Hybrid Conference

Quantum graphs
in
Mathematics, Physics and Applications

QGraph Network Meeting

under **COST Action CA18232**
"Mathematical models for interacting dynamics on networks"

Stockholm University, 8–9 December 2022

Programme

All lectures of the meeting will take place at Stockholm University (Cramèr room), and on Zoom via

`https://stockholmuniversity.zoom.us/j/62899046825`

The schedule refers to Stockholm time (CET).

Thursday, 8 December

09.00-11.00 Nobel lectures in physics (Aspect, Clauser, Zeilinger) via `https://www.kva.se/en/event/the-nobel-lectures-2022-2/`

14.30-14.55 D. Mugnolo

15.00-15.25 V. Pivovarchik

15.30-15.55 M.-E. Pistol

16.00-16.30 **Break**

16.30-16.55 G. Sofer

17.00-17.25 L. Alon

17.30-17.55 M. Ettehad

Friday, 9 December

14.30-14.55 A. Kaplun

15.00-15.25 P. Bifulco (to be confirmed)

15.30-15.55 L. Sirko

16.00-16.30 **Break**

16.30-16.55 Yu. Latushkin

17.00-17.25 N. Levi

17.30-17.55 D. Borthwick

Quantum graphs, quasicrystals and stable polynomials.

L. Alon (MIT Cambridge)

Roughly speaking, Smilansky's trace formula says that the countable sum of $\exp(ikt)$, over all k square-root eigenvalues of a given quantum graph, vanishes whenever t is not the length of some closed path on the graph. Kurasov and Sarnak showed that (up to some technicalities) this means that the spectral measure of a quantum graph is a special type of crystalline measure (a term coined by Meyer) which is called Fourier quasicrystal (a term coined by Lev and Olevskii). Crystalline measures are of interest, partially due to their surprising connection to both physics (quasicrystals) and number theory (the zeros of the Zeta and L functions under GRH). Kurasov and Sarnak showed that the underlying reason for the spectral measure of a quantum graph to have such property is that the secular polynomial is a stable polynomial and that given any stable polynomial and a set of positive frequencies (in analog to the edge lengths) the zeros measure of the resulting trigonometric polynomial is a Fourier quasicrystal. Stable polynomials are multivariate polynomials that are closed under certain operations and were previously used in proving the Lee-Yang circle theorem and the Kadison-Singer conjecture. After providing the needed background, I will discuss a recent work in progress with Cynthia Vinzant on Kurasov-Sarnak quasicrystals. We show that such quasicrystals have well-defined gap distributions, by extending the Barra-Gaspard argument. The implication for quantum graphs is that any quantum graph with scaling invariant vertex conditions has a well-defined gap distribution, for any choice of edge lengths. If time permits, I will also explain how to construct random Kurasov-Sarnak measures with gap distribution as close as we want to the eigenvalues spacing of a random unitary matrix.

Cheeger-type lower bounds on the eigenvalue gap for metric graphs

D. Borthwick (Emory University, Atlanta)

For a metric graph with a mixture of standard and Dirichlet vertex conditions, we establish a lower bound for the gap between the first two eigenvalues in terms of a weighted Cheeger constant. This is used to establish a lower bound on the gap depending only on the total length and minimum edge length of the graph. This is a report on work in progress, joint with Evans Harrell and Haozhe Yu.

On vertex conditions in elastic beam frames: analysis on compact graphs

M. Ettehad (Boulder)

We consider three-dimensional elastic frames constructed out of Euler-Bernoulli beams and describe extension of matching conditions by relaxing the vertex-rigidity assumption and the case in which concentrated mass may exists. This generalization is based on coupling an (elastic) energy functional in terms of field's discontinuities at a vertex along with purely geometric terms derived out of first principles. The corresponding differential operator is shown to be self-adjoint. Application of theoretical results is then discussed in details for compact frames along with road map towards periodic graphs. Moreover, derivation of characteristic equation based on the idea of geometric-free local spectral basis and enforcing geometry of the graph into play by an appropriate choice of the coefficient set will be discussed. Finally, we show the limit conditions in parameter space which results in decomposing of vector-valued beam Hamiltonian to a direct sum of scalar-valued ones. This is a joint work with Soohee Bae from Northeastern University.

