Mack's estimator motivated by large exposure asymptotics in a compound Poisson setting

Filip Lindskog
Stockholm University

Based on joint work with Nils Engler in:
Mack's estimator motivated by large exposure asymptotics in a compound Poisson setting
ASTIN Bulletin (2024), 54, 310-326
Open access: DOI: https://doi.org/10.1017/asb.2024.11

Astin Webinar, 11 June 2024

Data and task

Incremental claim amounts $X_{i,j}$. Claim amounts during 2023 in red.

		1	2	3	4	5
2019	1	$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$	$X_{1,5}$
2020	2	$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	$X_{2,4}$	$X_{2,5}$
2021	3	$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	$X_{3,4}$	$X_{3,5}$
2022	4	$X_{4,1}$	$X_{4,2}$	$X_{4,3}$	$X_{4,4}$	$X_{4,5}$
2023	5	$X_{5,1}$	$X_{5,2}$	$X_{5,3}$	$X_{5,4}$	$X_{5,5}$

Outstanding claim amounts from old accident years in blue.

Task: Predict these amounts. Estimate the prediction uncertainty.

Data and task

Accumulated claim amounts $C_{i,j} = X_{i,1} + \cdots + X_{i,j}$.

		1	2	3	4	5
2019	1	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	$C_{1,4}$	$C_{1,5}$
2020	2	$C_{2,1}$	$C_{2,2}$	$C_{2,3}$	$C_{2,4}$	$C_{2,5}$
2021	3	$C_{3,1}$	$C_{3,2}$	$C_{3,3}$	$C_{3,4}$	$C_{3,5}$
2022	4	$C_{4,1}$	$C_{4,2}$	$C_{4,3}$	$C_{4,4}$	$C_{4,5}$
2023	5	$C_{5,1}$	$C_{5,2}$	$C_{5,3}$	$C_{5,4}$	$C_{5,5}$

Observed data $\mathcal{D} = \{C_{i,j} : i + j \leq 6\}$. Predict $C_{i,5}$ by predictor $\widehat{C}_{i,5}$. Estimate conditional prediction error $\mathbb{E}[(C_{i,5} - \widehat{C}_{i,5})^2 \mid \mathcal{D}]$.

The chain ladder predictor

The (ancient) chain ladder (CL) predictor of $C_{3,5}$ is

$$\widehat{C}_{3,5} = C_{3,3}\widehat{f}_3\widehat{f}_4 \quad \text{where} \quad \widehat{f}_3 = \frac{C_{1,4}+C_{2,4}}{C_{1,3}+C_{2,3}}, \quad \widehat{f}_4 = \frac{C_{1,5}}{C_{1,4}}$$

		1	2	3	4	5
2019	1	$C_{1,1}$	$C_{1,2}$	$C_{1,3}$	$C_{1,4}$	$C_{1,5}$
2020	2	$C_{2,1}$	$C_{2,2}$	$C_{2,3}$	$C_{2,4}$	$C_{2,5}$
2021	3	$C_{3,1}$	$C_{3,2}$	$C_{3,3}$	$C_{3,4}$	$C_{3,5}$
2022	4	$C_{4,1}$	$C_{4,2}$	$C_{4,3}$	$C_{4,4}$	$C_{4,5}$
2023	5	$C_{5,1}$	$C_{5,2}$	$C_{5,3}$	$C_{5,4}$	$C_{5,5}$

Mack's distribution-free chain ladder

There exist constants $f_{\mathrm{MCL}t} > 0$ and $\sigma^2_{\mathrm{MCL}t} \geq 0$ such that

(1) $\mathbb{E}[C_{i,t+1} \mid C_{i,1}, \dots, C_{i,t}] = f_{\mathrm{MCL}t} C_{i,t}$, $t = 1, \dots, T-1$,
(2) $\mathrm{var}(C_{i,t+1} \mid C_{i,1}, \dots, C_{i,t}) = \sigma^2_{\mathrm{MCL}t} C_{i,t}$, $t = 1, \dots, T-1$,
(3) $(C_{1,1}, \dots, C_{1,T}), \dots, (C_{T,1}, \dots, C_{T,T})$ are independent.

