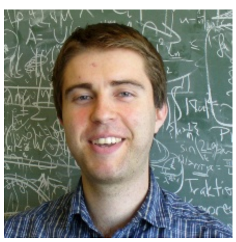
ZOOM Conference

Quantum graphs
in
Mathematics, Physics and Applications

QGraph Network Meeting

under **COST Action CA18232**
"Mathematical models for interacting dynamics on networks"

Celebrating promotion of



James Kennedy and Jonathan Rohleder

Stockholm University, 8–9 December 2020

Programme

All lectures of the meeting will take place on Zoom via

`https://stockholmuniversity.zoom.us/j/67081461604`

The schedule refers to Stockholm time (CET).

Tuesday, 8 December

09.00-11.00 Nobel lectures in physics via `https://www.nobelprize.org/`

13:00-13:05 **Opening**

13.05-13.30 U. Smilansky
13.35-14.00 S. Gnutzmann
14.05-14.30 L. Sirko
14.35-15.00 V. Pivovarchik
15.05-15.30 D. Pelinovsky

15.35-16.00 **Coffee break**

16.00-16.25 G. Berkolaiko
16.30-16.55 M. Ettehad
17.00-17.25 S. Avdonin
17.30-17.55 R. Carlson

Wednesday, 9 December

09.00-9.25	A. Hussein
09.30-9.55	M.-E. Pistol
10.00-10.25	J. Kerner
10.30-11.00	**Coffee break**
11.00-11.25	I. Popov
11.30-11.55	J. Lipovský
12.00-13.00	**Lunch break**
13.00-13.25	D. Mugnolo
13.30-13.55	M. Plümer
14.00-14.25	A. Serio
14.30-14.55	J. Kennedy
15.00-15.25	M. Hofmann
15.30-16.00	**Coffee break**
16.00-16.25	S. Sukhtaiev
16.30-16.55	L. Alon
17.00-17.25	K. Jones
17.30-17.55	E. Harrell
18.30-	**Digital conference dinner**

Towards universality of the nodal statistics on metric graphs

Lior Alon (Princeton)

The study of nodal sets of Laplace eigenfunctions has intrigued many mathematicians over the years. The nodal count problem has its origins in the works of Strum (1936) and Courant (1923) which led to questions that remained open to this day. One such question was the universal behavior of the nodal statistics. In 2002 Blum, Gnutzmann and Smilansky observed numerically that the statistical behaviors of the (properly rescaled) nodal count on planar billiards, exhibit two types of behaviors. These behaviors seemed to be independent of the specific shape of the billiards, hence universal and were determined by the system's classification to chaotic or integrable. Proving this universality is still open.

We study an analog problem of nodal statistics on metric graphs. In this talk, I will present a conjecture of a universal nodal statistics behavior for metric graphs and will provide new experimental and analytic results supporting the conjecture. I will also describe a new method for calculating the nodal statistics by integrating over the family of unitary evolution matrices. associated with the discrete underlying graph.

This is a joint work with R. Band (Technion) and G. Berkolaiko (Texas A&M).

An inverse problem for quantum trees with internal observations

Sergei Avdonin (Alaska Fairbanks)

In this talk we discuss a non-standard dynamical inverse problem for the wave equation on a metric tree graph. We assume that positive masses may be attached to the internal vertices of the graph. Another specific feature of our investigation is that we use only one boundary actuator and one boundary sensor, all other observations being internal. Using the Dirichlet-to-Neumann map (acting from one boundary vertex to one boundary and all internal vertices) we recover the topology and geometry of the graph, the coefficients of the equations and the masses at the vertices. The talk is based on joint work with Julian Edward.

Spectral shift via "lateral" perturbation

Gregory Berkolaiko (Texas A&M)

The first step in the proofs of several spectral geometry theorems is perturbing the operator "along" a given eigenfunction f, i.e. adding a perturbation P that vanishes on f and therefore leaves the corresponding eigenvalue λ_0 in its place. But such perturbation may still affect the sequential number of λ_0 in the spectrum, creating a spectral shift. We will discuss a general theorem that recovers the value of the spectral shift by looking at the stability of λ_0 with respect to small variations of the perturbation P. Time permitting, we will discuss applications of this result to counting zeros of eigenfunctions and searching for band edges in the spectrum of periodic graphs. Based on joint work with Y.Canzani, G.Cox, P.Kuchment and J.Marzuola.

