On Rank One H_{-3}-Perturbations of Positive Self–adjoint Operators

P. Kurasov and K. Watanabe

Dedicated with great admiration to S.Albeverio on the occasion of his 60th birthday

Abstract. Rank one H_{-3} perturbations of positive self–adjoint operators are constructed using a certain extended Hilbert space and regularization procedures. Applications to Schrödinger operators with point interactions are discussed.

1. Introduction

Self–adjoint operators with complicated spectral structure can be obtained using finite rank perturbations of well studied operators with rather simple structure of the spectrum. Nevertheless these perturbed operators are exactly solvable in the sense that all eigenfunctions, spectral decomposition and scattering matrix can be written explicitly, sometimes using analytic functions. Suppose that A is a certain self–adjoint operator acting in the Hilbert space H. Then a finite rank perturbation of it is formally defined by the formula

$$A_V = A + V, \tag{1.1}$$

where V is a finite dimensional operator. The domains of the perturbed and unperturbed operators coincide if the perturbation is a bounded operator in the Hilbert space $V \in \mathcal{B}(H)$. But it is possible to consider more general perturbations determined by operators V acting in the scale of Hilbert spaces associated with the original operator A

$$V : H_2(A) \to H_{-2}(A). \tag{1.2}$$

In the latter case the perturbed operator can be defined using the form perturbation technique and the extension theory for symmetric operators. Really the operators A and A_V restricted to the domain

$$\mathrm{Dom}(A^0) = \{\psi \in \mathrm{Dom}(A) : \psi \in \mathrm{Ker}(V)\}$$

1991 *Mathematics Subject Classification.* Primary 47B25, 47A20, 81Q10; Secondary 35P05.

Supported by The Royal Swedish Academy of Sciences, Sweden-Japan Sasakawa Foundation and Gakushuin Univ. Foundation.

Published in: F.Gesztesy, H.Holden, J.Jost, S.Paycha, M.Röckner and S.Scarlatti (eds.), Stochastic Processes, Physics and Geometry: New Interplays. II, (CNS Conference proceedings, **29**), AMS, Providence, 2000, 413–422.

coincide. If the operator V is symmetric and finite dimensional then the restricted operator has finite deficiency indices and the resolvents of the perturbed and original operators are related via Krein's formula [**19, 20, 21, 24**]. This approach has been used to construct the so-called operators with point interactions [**2, 7, 9, 25, 26**]. Perturbations with the support on a discrete set of points or on low dimensional manifolds have been studied in detail, especially in applications to Schrödinger operators. More general finite rank perturbations of self–adjoint operators have been investigated recently [**3, 4, 14, 18, 22, 23, 32**]. It has been shown that standard perturbation theory allows one to define only perturbations determined by operators acting as $H_1(A) \to H_{-1}(A)$, the so-called H_{-1} perturbations. It is possible to generalize this approach to include more general perturbations (1.2). The perturbed operator cannot be determined uniquely in this case only a finite parameter family corresponding to the formal expression (1.1) can be established. To determine a unique operator one needs to use extra assumptions, for example scaling properties of the original operator and the perturbation [**3, 4, 5**]. This approach does not work for H_{-3} perturbations determined by the operators acting as

$$V : H_3(A) \to H_{-3}(A). \tag{1.3}$$

Really consider the one dimensional perturbation formally determined by

$$A_\alpha = A + \alpha\langle\varphi, \cdot\rangle\varphi, \tag{1.4}$$

where $\varphi \in H_{-3}(A) \setminus H_{-2}(A)$. The restriction of the operator A to the domain $\mathrm{Dom}(A^0) = \{\psi \in H_3(A) : \langle\varphi, \psi\rangle = 0\}$ is essentially self–adjoint. Every self–adjoint extension of this symmetric operator coincides with the original operator A. Therefore the operator corresponding to (1.4) cannot be defined in the original Hilbert space using standard methods of extension theory. The aim of the current paper is to define the operator A_α as a self–adjoint operator in a certain extended Hilbert space.