Application of C^*-algebra to inverse problem on graph

A. Kaplun (St. Petersburg)

C^*-algebra of eikonals $\mathfrak{E}$ is an operator algebra associated with the dynamical system on the metric graph. It has been studied since work [Wad15]. The motivation for the application of this algebra (so-called algebraic version of Boundary Control method) to the inverse problem on a graph is the fact that inverse data (dynamical and/or spectral) determine some isomorphic copy of such algebra. Recent results from [Kap22] show that algebra $\mathfrak{E}$ allows realization as an algebra of semi-continuous sections of C-bundle with some metric graph (frame) as a base. This graph is the factor-space of the spectrum (set of classes of equivalence of irreducible representations) of algebra $\mathfrak{E}$ and at the same time is related to the original metric graph.

[Kap22] M.I. Belishev; A.V. Kaplun. Canonical forms of metric graph eikonal algebra and graph geometry. *arXiv*: 2210.13246, 2022.

[Wad15] M.I. Belishev; N. Wada. A C^*-algebra associated with dynamics on a graph of strings. *J. Math. Soc. Japan*, 67(3):1239–1274, 2015.

Hamiltonian spectral flows, the Maslov index, and the stability of standing waves in the nonlinear Schrödinger equation

Yu. Latushkin (Missouri)

This is a joint work with G. Cox, M. Curran, and R. Marangell.

We use the Maslov index to study the spectrum of a class of linear Hamiltonian differential operators. We provide a lower bound on the number of positive real eigenvalues, which includes a contribution to the Maslov index from a non-regular crossing. A close study of the eigenvalue curves, which represent the evolution of the eigenvalues as the domain is shrunk or expanded, yields formulas for their concavity at the non-regular crossing in terms of the corresponding Jordan chains. This, along with homotopy techniques, enables the computation of the Maslov index at such a crossing. We apply our theory to study the spectral (in)stability of standing waves in the nonlinear Schrödinger equation on a compact spatial interval. We derive new stability results in the spirit of the Jones–Grillakis instability theorem and the Vakhitov–Kolokolov criterion, both originally formulated on the real line.

Subordinacy Theory on Star-Like Graphs

N. Levi (Jerusalem)

The notion of subordinacy was introduced by Gilbert and Pearson, and it enables one to separate the singular and absolutely continuous parts of the spectrum of Schroedinger operators on the line via asymptotic properties of solutions to the eigenvalue equation. Informally speaking, a solution is called subordinate if it decays faster than any other linearly independent solution. We present a generalization of the Gilbert-Pearson subordinacy theory to Schroedinger operators on star-like graphs, which are graphs that consist of a compact component C, to which a finite number of half-lines are attached. We use our result to draw conclusions on the multiplicity of the singular spectrum of such operators.

Global bounds for eigenfunctions of magnetic quantum graphs by control sets

D. Mugnolo (Hagen)

In this talk, I will show how the Logvinenko-Sereda approach can be efficiently used to deliver upper bounds on L^2-norms of eigenfunctions of quantum graphs in terms of their L^2-norms on small, measurable control sets. An essential step consists in proving a Bernstein-type estimate for quantum graphs. If time allows, I will also report about a recent extension of Bernstein-type estimates to magnetic Schrödinger operators.

This is joint work with Michela Egidi (Rostock) and Albrecht Seelman (Dortmund).