Metric Graphs with Totally Disconnected Boundary

Robert Carlson (Colorado)

Boundary analysis is developed for a rich class of generally infinite weighted graphs with compact metric completions. These graph completions have totally disconnected boundaries. The classical notion of ϵ-components and the existence of suitable measures are used to construct generalized Haar bases and Hilbert spaces of functions on the boundaries. Suitable exit measures are constructed and analyzed using harmonic functions.

Three Dimensional Elastic Frames: Rigid Joint Conditions In Variational And Differential Formulation

Mahmood Ettehad (Texas A&M)

We consider elastic frames constructed out of Euler-Bernoulli beams. Correct vertex conditions corresponding to rigid joints have been a subject of active interest in both mathematical and structural engineering literature, with consideration usually limited to planar frames. In this paper we will describe a simple process of generating joint conditions out of the geometric description of an arbitrary three-dimensional frame. The corresponding differential operator is shown to be self-adjoint. Furthermore, in the presence of symmetry, one can restrict the operator onto reducing subspaces corresponding to irreducible representations of the symmetry group. This decomposition is demonstrated in general planar frames and in a three dimensional example with rotational symmetry.

The trace formula for quantum graphs with piecewise constant potentials and multi-mode quantum graphs

Sven Gnutzmann (Nottingham)

It is well known that one may define a unitary quantum map for quantum graphs with free particle movement on all edges. This is a key ingredient in the derivation of the trace formula by Kottos and Smilansky. We will generalise this quantum map to the case of piecewise constant potentials. Due to the appearance of evanescent modes this is in general not a unitary map. We will show that there is however a reduced unitary map that only acts on the edges where the modes are propagating. We will use this to derive a trace formula and discuss some examples. The same technique may be applied to multi-mode quantum graphs which may be considered as a special case.

Localization of eigenfunctions on quantum graphs

Evans Harrell (Georgia)

I'll discuss ways to construct realistic "landscape functions" for eigenfunctions ψ of quantum graphs. This term refers to functions that are easier to calculate than exact eigenfunctions, but which dominate $|\psi|$ in a non-uniform pointwise fashion constraining how ψ can be localized. Our techniques include Sturm-Liouville analysis, a maximum principle, and Agmon's method.

This is based on joint work with Anna Maltsev of Queen Mary University, London.

Weyl asymptotics and estimates for spectral minimal partitions of metric graphs

Matthias Hofmann (Lisbon)

We consider spectral minimal partitions of metric graphs within the framework introduced by Kennedy et al (arXiv:2005.01126), built on Dirichlet and standard Laplacian eigenvalues. We show three properties of the associated minimal partition energies which strongly recall the Laplacian eigenvalues of the whole graph. Firstly, we provide sharp lower and upper estimates for minimal partition energies reminiscent of spectral estimates obtained in [Berkolaiko *et al* (2017), J. Phys. A: Math. Theor. **50** 365201]. Secondly, we show interlacing inequalities between these minimal partition energies depending on the combinatorial structure of the graph.

This chain of inequalities, which involve the first Betti number and the number of degree one vertices of the graph, recall both interlacing and other inequalities for the Laplacian eigenvalues of the whole graph. As a corollary we obtain an inequality between these energies and the actual Dirichlet and standard Laplacian eigenvalues, valid for all compact graphs, which generalizes and complements a result of [Rohleder (2017), Proc. Amer. Math. Soc. **145**, 2119–2129] for tree graphs, based on Friedlander's inequalities between Dirichlet and Neumann eigenvalues of a domain. Thirdly, the spectral minimal partition energies satisfy the same Weyl asymptotics as the respective Laplacian eigenvalues, a result that follows from each of the former properties.

This is based on joint works with James Kennedy, Delio Mugnolo and Marvin Plümer.