Operators formally defined by (1.4) are important in different applications. For example, consider the Laplace operator with one point interaction acting in $L_2(\mathbf{R}^3)$. The deficiency indices of Laplace operator restricted to the set of functions vanishing at one point are equal to $(1, 1)$. All self–adjoint extensions of this symmetric operator are rotationally invariant. Therefore no non-spherically symmetric point interaction for the Laplace operator in $\mathbf{R}^3$ can be constructed. Different modifications of this approach have been suggested. The first mathematically rigorous approach uses perturbations in Pontryagin spaces, where the perturbed operator has been defined in a certain extension of the original Hilbert space with non positive definite scalar product [**12, 17, 27, 28, 29, 30, 31**].[1] Another approach has been used by physicists who suggested considering certain abstract boundary conditions connecting coefficients in the asymptotics of wave functions [**10, 11**] without paying any attention to the fact that the wave functions corresponding to the p-state have non-normalizable singularities of order r^{-2}. Therefore no scalar product space has been taken into consideration. Another constructive approach has been suggested by Yu.E. Karpeshina [**16**]. In the first step a set of functions having the form of p-scattering states is constructed. Then the self–adjoint operator and the Hilbert space are defined using this set of scattering waves. Recently I. Andronov

[1] After finishing this manuscript the authors learned about the recent paper [**13**] where the idea of using Pontryagin spaces to define singular perturbations is developed.

suggested using extended Hilbert spaces to model non-spherically symmetric scatterers in $\mathbf{R}^2$ [**6**]. This model is very similar to the generalized point interactions introduced by B. Pavlov [**5, 25**]. We are going to use some of the ideas suggested by I. Andronov to construct rank one H_{-3} perturbations. Nonspherically symmetric point interactions appear naturally in the studies of the Aharonov-Bohm effect but no modification of the Hilbert space is needed [**1, 8**].

In [**3, 4**] it was proven that H_{-2} perturbations can be defined uniquely using a certain regularization procedure. It appears that a similar approach works for H_{-3}-perturbations. In our paper we are going to show how to define arbitrary rank one H_{-3}-perturbations of self–adjoint operators restricting our consideration to the case of semibounded (positive) operators. The approach suggested can easily be modified to include finite rank and not necessarily semibounded operators. We are not going to discuss how to construct regularizations and relations to symmetry properties of the problem. We are planning to present those results in one of our forthcoming publications.

2. The Hilbert Space

Our aim is to define the self–adjoint operator corresponding to (1.4), where A is a certain positive self–adjoint operator acting in the Hilbert space H and φ is an element from the space H_{-3} from the scale of Hilbert spaces associated with the operator A. In what follows we are going to consider the case where

$$\varphi \in H_{-3} \setminus H_{-2}, \tag{2.1}$$

since rank one perturbations from the class H_{-2} have already been studied in detail in [**3, 4, 22, 23, 32**].

It is natural to determine the operator A_α using the restriction-extension method. The operator A_α coincides with one of the extensions of the operator A^0, which is the restriction of the operator A to the set of functions u from the domain $\mathrm{Dom}(A)$ of the operator A satisfying the condition

$$\langle \varphi, u \rangle = 0. \tag{2.2}$$

Suppose that the operator A is considered as an operator in the original Hilbert space H. Then the domain of the operator coincides with the space H_2, $\mathrm{Dom}(A) = H_2$ and the operator A^0 is essentially self–adjoint if φ satisfies (2.1). The operator A^0 is not essentially self–adjoint for such φ only if the domain of the unperturbed operator A is a subset of H_3. Therefore let us consider the operator A as a self–adjoint operator acting in the Hilbert space H_1 equipped with the scalar product

$$\langle u, v \rangle_1 = \langle u, (1 + bA)v \rangle, \tag{2.3}$$

where b is a positive real number,

$$b > 0. \tag{2.4}$$

The norm determined by the latter scalar product is equivalent to the standard norm in the space H_1 and is given by

$$\| u \|_1^2 = \langle u, (1 + bA)u \rangle.$$

Then the domain of the operator A coincides with the space H_3 and the operator is self–adjoint on this domain. The operator A^0 being the restriction of the operator