Generating isospectral but not isomorphic quantum graphs

M.-E. Pistol (Lund)

Quantum graphs are defined by having a Laplacian defined on the edges a metric graph with boundary conditions on each vertex such that the resulting operator, L, is self-adjoint. We use Neumann boundary conditions. The spectrum of L does not determine the graph uniquely, that is, there exist non-isomorphic graphs with the same spectra. There are few known examples of pairs of non-isomorphic but isospectral quantum graphs. In this paper we rectify this situation by finding hundreds of isospectral sets. Using computer algebra we have found all 364 sets of isospectral but nonisomorphic equilateral connected quantum graphs with at most nine vertices. This includes thirteen isospectral triplets and one isospectral set of four. One of the isospectral triplets involves a loop where we could prove isospectrality. We also present three different combinatorial methods to generate arbitrarily large sets of isospectral graphs, where one method involves star graphs having pumpkin graphs as leaves. We will show that superlattices of graphs can have the same spectrum as not periodic chains of graphs. This work is based on `https://arxiv.org/abs/2104.12885`, but will also include new results.

Distinguishing co-spectral quantum graphs by scattering.

V. Pivovarchik (Odessa)

We propose a simple method for resolution of co-spectrality of Schrödinger operators on metric graphs. Our approach consists of attaching a lead to them and comparing the S-functions of the corresponding scattering problems on these (non-compact) graphs.

We show that in several cases – including general graphs on at most 6 vertices, general trees on at most 9 vertices, and general fuzzy balls – eigenvalues and scattering data are together sufficient to distinguish co-spectral metric graphs.

These results were obtained in collaboration with Delio Mugnolo.

Investigations of microwave networks with symplectic and orthogonal symmetries.

L. Sirko (Institute of Physics, Polish Academy of Sciences, Warszawa)

We report on experimental studies of the distribution of the reflection coefficients, and the imaginary and real parts of Wigner's reaction (K) matrix using open microwave networks with symplectic symmetry and varying absorption. We also analyze the situation when the original graph with time reversal symmetry is split at edges into two disconnected subgraphs [1-2]. We show that there is a relationship between the generalized Euler characteristic of the original graph and the generalized Euler characteristics of two disconnected subgraphs. The theoretical predictions are verified experimentally using microwave networks which simulate quantum graphs.

This is joint work of Omer Farooq, Afshin Akhshani, Małgorzata Białous, Szymon Bauch, Michał Ławniczak, and Leszek Sirko.

[1] Oleh Hul, Szymon Bauch, Prot Pakoǹski, Nazar Savytskyy, Karol Życzkowski, and Leszek Sirko, Phys. Rev. E **69 (5)**, 056209 (2004).

[2]. Omer Farooq, Afshin Akhshani, Małgorzata Białous, Szymon Bauch, Michał Ławniczak, and Leszek Sirko, Mathematics **10(20)**, 3785 (2022).

Time dependent quantum graphs and the geometric phase

G. Sofer (Haifa)

Consider a metric graph whose edge lengths are functions of time. While this system is relatively simple to describe, it turns out that presenting it as a well posed boundary value problem holds several difficulties. For instance, the time dependence of the edges gives rise to non-unitary time evolution. This problem has been discussed by Duca and Joly in [1] in the context of the SchrÃűdinger equation on moving domains. We attempt to solve this problem by suggesting an alternative description of the system. We show that the original problem can be translated into the time independent problem of a harmonic oscillator on a non-homogeneous quantum graph, along with a magnetic potential. One can then derive an expression for the geometric phase (defined in [2]) accumulated by the wave function as the edge lengths complete a cycle in the parameter space. We study the relation between the geometric phase and the corresponding cycles in parameter space when they are taken to be Lissajous knots in 3-dimensional space. The talk is based on a joint work with Uzy Smilansky.

[1] R. Joly A. Duca. Schrödinger equation in moving domains. *Ann. Henri Poincare*, 2021.

[2] M. Berry. Quantal phase factors accompanying adiabatic changes. *Proceedings of the Royal Society A*, 1984.