If time were a graph, what would evolution equations look like? Science fiction, popular science, analysis

Amru Hussein (Kaiserslautern)

Linear evolution equations are considered usually for the time variable being defined on an interval where initial conditions or time periodicity of solutions are required. Going beyond this, there are different perceptions of time expressed in the multiverse interpretation of quantum mechanics or in the discussions on closed timelike curves in general relativity. Here we would like to make a point of allowing time to be defined on a metric graph or network where on the branching points coupling conditions are imposed such that time can have ramifications and even loops. For this setting questions of well-posedness for parabolic time-graph Cauchy problems are studied. This generalizes the classical setting and admits more freedom in the modeling of coupled and interacting systems of evolution equations. This and the impact of the graph's geometry is exemplified by discussing possible perceptions of fictional time-travel and by the more real-world case of coupled systems with constraints which are non-local in time akin to periodicity.

The tals is based on the preprint https://arxiv.org/abs/2001.06868

Finding an Upper Bound for the Heat Kernel on a Quantum Graph using the Heat Kernel on a Neumann Interval

Kenny Jones (Emory University)

This seminar will investigate a new bound for the heat kernel on an arbitrary quantum graph. Using a representation of the heat kernel as a sum of gaussians, we created an upper bound for the heat kernel along an edge of a quantum graph by the Neumann interval with the same length as the edge. The bound uses a technique for partitioning the paths along a quantum graph that we believe could be applied to other areas of quantum graph research.

Spectral partitions of metric graphs

James Kennedy (Lisbon)

We introduce a theory of partitions of metric graphs via spectral-type functionals, associating with any given partition a spectral energy built around eigenvalues of differential operators like the Laplacian, and then minimise (or maximise) this energy over all admissible partitions.

We first sketch a general existence theory for optimisers of such partition functionals, and discuss a number of natural functionals and optimisation problems.

We also illustrate how changing the functionals and the classes of partitions under consideration – for example, imposing Dirichlet versus standard conditions at the cut vertices or considering min-max versus max-min type functionals – may lead to qualitatively different optimal partitions which seek out different features of the graph.

This is based on joint work with Pavel Kurasov, Corentin Léna and Delio Mugnolo.

On the spectral gap of one-dimensional Schrödinger operators on large intervals

Joachim Kerner (Hagen)

In this talk we focus on the spectral gap of one-dimensional Schrödinger operators. In particular, we are interested in understanding the influence of non-negative potentials on the spectral gap in the limit of large intervals. We present some classical and some new results (joint work with Matthias Täufer (Hagen)).

Graphs with preferred-orientation coupling and their spectral properties

Jiří Lipovský (Hradec Králové)

We investigate quantum graphs with the preferred-orientation coupling conditions suggested by Exner and Tater [1]. In particular, we are interested in the high-energy limit of their spectra. These coupling conditions violate the time-reversal symmetry, for a particular energy, the particle approaching the vertex from a given edge leaves it through the neighbouring edge (for instance, to the left of the incoming edge) and this property is cyclical. It was previously shown that the vertex scattering matrix depends on the degree of the vertex; for an odd-degree vertex, the scattering matrix converges in the high-energy limit to the identity matrix, while even-degree vertices behave differently. This behaviour affects the transport properties of these graphs.

We study two models. The first one is a finite graph consisting of edges of Platonic solids. We find that the asymptotical distribution of the eigenvalues for the octahedron graph (having even degrees of vertices) is different from the other Platonic solids (having odd degrees of vertices), for which the eigenvalues approach the spectrum of the Dirichlet Laplacian on an interval. The second model consists of two types of infinite lattices. For one of them, the transport at high energies is possible in the middle of the strip and is suppressed at the edges. For the other one, the transport is possible at the edge only.

The talk will be based on two papers in collaboration with P. Exner [2, 3].

References:

[1] P. Exner, M. Tater, Quantum graphs with vertices of a preferred orientation, Phys. Lett. A 382 (2018) 283âĂŞ287.

[2] P. Exner, J. Lipovský, Spectral asymptotics of the Laplacian on Platonic solids graphs, J. Math. Phys. 60 (2019), 122101.

[3] P. Exner, J. Lipovský, Topological bulk-edge effects in quantum graph transport, Phys. Lett. A 384 (2020), 126390.