A to the domain

$$\mathrm{Dom}(A_0) = \{u \in H_3 : \langle \varphi, u\rangle = 0\} \tag{2.5}$$

is a densely defined symmetric operator, since φ satisfies (2.1). The domain of the self–adjoint operator corresponding to the formal expression (1.4) necessarily contains the element $g_1 = \frac{1}{A+a_1}\varphi \in H_{-1}$. Therefore the extension of the operator A^0 corresponding to formal expression (1.4) cannot be constructed in the Hilbert space H_1. Let us consider the one dimensional extension $\mathcal{H}$ of this space,

$$\mathcal{H} = \mathrm{Dom}(A^0)\dot{+}\mathbf{C} \ni \mathcal{U} = (u, u_1). \tag{2.6}$$

Note that $\mathcal{H} \subset H_3\dot{+}\mathbf{C}$. We define the following natural embedding ρ of the space $\mathcal{H}$ into the space H_{-1}:

$$\begin{array}{rcl} \rho & : & \mathcal{H} \to H_{-1} \\ & & (u, u_1) \mapsto u + u_1 g_1. \end{array} \tag{2.7}$$

Then the scalar product in the space $\mathcal{H}$ can be introduced using the following formal calculations where b is a certain positive constant:

$$\begin{array}{rcl} \ll \mathcal{U}, \mathcal{V} \gg_{\mathcal{H}} & = & \langle \rho\mathcal{U}, \rho\mathcal{V}\rangle + b\langle \rho\mathcal{U}, A\rho\mathcal{V}\rangle \\ & = & \langle u + u_1 g_1, v + v_1 g_1\rangle + b\langle u + u_1 g_1, A(v + v_1 g_1)\rangle \\ & = & \langle u, v\rangle + b\langle u, Av\rangle + \bar{u}_1 v_1 \left(\| g_1 \|^2 + b\langle g_1, Ag_1\rangle\right) \\ & & +\bar{u}_1 \left(\langle g_1, v\rangle + b\langle Ag_1, v\rangle\right) + v_1 \left(\langle u, g_1\rangle + b\langle u, Ag_1\rangle\right). \end{array}$$

The last two terms can be simplified taking into account that

$$Ag_1 = -a_1 g_1 + \varphi$$

and the fact that the functions $u, v \in H_3$ satisfy (2.2). Then the scalar product is given by the expression

$$\begin{array}{rcl} \ll \mathcal{U}, \mathcal{V} \gg_{\mathcal{H}} & = & \langle u, v\rangle + b\langle u, Av\rangle + \bar{u}_1 v_1 \left(\| g_1 \|^2 + b\langle g_1, Ag_1\rangle\right) \\ & & +(1 - ba_1)\left(u_1\langle g_1, v\rangle + v_1\langle u, g_1\rangle\right), \end{array}$$

which can be considered only formally, since the scalar product $\langle g_1, Ag_1\rangle$ and the norm $\| g_1 \|^2$ are not defined (since φ is an element from $H_{-3} \setminus H_{-2}$). To define the scalar product we extend φ as a bounded linear functional using the equalities

$$\langle g_1, g_1\rangle = c_1, \quad \langle g_1, Ag_1\rangle = c_2, \tag{2.8}$$

where c_1 and c_2 are arbitrary positive real constants.[2] In what follows we are going to use thenotation

$$d = c_1 + bc_2 \in \mathbf{R}_+. \tag{2.9}$$

The scalar product determined by the following expression will also be considered:

$$\ll \mathcal{U}, \mathcal{V} \gg_{\mathcal{H}} \;=\; \langle u, v\rangle + b\langle u, Av\rangle + d\bar{u}_1 v_1 + (1 - ba_1)\left\{\bar{u}_1\langle g_1, v\rangle + v_1\langle u, g_1\rangle\right\}. \tag{2.10}$$

This formula defines a sesquilinear form on the domain $\mathrm{Dom}(A^0)\dot{+}\mathbf{C}$. This form defines a scalar product only if it is positive definite.

