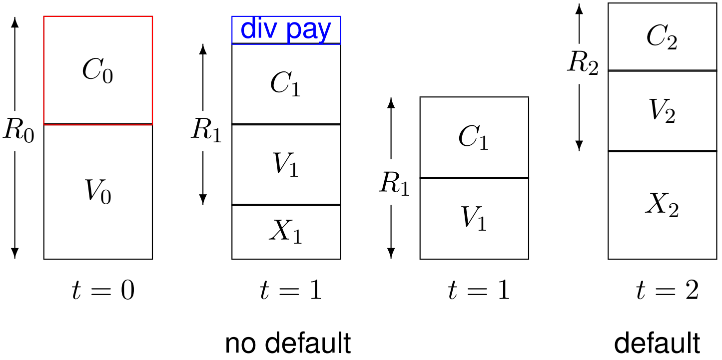
Dividend payment followed by capital injection

div pay $= R_0 - R_1 - X_1$ cap inj $= -(R_1 - R_2 - X_2)$



$V_t := R_t - C_t$ for all t

Dividend payment followed by default



Why default instead of a capital injection here?

Value of ownership

- Suppose owner assigns monetary value at time t to discounted future cash flow using valuation operator $Y \mapsto \mathrm{VALUE}_t(Y)$
- Value of ownership at time $t = T$ is $C_T := 0$ and

$$C_t := \operatorname{ess\,sup}_{\tau > t} \mathrm{VALUE}_t\left(\sum_{u=t+1}^{\tau-1}(R_{u-1} - R_u - X_u)\right), \quad t < T$$

Value of ownership

- Choose a set $\mathcal{Q}$ of probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that each $\mathbb{Q} \in \mathcal{Q}$ correctly prices traded cash flows and may correspond to a pessimistic view of nontraded insurance risk
- Choose

$$\mathrm{VALUE}_t(Y) := \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}}[Y], \quad \mathbb{E}_t^{\mathbb{Q}}[\cdot] := \mathbb{E}^{\mathbb{Q}}[\cdot \mid \mathcal{F}_t]$$

- $\mathcal{Q} = \{\mathbb{Q}_0\}$ is a possible but not necessarily suitable choice
- Value of ownership at time $t < T$ is

$$C_t := \operatorname{ess\,sup}_{\tau > t} \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}}\Bigg[\sum_{u=t+1}^{\tau-1} \big(R_{u-1} - R_u - X_u\big)\Bigg]$$

assuming $\mathcal{Q}$-uniform integrability of the involved variables

Value of residual liabilities

- Value of the residual liability at time t is $V_t := R_t - C_t$, implies

$$V_t = \operatorname{ess\,inf}_{\tau > t} \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}} \Big[\sum_{u=t+1}^{\tau-1} X_u + R_{\tau-1} \Big]$$

- Note:

$$V_t \leq \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}} \Big[\sum_{u=t+1}^{T} X_u \Big] \text{ (no stopping) },$$

$$V_t \geq \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \operatorname{ess\,inf}_{\tau > t} \mathbb{E}_t^{\mathbb{Q}} \Big[\sum_{u=t+1}^{\tau-1} X_u + R_{\tau-1} \Big] \; (\sup \inf \leq \inf \sup)$$

Value of original liabilities

Recall that all $\mathbb{Q} \in \mathcal{Q}$ correctly prices traded cash flows. Hence, for any $\mathbb{Q}' \in \mathcal{Q}$,

$$\begin{aligned}
L_0 &= \mathbb{E}_0^{\mathbb{Q}'}\left[\sum_{u=1}^{T} X_u^r\right] + V_0 \\
&= \mathbb{E}_0^{\mathbb{Q}'}\left[\sum_{u=1}^{T} X_u^r\right] + \inf_{\tau>0} \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{u=1}^{\tau-1} X_u + R_{\tau-1}\right] \\
&\leq \mathbb{E}_0^{\mathbb{Q}'}\left[\sum_{u=1}^{T} X_u^r\right] + \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{u=1}^{T} (X_u^o - X_u^r)\right] \\
&= \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{u=1}^{T} X_u^o\right]
\end{aligned}$$