The conditions (1)-(3) together are referred to as Mack's distribution-free chain ladder model. The parameters $f_{\mathrm{MCL}t}$ and $\sigma^2_{\mathrm{MCL}t}$ are estimated by (weighted least squares)

$$\widehat{f}_t = \frac{\sum_{i=1}^{T-t} C_{i,t+1}}{\sum_{i=1}^{T-t} C_{i,t}} \quad \text{and} \quad \widehat{\sigma}^2_t = \frac{1}{T-t-1} \sum_{i=1}^{T-t} C_{i,t} \left(\frac{C_{i,t+1}}{C_{i,t}} - \widehat{f}_t \right)^2,$$

Mack's CL - estimator of conditional MSEP

The main contribution of Mack (1993) is the explicit estimator of $\mathbb{E}\big[(C_{i,T} - \widehat{C}_{i,T})^2 \mid \mathcal{D}\big]$:

$$(\widehat{C}_{i,T})^2 \sum_{t=T-i+1}^{T-1} \frac{\widehat{\sigma}_t^2}{\widehat{f}_t^2} \left(\frac{1}{\widehat{C}_{i,t}} + \frac{1}{\sum_{j=1}^{T-t} C_{j,t}} \right)$$

which was derived using properties (1)-(3) of Mack's CL.

Mack's estimator is implemented in all standard reserving software

Mack's estimator has been analysed and derived, using different approaches, in a series of works by several insurance mathematicians including (in alphabetic order) A. Gisler, M. Merz, A. Röhr, M. Wüthrich

Mack's CL - estimator of conditional MSEP

The conditional variance property of Mack's distribution-free chain ladder

$$\mathrm{var}(C_{i,t+1} \mid C_{i,1}, \ldots, C_{i,t}) = \sigma^2_{\mathrm{MCL}t} C_{i,t}$$

is hard to test and natural stochastic models typically do not satisfy this property.

Still, the CL predictor and Mack's estimator of conditional MSEP have been (ab)used extensively and have been observed to perform surprisingly well.

A simple Poisson model

Renshaw and Verrall (1998) considered independent incremental claim amounts $X_{i,j} \sim \text{Pois}(\lambda_i q_j)$, $\sum_{j=1}^{T} q_j = 1$.

The natural predictor of $C_{i,T}$ for this model is (independent Poisson-distributed incremental claim amounts)

$$\widehat{C}_{i,T} = C_{i,T-i+1} + \widehat{\lambda}_i \big(\widehat{q}_{T-i+2} + \cdots + \widehat{q}_T \big)$$

Estimating parameters λ_i and q_j by Maximum Likelihood gives rise to the CL predictor.

A simple Poisson model

Except for independent accident years, the properties of Mack's distribution-free CL are not satisfied by the model considered by Renshaw and Verrall (1998):

$$\mathbb{E}[C_{i,t+1} \mid C_{i,1}, \ldots, C_{i,t}] = C_{i,t} + \mathbb{E}[X_{i,t+1}] = C_{i,t} + \lambda_i q_{t+1}$$

and

$$\text{var}(C_{i,t+1} \mid C_{i,1}, \ldots, C_{i,t}) = \text{var}(X_{i,t+1}) = \lambda_i q_{t+1}$$

are both inconsistent with Mack's distribution-free chain ladder.

A compound Poisson model

From now on: consider independent incremental claim amounts

$$X_{i,j}^{\alpha} = \sum_{k=1}^{N_{i,j}^{\alpha}} Z_{i,j,k}, \quad N_{i,j}^{\alpha} \sim \text{Pois}(\alpha\lambda_i q_j),$$

where

- $(Z_{i,j,k})_{k\geq 1}$ i.i.d. sequence with generic element Z with $\mathbb{E}[Z] > 0$ and $\text{var}(Z) < \infty$
- all $N_{i,j}^{\alpha}$ and all sequences $(Z_{i,j,k})_{k\geq 1}$ are independent

$Z \equiv 1$ gives the model considered by Renshaw and Verrall (1998)

Parameter α is a measure of exposure/volume (e.g. α number of contracts, λ_i the fraction of contracts for accident year i) which is a large number

Large exposure limits

Let $C_{i,j}^{\alpha} = X_{i,1}^{\alpha} + \cdots + X_{i,j}^{\alpha}$. Let $(N_{i,j}^{\alpha})_{\alpha \geq 0}$ be a homogeneous Poisson process with intensity $\lambda_i q_j$. Then

$$\widehat{f}_j := \frac{\sum_{i=1}^{T-j} C_{i,j+1}^{\alpha}}{\sum_{i=1}^{T-j} C_{i,j}^{\alpha}} \overset{a.s.}{\to} \frac{\sum_{s=1}^{j+1} q_s}{\sum_{s=1}^{j} q_s} =: f_j \quad \text{as } \alpha \to \infty$$