Random evolution equations on graphs and beyond

Delio Mugnolo (Hagen)

We begin our talk by studying diffusion-type equations supported on combinatorial and metric graphs that are randomly varying in time. We hence follow the evolution of a system along the path of a random walk whose states are diffusion equations driven by different graph Laplacians. After settling the issue of well-posedness, we focus on the asymptotic behavior of solutions and show convergence of the propagator towards a deterministic steady state. In the second part of our talk, we turn to a different viewpoint and follow the evolution of a system not anymore along a tree-like time structure corresponding to all possible paths of the Markov chain, but rather along a time structure given by a general network. In this rather general setting we can prove well-posedness and certain qualitative properties of the solution. This talk is based on joint articles with Stefano Bonaccorsi (Trento), Francesca Cottini (Milano-Bicocca) and Amru Hussein (Kaiserslautern).

Existence and stability of standing waves on quantum graphs

Dmitry Pelinovsky (McMaster University)

Positive single-lobe standing waves on quantum graphs are considered in the framework of the nonlinear Schrodinger equation. Examples include graphs with cycles such as tadpole and flower graphs. I will explain how the period function for second-order differential equations can be used towards rigorous analysis of the symmetries, bifurcations, and stability of such standing waves. These results are compared with analysis of the variational formulation of the ground state on the quantum graphs.

Finding eigenvalues of quantum graphs and applying this to growing quantum graphs

Mats-Erik Pistol (Lund)

Metric graphs that have a Laplacian associated with their edges and suitably imposed boundary conditions at their vertices define an eigenvalue problem.

We have developed a computer program (in Mathematica) that solves this eigenvalue problem, in case the edge-lengths of the graph are rationally dependent, where we use Neumann boundary conditions. We find all eigenvalues of the graphs analytically, although the computing time may become excessive for complicated graphs as well as for graphs where two edges have similar lengths. The analytical solution often involves roots of high-degree polynomials. This solves the forward problem for these types of graphs.

The inverse problem of finding graphs having a prescribed spectrum is numerically solved by evolving graphs from simple graphs to more complicated graphs such that the resulting graph has a spectrum which is close to a prescribed spectrum. We find that this procedure most often âĂIJconvergesâĂİ. We will illustrate everything using a live demonstration.

On 'geometric' Ambarzumian's theorem

Vyacheslav Pivovarchik (Odessa)

Sturm-Liouville problems on simple connected equilateral graphs of < 6 vertices and trees of < 9 vertices are considered with Kirchhoff's and continuity conditions at the interior vertices and Neumann conditions at the pendant vertices and real L_2 potential on the edges. It is proved that if the spectrum of such a problem is unperturbed (such as in case of zero potential) then this spectrum uniquely determines the shape of the graph and the zero potential. This is a generalization of the 'geometric' Ambarzumian's theorem of [Boman, Kurasov, Suhr. 2018].

On Pleijel's theorem for quantum graphs

Marvin Plümer (Hagen)

We discuss a metric graph counterpart of Pleijel's theorem on the asymptotics of the nodal counting ν_n of the nth eigenfunction of the Laplacian. Our focus lays on the generality of our setting: unlike in previous investigations, we do not impose any assumption on the lengths of the graph's edges: while the behaviour $\nu \sim n$ is *generically* true, we can exhibit graphs, and sequences of Laplacians eigenfunctions thereupon, such that $\nu_n \not\sim n$. Additionally, we are able to extend our findings to nodal domains of two classes of operators that seem to have been previously seldom considered in this context: Schrödinger operators generating positive semigroups and p-Laplacians.

A few experiments with models of coupled rings in a magnetic field: spectrum, scattering, resonances

Igor Popov (St. Petersburg)

We consider electron transmission through two orthogonal coupled rings in a magnetic field and possibility of the transmission control by the magnetic field direction. Spectrum, transmission and resonances for coupled rings with spin-orbit interaction are also discussed.

(Igor Y. Popov, Maria O. Smolkina, Irina V. Blinova)

Optimal potentials for quantum graphs

Andrea Serio (Stockholm)

We present a study of Schrödinger operators on metric graphs with delta couplings at the vertices. In particular, we show which potential and which distribution of delta couplings on a given graph maximise the ground state energy, provided the integral of the potential and the sum of strengths of the delta couplings are fixed. This is based on a joint work with Pavel Kurasov.