Comment on Cost-of-Capital Valuation

For cash flow $(Y_t)_{t=1}^{T}$, if instead of

$$\mathrm{VALUE}_t\left(\sum_{u=t+1}^{T} Y_u\right) := \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{u=t+1}^{T} Y_u\right]$$

we choose

$$\mathrm{VALUE}_t\left(\sum_{u=t+1}^{T} Y_u\right) := \mathbb{E}_t^{\mathbb{P}}\left[\sum_{u=t+1}^{T} \frac{Y_u}{B_{t,u}}\right], \quad B_{t,u} = \prod_{s=t}^{u-1}(1+\eta_s),$$

where η_s are cost-of-capital rates, and $R_t = \mathrm{VaR}_{t,0.005}(-X_{t+1} - V_{t+1})$, then we obtain the valuation framework in (Möhr -11, ASTIN Bulletin) explaining the valuation principle of Solvency 2

Optimal stopping with multiple priors

- Conditions on $\mathcal{Q}$ necessary to express V_t in terms of a backward recursion (Riedel -09) (hold automatically for $\mathcal{Q} = \{\mathbb{Q}_0\}$)
- $\mathcal{Q}$ is stable (under pasting) if for a stopping time τ and $\mathbb{Q}^{(1)}, \mathbb{Q}^{(2)} \in \mathcal{Q}$ with density processes $D^{(1)}, D^{(2)}$ such that

$$D_t^{(1)} = \frac{\mathrm{d}\mathbb{Q}^{(1)}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t}, \quad D_t^{(2)} = \frac{\mathrm{d}\mathbb{Q}^{(2)}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t},$$

  the density process $D^{(3)}$ given by

$$D_t^{(3)} = \mathbb{I}_{\{t \leq \tau\}} D_t^{(1)} + \mathbb{I}_{\{t > \tau\}} \frac{D_\tau^{(1)} D_t^{(2)}}{D_\tau^{(2)}}$$

  corresponds to $\mathbb{Q}^{(3)} \in \mathcal{Q}$. (Note: $\{t \leq \tau\}, \{t > \tau\} \in \mathcal{F}_{t-1}$)

Multiple priors optimal stopping and recursions

- Suppose $\mathcal{Q}$ is stable under pasting and the set $\mathcal{D}_T$ of Radon Nikodym densities $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ corresponding to $\mathcal{Q}$ are weakly compact in $L^1(\mathcal{F},\mathbb{P})$ (automatically fulfilled for $\mathcal{Q}=\{\mathbb{Q}_0\}$)
- Seen from time t, the optimal stopping time is

$$\tau_t := \inf\left\{u \in \{t+1,\dots,T\} : R_{u-1} < X_u + V_u\right\} \wedge (T+1)$$

- $(C_t)_{t=0}^T$ and $(V_t)_{t=0}^T$ are determined by

$$\begin{aligned} R_T &= C_T = 0, \\ C_t &= \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}}[(R_t - X_{t+1} - V_{t+1})^+], \\ V_t &= R_t - C_t, \end{aligned}$$

where $(x)^+ := \max(x,0)$

Comment on Cost-of-Capital Valuation

If the owner uses the valuation operator

$$\mathrm{VALUE}_t\left(\sum_{u=t+1}^{T} Y_u\right) := \mathbb{E}_t^{\mathbb{P}}\left[\sum_{u=t+1}^{T} \frac{Y_u}{B_{t,u}}\right], \quad B_{t,u} = \prod_{s=t}^{u-1}(1+\eta_s)$$

to assess value of continued ownership, then the analogous recursion holds:

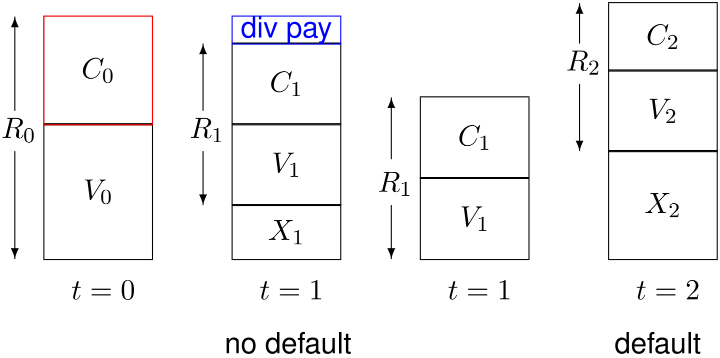
$$V_t = R_t - \frac{1}{1+\eta_t}\mathbb{E}_t^{\mathbb{P}}[(R_t - X_{t+1} - V_{t+1})^+]$$