Let us denote by $\| \mathcal{U} \|^2_{\mathcal{H}} = \ll \mathcal{U}, \mathcal{U} \gg_{\mathcal{H}}$ the norm associated with the previously introduced scalar product. The space $\mathcal{H}$ with this norm is not complete, and the following lemma describes its completion with respect to this norm.

[2] One can use some regularization procedure similar to those introduced for H_{-2}-pertubations in [**3, 4**].

LEMMA 2.1. *Let the following inequality be satisfied*

$$d > |1 - ba_1|^2 \parallel g_1 \parallel_{-1}^2 . \tag{2.11}$$

Then the norm $\| \cdot \|_{\mathcal{H}}$ *is equivalent to the standard norm in the Hilbert space* $H_1 \oplus \mathbf{C}$

$$\| \mathcal{U} \|^2 \equiv \| (u, u_1) \|^2 = \langle u, (1 + bA)u \rangle + |u_1|^2.$$

The completion of the space $\mathcal{H} = \mathrm{Dom}(A^0) \dot{+} \mathbf{C}$ *with respect to the norm* $\| \cdot \|_{\mathcal{H}}$ *coincides with the space* $H_1 \dot{+} \mathbf{C}$.

Proof. We prove first that the norm $\| \cdot \|_{\mathcal{H}}$ can be estimated from above by the standard norm. The last term in (2.10) for $\mathcal{U} = \mathcal{V}$ can be estimated as follows

$$\begin{aligned} |(1 - ba_1) \{\bar{u}_1 \langle g_1, u \rangle + u_1 \langle u, g_1 \rangle\}| &\leq 2|1 - ba_1||u_1| \parallel u \parallel_1 ||g_1||_{-1} \\ &\leq |1 - ba_1| \parallel g_1 \parallel_{-1} \{|u_1|^2 + \parallel u \parallel_1^2\}. \end{aligned} \tag{2.12}$$

It follows that

$$\begin{aligned} \ll \mathcal{U}, \mathcal{U} \gg_{\mathcal{H}} &\leq \parallel u \parallel_1^2 + d|u_1|^2 + |1 - ba_1| \{\parallel u \parallel_1^2 + |u_1|^2\} \\ &\leq (\max\{1, d\} + |1 - ba_1|) \parallel \mathcal{U} \parallel^2 . \end{aligned} \tag{2.13}$$

This upper estimate is proven without using any assumption on the parameters d, b, a_1.

Let us prove the lower estimate provided that condition (2.11) is satisfied. Using estimate (2.12) we can estimate the norm as follows

$$\begin{aligned} \ll \mathcal{U}, \mathcal{U} \gg &\geq \|u\|_1^2 + d|u_1|^2 - 2|1 - ba_1| \parallel g_1 \parallel_{-1} |u_1| \parallel u \parallel_1 \\ &= (\parallel u \parallel - |1 - ba_1| \parallel g_1 \parallel_{-1})^2 + \left(d - |1 - ba_1||^2 \parallel g_1 \parallel_{-1}^2\right) |u_1|^2 \\ &\geq \epsilon \parallel \mathcal{U} \parallel^2 \end{aligned}$$

where $\epsilon = \min\{\frac{d - |1 - ba_1|^2 \|g_1\|_{-1}^2}{2}, \frac{d - |1 - ba_1|^2 \|g_1\|_{-1}^2}{d + |1 - ba_1|^2 \|g_1\|_{-1}^2}\} > 0$. This completes the proof of the Lemma.

Note that the scalar product in the space $\overline{\mathcal{H}}$ calculated on the vectors with the component u_1 equal to zero is equivalent to the scalar product in the space H_1. But the decomposition of the space $\overline{\mathcal{H}} = H_1 \dot{+} \mathbf{C}$ is not orthogonal. Only if $ba_1 = 1$ does the decomposition become orthogonal. We are going to use the same notation $\mathcal{H}$ for the completed space.