(CL development factor estimators converge) and

$$\frac{C_{i,T-i+1}^{\alpha} \widehat{f}_{T-i+2} \cdots \widehat{f}_{T-1}}{C_{i,T}^{\alpha}} \overset{a.s.}{\to} 1 \quad \text{as } \alpha \to \infty$$

(CL predictor is consistent)

Conclusion: CL development factors and CL prediction make sense

Large exposure limits

The estimator of the variance parameter in Mack's distribution-free CL converges in distribution: as $\alpha \to \infty$, for $t < T-1$,

$$\widehat{\sigma}_t^2 = \frac{1}{T-t-1}\sum_{i=1}^{T-t} C_{i,t}^{\alpha}\left(\frac{C_{i,t+1}^{\alpha}}{C_{i,t}^{\alpha}} - \widehat{f}_t\right)^2 \xrightarrow{d} \sigma_t^2 \frac{\chi^2_{T-t-1}}{T-t-1},$$

where χ^2_ν is χ^2-distributed with ν degrees of freedom, and

$$\sigma_t^2 = (f_t - 1) f_t \frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]}, \quad \mathrm{var}\left(\frac{\chi^2_{T-t-1}}{T-t-1}\right) = \frac{2}{T-t-1}$$

Since the limit is a random variable rather than a constant, the Mack CL property

$$\mathrm{var}(C_{i,t+1} \mid C_{i,1}, \ldots, C_{i,t}) = \sigma^2_{\mathrm{MCL}t} C_{i,t}$$

has no obvious interpretation in our model setting.

Conditional mean squared error of prediction

Clearly,

$$\mathbb{E}\Big[(C^{\alpha}_{i,T} - \widehat{C}^{\alpha}_{i,T})^2 \,\Big|\, \mathcal{D}^{\alpha}\Big] = C^{\alpha}_{i,T-i+1}\mathbb{E}\left[\frac{(C^{\alpha}_{i,T} - \widehat{C}^{\alpha}_{i,T})^2}{C^{\alpha}_{i,T-i+1}} \,\middle|\, \mathcal{D}^{\alpha}\right]$$

We show that

$$\mathbb{E}\left[\frac{(C^{\alpha}_{i,T} - \widehat{C}^{\alpha}_{i,T})^2}{C^{\alpha}_{i,T-i+1}} \,\middle|\, \mathcal{D}^{\alpha}\right] \xrightarrow{d} L \quad \text{as } \alpha \to \infty$$

so replacing L by its estimator $\widehat{L}^{\alpha}$ gives

$$\mathbb{E}\Big[(C^{\alpha}_{i,T} - \widehat{C}^{\alpha}_{i,T})^2 \,\Big|\, \mathcal{D}^{\alpha}\Big] \approx C^{\alpha}_{i,T-i+1}\widehat{L}^{\alpha}$$

Surprisingly, $C^{\alpha}_{i,T-i+1}\widehat{L}^{\alpha}$ coincides with Mack's estimator of conditional MSEP

Rediscovering Mack's estimator of conditional MSEP

It can be shown that

$$\mathbb{E}\left[\frac{(C^{\alpha}_{i,T} - \widehat{C}^{\alpha}_{i,T})^2}{C^{\alpha}_{i,T-i+1}} \,\middle|\, \mathcal{D}^{\alpha}\right] \stackrel{d}{\to} L \quad \text{as } \alpha \to \infty$$

where $\mathbb{E}[L]$ is an expression in terms of

$$f_t, \quad \sigma_t^2, \quad \mathbb{E}[Z]\lambda_j(q_1 + \cdots + q_t)$$

for certain values of j and t. Since

$$\widehat{f}_t \stackrel{a.s.}{\to} f_t, \quad \widehat{\sigma}_t^2 \stackrel{d}{\to} \sigma_t^2 \frac{\chi^2_{T-t-1}}{T-t-1}, \quad \frac{C^{\alpha}_{j,t}}{\alpha} \stackrel{a.s.}{\to} \mathbb{E}[Z]\lambda_j(q_1 + \cdots + q_t)$$

$\widehat{f}_t$, $\widehat{\sigma}_t^2$, $C^{\alpha}_{j,t}/\alpha$ are natural estimators of the above constants. This yields Mack's estimator.