which can be expressed ($R_t - X_{t+1} - V_{t+1} = C_{t+1} + R_t - R_{t+1} - X_{t+1}$)

$$\frac{\mathbb{E}_t^{\mathbb{P}}[(C_{t+1} + \text{div pay}_{t+1})^+]}{C_t} = 1 + \eta_t$$

in terms of expected excess return on equity

Optimal stopping: illustration



Optimal to terminate ownership first time τ such that $R_{\tau-1} < X_\tau + V_\tau$

Capital requirements by risk measures

Natural to consider $R_t := \rho_t(-X_{t+1} - V_{t+1})$ so that

$$\begin{aligned} V_t &= \rho_t(-X_{t+1} - V_{t+1}) \\ &\quad - \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}}[(\rho_t(-X_{t+1} - V_{t+1}) - X_{t+1} - V_{t+1})^+] \end{aligned}$$

for conditional risk measures $\rho_t : L^1(\mathcal{F}_{t+1}, \mathbb{P}) \to L^1(\mathcal{F}_t, \mathbb{P})$ such as

$$\begin{aligned} \mathrm{VaR}_{t,0.005}(Y) &= F^{-1}_{-Y|\mathcal{F}_t}(0.995), \\ \mathrm{ES}_{t,0.01}(Y) &= \frac{1}{0.01}\int_0^{0.01} \mathrm{VaR}_{t,u}(Y)\mathrm{d}u \end{aligned}$$

Notice: $\rho_t = \mathrm{VaR}_{t,q}$ or $\rho_t = \mathrm{ES}_{t,q}$ with $q \downarrow 0$ implies

$$V_0 \uparrow \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X_1 + \cdots + X_T]$$

Example: insurance liability cash flow

$C_{i,k}$ denotes the cumulative amount paid to policyholders during development year $1, 2$ due to accidents in accident year i.
$C_{i,k}$ is observed at calendar time $i + k$.

	1	2
-10	$C_{-10,1}$	$C_{-10,2}$
$\vdots$	$\dots$	$\dots$
-3	$C_{-3,1}$	$C_{-3,2}$
-2	$C_{-2,1}$	$C_{-2,2}$
-1	$C_{-1,1}$	$C_{-1,2}$
0	$C_{0,1}$	$C_{0,2}$

Liability cash flow: $(X_1, X_2) = (C_{-1,2} - C_{-1,1} + C_{0,1}, C_{0,2} - C_{0,1})$

Example - continued

Consider the development year dynamics of exposure adjusted cumulative amounts

$$C_{i,1} = \frac{2}{3} + \frac{1}{5}\varepsilon_{i,1}, \quad C_{i,2} = \frac{3}{2}C_{i,1} + \frac{1}{5}\varepsilon_{i,2},$$

where all $\varepsilon_{i,k}$ are independent and $N(0,1)$ with respect to $\mathbb{P}$.

Suppose we want $\mathbb{Q}_\theta$, $\theta = (f_0, f_1, \sigma_0, \sigma_1)$, such that

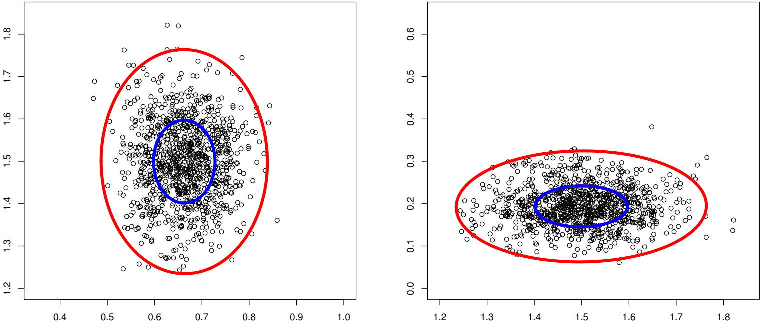
$$C_{i,1} = f_0 + \sigma_0\varepsilon^\theta_{i,1}, \quad C_{i,2} = f_1 C_{i,1} + \sigma_1\varepsilon^\theta_{i,2},$$

where all $\varepsilon^\theta_{i,k}$ are independent and $N(0,1)$ with respect to $\mathbb{Q}_\theta$.