3. The Operator

We define the operator $\mathcal{A}$ on the set of regular elements $\mathrm{Dom}_r \subset \mathcal{H}$ which possess the representation

$$\mathcal{U} = (u, u_1) = (u_r + u_2 g_2, u_1),$$

where $u_r \in H_3, u_2 \in \mathbf{C}$. The vector

$$g_2 = \frac{1}{A + a_2} g_1 = \frac{1}{A + a_2} \frac{1}{A + a_1} \varphi \in H_1$$

is defined using another one positive parameter, $a_2 > 0$. The embedding operator ρ maps every such element to a vector from H_{-1} as follows:

$$\rho(u_r + u_2 g_2, u_1) = u_r + u_2 g_2 + u_1 g_1.$$

Then the operator $\mathcal{A}$ in $\mathcal{H}$ is defined on Dom_r in such a way that the following equality holds:

$$A\rho\mathcal{U} \equiv \rho\mathcal{A}\mathcal{U} (\mathrm{mod} \varphi), \tag{3.1}$$

where this equality in H_{-3} holds if and only if the difference between the left and right hand sides is proportional to $\varphi \in H_{-3}$. In other words, the operator $\mathcal{A}$ acts as formal adjoint operator. There exists a unique operator $\mathcal{A}$ in $\mathcal{H}$ satisfying (3.1)

$$\mathcal{A}\mathcal{U} = \mathcal{A}\begin{pmatrix} u_r + u_2 g_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} Au_r - a_2 u_2 g_2 \\ u_2 - a_1 u_1 \end{pmatrix}. \tag{3.2}$$

The operator $\mathcal{A}$ is not self–adjoint. In fact it is not even symmetric. The boundary form of the operator can be calculated explicitly. We present here the result of these tedious but otherwise straightforward calculations

$$\begin{aligned} \ll \mathcal{A}\mathcal{U}, \mathcal{V} \gg_{\mathcal{H}} - \ll \mathcal{U}, \mathcal{A}\mathcal{V} \gg_{\mathcal{H}} = & < u_r, \varphi > (bv_2 + (1 - ba_1)v_1) \\ & - (b\bar{u}_2 + (1 - ba_1)\bar{u}_1) < \varphi, v_r > + a\left\{\bar{u}_2 v_1 - \bar{u}_1 v_2\right\}, \end{aligned} \tag{3.3}$$

where we used the notation

$$a = d + (ba_1 - 1) < g_1, g_2 > (a_2 - a_1).$$

This formula defines the following sesquilinear form in $\mathbf{C}^3$:

$$\begin{aligned} & \ll \mathcal{U}, \mathcal{A}\mathcal{V} \gg_{\mathcal{H}} - \ll \mathcal{A}\mathcal{U}, \mathcal{V} \gg_{\mathcal{H}} = \\ & = \left\langle \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} < \varphi, u_r > \\ u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} < \varphi, v_r > \\ v_1 \\ v_2 \end{pmatrix} \right\rangle, \end{aligned} \tag{3.4}$$

where $c = ba_1 - 1 \in \mathbf{R}$. The rank of the 3×3 matrix appeared in this formula is equal to 2 if at least one of the parameters a, b, c is different from zero, since the characteristic polynomial for the matrix is given by $-\lambda(\lambda^2 + a^2 + b^2 + c^2)$. Then all symmetric restrictions of the operator $\mathcal{A}$ can be defined by certain boundary conditions imposed on the functions from the domain of the operator. The problem of defining a symmetric restriction of $\mathcal{A}$ is equivalent to the problem of finding a Lagrangian plane of the boundary form.