Experimental investigations of the Euler characteristic and other peculiar properties of microwave networks and graphs

Leszek Sirko (Polish Academy of Sciences)

The problem of seven bridges of Koenigsberg considered by Leonhard Euler in 1736 was one ofthe most notable mathematical problems which laid the foundations of graph theory and topology. This idea was applied later by Linus Pauling to discuss a quantum particle on a physical network by a model of a quantum graph. A graph is a union of $|V|$ vertices connected by $|E|$ edges. The difference between the number of vertices and edges of a graph $\chi = |V| - |E|$, called the Euler characteristic, is the most important topological characteristic of a graph. The Euler characteristic plays a crucial role in determining graphs' properties. For example it defines a famous Euler invariance $I = \chi + |F| = 2$ fulfilledfor the polyhedral surfaces and plane graphs, where $|F|$ is the number of faces in the graph. In this talk we report on experimental investigations of the Euler characteristic using microwave networks [1]. We willalsodiscuss some peculiar properties of open quantum graphs and microwave networks [2,3]. In particular, we will show thatthere exist graphs which do not obey the Weyl's law $N(R) = LR/\pi$.The Weyl's law directly links the counting function $N(R)$ of the number of resonances with the square root of energy k, $0 < k < R$,and the total length of a graph L. Such graphs will be called non-Weyl graphs. We demonstratethat for standard coupling conditions the transition from a Weyl graph to a non-Weyl graph occurs if we introduce a balanced vertex. A vertex of a graph is called balanced if the numbers of infinite leads and internal edges meeting at a vertex are the same. We show that the experimental results demonstrating the existence of non-Weyl networks are in agreement with the theoretical predictions [3].

Acknowledgements: This work was supported in part by the National Science Centre, Poland,Grant No. 2016/23/B/ST2/03979.

[1] M. Lawniczak, P.Kurasov, S.Bauch, M. Bialous, V. Yunko, and L.Sirko, Phys. Rev. E 101, 052320 (2020).
[2] M. Bialous, V. Yunko, S. Bauch, M. Lawniczak, B. Dietz, and L. Sirko, Phys. Rev. Lett. 117, 144101 (2016).
[3] M. Lawniczak, J. Lipovsky, and L. Sirko, Phys. Rev. Lett. 122, 140503 (2019).

(Leszek Sirko, Malgorzata Bialous, Szymon Bauch, Pavel Kurasov, Jiri Lipovsky, Barbara Dietz, and Michal Lawniczak)

Multi-mode Quantum graphs – spectral theory, trace formulae and scattering resonances and anomalies

Uzy Smilansky (Weizmann Institute)

Multi-mode graphs are encountered whenever the Shroedinger operator consists of the standard graph 1-D Laplacian coupled to other (âĂIJinternalâĂİ orâĂİ transversalâĂİ) degrees of freedom. The total wave function is not separable which introduces several new effects and difficulties in the theory: Appropriate generalization of the boundary conditions at vertices; The existence of evanescent modes and thresholds; Periodic orbits where the total energy is shared between the translational and the internal degrees of freedom, and extending the secular equation and the trace formula to incorporate these effects. Attaching the graphs to leads where the coupling vanishes, one should develop a new scattering theory which should give appropriate account of resonances, thresholds and trapping in this inelastic scattering setup. We present two types of models which differ in the form of and the coupling to the internal or traversal degrees of freedom. Juxtaposing them side by side will demonstrate the richness of this problem and the broad range of mathematical tools needed for their discussion. We shall also point out several experimental setups where multi-mode effects might be observes and measured.

Limits of quantum graph operators with shrinking edges

Selim Sukhtaiev (Rice University)

In this talk, we will discuss the question of convergence of Schrödinger operators on metric graphs with general self-adjoint vertex conditions as lengths of some of graph's edges shrink to zero. Using a combination of functional-analytic bounds on the edges of the graph and Lagrangian geometry considerations for the vertex conditions we will establish a sufficient condition for resolvent convergence. This condition encodes an intricate balance between the topology of the graph and its vertex data. This is join work with Y. Latushkin and G. Berkolaiko.