Choose a compact set $\Theta \subset (0,\infty)^4$ describing parameter uncertainty.

Example - continued

Illustration of (f_0, f_1)-projection and (f_1, σ_1)-projection of boundary of Θ together with parameter estimates



$\mathbb{P}(\widehat{\theta} \in \Theta^{\text{red}}) = 0.9$ and $\mathbb{P}(\widehat{\theta} \in \Theta^{\text{blue}}) = 0.1$

Example - continued

- $\mathcal{Q}_\Theta = \{\mathbb{Q}_\theta : \theta \in \Theta\}$ is not stable under pasting
- The "stable hull" $\widetilde{\mathcal{Q}}_\Theta$ obtained by considering random switching among the probability measures in $\mathcal{Q}_\Theta$ according to predictable $\mathcal{P}(\Theta)$ processes is stable under pasting (and satisfies other required conditions)
- $\widetilde{\mathcal{Q}}_\Theta$ is considerably larger than $\mathcal{Q}_\Theta$. However,

$$\operatorname{ess\,inf}_{\mathbb{Q}\in\widetilde{\mathcal{Q}}_\Theta} \mathbb{E}_t^{\mathbb{Q}}[Y_{t+1}] = \operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_\Theta} \mathbb{E}_t^{\mathbb{Q}}[Y_{t+1}]$$

  for $\mathcal{F}_{t+1}$-measurable Y_{t+1} which means that we can take one step in the backward recursion optimizing only over the smaller set $\mathcal{Q}_\Theta$

Example - continued

Assume $C_{-1,1} = 2/3$, choose $R_t = \mathrm{VaR}_{t,q}$ and Θ containing the outcome of an estimator of $(f_0, f_1, \sigma_0, \sigma_1)$ with probability 0.1 or 0.9.

$$C_{i,1} = \frac{2}{3} + \frac{1}{5}\varepsilon_{i,1}, \quad C_{i,2} = \frac{3}{2}C_{i,1} + \frac{1}{5}\varepsilon_{i,2},$$

where all $\varepsilon_{i,k}$ are independent and $N(0,1)$ with respect to $\mathbb{P}$,

$$C_{i,1} = f_0 + \sigma_0\varepsilon^{\theta}_{i,1}, \quad C_{i,2} = f_1 C_{i,1} + \sigma_1\varepsilon^{\theta}_{i,2},$$

where all $\varepsilon^{\theta}_{i,k}$ are independent and $N(0,1)$ with respect to $\mathbb{Q}_\theta$.
$\mathbb{E}^{\mathbb{P}}[X_1 + X_2] = 4/3$ and we want to compute

$$\underline{V}_0 \leq V_0 \leq \overline{V}_0 = \sup_{\theta \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X_1 + X_2]$$

Example - continued

$\mathcal{Q} = \mathcal{Q}_\Theta$	$p = 0.1$	$p = 0.9$
$q = 0.10$	$(1.452, 1.491)$	$(1.686, 1.787)$
$q = 0.05$	$(1.473, 1.491)$	$(1.730, 1.787)$
$q = 0.01$	$(1.490, 1.491)$	$(1.772, 1.787)$
$q = 0.005$	$(1.491, 1.491)$	$(1.780, 1.787)$
$\mathcal{Q} = \widetilde{\mathcal{Q}}_\Theta$	$p = 0.1$	$p = 0.9$
$q = 0.10$	$(1.470, 1.513)$	$(1.734, 1.856)$
$q = 0.05$	$(1.491, 1.513)$	$(1.786, 1.856)$
$q = 0.01$	$(1.509, 1.513)$	$(1.835, 1.856)$
$q = 0.005$	$(1.511, 1.513)$	$(1.845, 1.856)$

Table: Lower and upper bounds $(\underline{V}_0, \overline{V}_0)$ using $\rho_t = \mathrm{VaR}_{t,q}$. Empirical estimates based on 10^5 iid samples. $\overline{V}_0 = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[X_1 + X_2]$ easily computed

main references

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