Numerical illustration in Poisson setting

We explicitly compute

$$\mathbb{E}\left[\frac{(C_{i,T}^{\alpha} - \widehat{C}_{i,T}^{\alpha})^2}{C_{i,T-i+1}^{\alpha}} \,\middle|\, \mathcal{D}^{\alpha}\right]$$

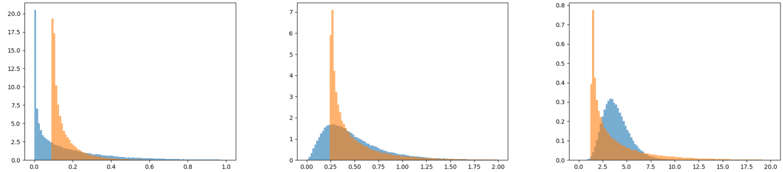
and

$$\widehat{L}^{\alpha} = \frac{(\widehat{C}_{i,T}^{\alpha})^2}{C_{i,T-i+1}^{\alpha}} \sum_{t=T-i+1}^{T-1} \frac{\widehat{\sigma}_t^2}{\widehat{f}_t^2}\left(\frac{1}{\widehat{C}_{i,t}^{\alpha}} + \frac{1}{\sum_{j=1}^{T-t} C_{j,t}^{\alpha}}\right)$$

which both are functions of claims triangle data $\mathcal{D}^{\alpha}$.

By simulating many triangles (based on first fitting the Poisson model to Table 1 in Mack (1993)) we may compare the histograms.

Numerical illustration in Poisson setting



Blue histograms: standardized Mack's estimator of conditional mean squared error. Orange histograms: true standardized conditional mean squared error. Plots correspond to accident years $i = 3, 5, 8$. For each i, the empirical means differ by less than 0.01.

The histograms are insensitive to varying α

Numerical illustration in Poisson setting

1	2	3	4	5	6	7	8	9	10
357848	1124788	1735330	2218270	2745596	3319994	3466336	3606286	3833515	3901463
352118	1236139	2170033	3353322	3799067	4120063	4647867	4914039	5339085	
290507	1292306	2218525	3235179	3985995	4132918	4628910	4909315		
310608	1418858	2195047	3757447	4029929	4381982	4588268			
443160	1136350	2128333	2897821	3402672	3873311				
396132	1333217	2180715	2985752	3691712					
440832	1288463	2419861	3483130						
359480	1421128	2864498							
376686	136329								
344014									

Example: For accident year 5, Mack's estimator of conditional MSEP is $3873311 \cdot \widehat{L}^{\alpha}$

Blue histograms are generated by repeatedly simulating new (upper left) triangles and computing an standardized estimator $\widehat{L}^{\alpha}$ for the simulated triangle.

A more general stochastic model

The stochastic model considered earlier corresponds to, for each accident year i, independently,

- $(M_i^\alpha)_{\alpha \geq 0}$ homogeneous Poisson process with intensity λ_i,
- M_i^α is the number of claims for accident year i,
- given M_i^α, allocate claims to development years according to a multinomial distribution with probabilities $q_1, \dots, q_T$,
- claim sizes are i.i.d., independent of the numbers of claims.

Can $(M_i^\alpha)_{\alpha \geq 0}$ be replaced by a more general renewal counting process satisfying $M_i^\alpha / \alpha \stackrel{a.s.}{\to} \lambda_i$?

Large exposure asymptotics for the general model

As in the compound Poisson setting,

- $\widehat{f}_t \xrightarrow{a.s.} f_t$ as $\alpha \to \infty$,
- $C^{\alpha}_{i,T-i+1}\widehat{f}_{T-i+2}\cdots\widehat{f}_{T-1}/C^{\alpha}_{i,T} \xrightarrow{a.s.} 1$ as $\alpha \to \infty$,
- $\widehat{\sigma}^2_t \xrightarrow{d} \sigma^2_t \frac{\chi^2_{T-t-1}}{T-t-1}$ as $\alpha \to \infty$.

However, although there is still convergence in distribution

$$\mathbb{E}\left[\frac{(C^{\alpha}_{i,T}-\widehat{C}^{\alpha}_{i,T})^2}{C^{\alpha}_{i,T-i+1}}\,\middle|\,\mathcal{D}^{\alpha}\right] \xrightarrow{d} L \quad \text{as } \alpha \to \infty,$$

the limit L may differ from that in the compound Poisson setting so the estimator $\widehat{L}^{\alpha}$ of L does not necessarily reproduce Mack's estimator of conditional MSEP.

Summary

- Mack's estimator of conditional MSEP is a convenient explicit measure of chain ladder prediction uncertainty
- The estimator was originally derived assuming the three properties of Mack's distribution-free chain ladder
- A stochastic model corresponding to independent compound Poisson distributed incremental claim amounts violates (two of) the conditions of the distribution-free chain ladder
- However, chain ladder prediction and prediction uncertainty assessment using Mack's estimator still make sense for such a model, under the assumption of reasonably large exposure