Suppose that the boundary conditions are written in the form

$$\alpha < \varphi, u_r > + \beta u_1 + \gamma u_2 = 0, \tag{3.5}$$

where $\alpha, \beta, \gamma \in \mathbf{C}$ are certain complex parameters, not all equal to zero simultaneously. Suppose that the parameter α is different from zero. Then the boundary form of the operator restricted to the linear set of functions satisfying the boundary conditions is given by

$$\begin{aligned} \ll \mathcal{A}\mathcal{U}, \mathcal{V} \gg_{\mathcal{H}} - \ll \mathcal{U}, \mathcal{A}\mathcal{V} \gg_{\mathcal{H}} \quad = \quad & \bar{u}_1 v_1 c\left(\tfrac{\beta}{\alpha} - \tfrac{\bar{\beta}}{\bar{\alpha}}\right) + \bar{u}_2 v_2 b\left(-\tfrac{\gamma}{\alpha} + \tfrac{\bar{\gamma}}{\bar{\alpha}}\right) \\ & + \bar{u}_1 v_2\left(a + b\tfrac{\bar{\beta}}{\bar{\alpha}} + c\tfrac{\gamma}{\alpha}\right) - \bar{u}_2 v_1 \overline{\left(a + b\tfrac{\bar{\beta}}{\bar{\alpha}} + c\tfrac{\gamma}{\alpha}\right)}. \end{aligned} \tag{3.6}$$

This expression vanishes for arbitrary $u_{1,2}, v_{1,2}$ if and only if the following three conditions are satisfied:

$$\frac{\beta}{\alpha} \in \mathbf{R}; \quad \frac{\gamma}{\alpha} \in \mathbf{R}; \quad a + b\frac{\bar{\beta}}{\bar{\alpha}} + c\frac{\gamma}{\alpha} = 0. \tag{3.7}$$

The first two conditions imply that the complex parameters α, β and γ have equal phase. Hence, without loss of generality, we can restrict our consideration to the case of real parameters, since the boundary condition (3.5) is linear. Then the first two conditions are fulfilled automatically. The third condition can be written as

$$\alpha a + \beta b + \gamma c = 0 \Leftrightarrow (\alpha, \beta, \gamma) \perp (a, b, c). \tag{3.8}$$

The symmetric restrictions of $\mathcal{A}$ have been described by three real parameters $(\alpha, \beta, \gamma) \in \mathbf{R}^3$ satisfying (3.8). Since the length of the vector (α, β, γ) does not play any role, all of the Lagrangian planes can be parameterized by one real parameter $\theta \in [0, 2\pi)$ as follows:

$$(\alpha, \beta, \gamma) = (b \sin\theta, -a \sin\theta - c \cos\theta, b \cos\theta), \tag{3.9}$$

where we have taken into account that b is not equal to zero due to (2.4). We are going to use the following definition in what follows

Definition 3.1. The operator $\mathcal{A}_\theta$, $\theta \in [0, 2\pi)$, is the restriction of the operator $\mathcal{A}$ defined by (3.2) to the domain of functions $\mathcal{U} = (u, u_1) \in \mathcal{H}$ possessing the representation

$$(u, u_1) = (u_r + u_2 g_2, u_1),\ u_r \in H_3, u_{1,2} \in \mathbf{C}$$

and satisfying the boundary condition

$$b \sin\theta \langle \varphi, u_r \rangle - (a \sin\theta + c \cos\theta) u_1 + b \cos\theta u_2 = 0. \tag{3.10}$$

4. Self–adjointness

Theorem 4.1. *The operator $\mathcal{A}_\theta$ is a self–adjoint operator in $\mathcal{H}$ with the scalar product $\ll \cdot, \cdot \gg_{\mathcal{H}}$.*

Proof. It has already been proven that the operator $\mathcal{A}_\theta$ is symmetric. We are going to prove that it is self–adjoint by calculating its resolvent for large negative $\lambda,\ \lambda \ll 0$.

We prove that the range of the operator $\mathcal{A}_\theta - \lambda$ coincides with the Hilbert space $\mathcal{H}$

$$\mathcal{R}(\mathcal{A}_\theta - \lambda) = \mathcal{H},$$

i.e. that for for any $\mathcal{V} = (v, v_1) \in \mathcal{H}$ there exits an element $\mathcal{U} = (u_r + u_2 g_2, u_1) \in \mathrm{Dom}(\mathcal{A}_\theta)$ such that

$$(\mathcal{A}_\theta - \lambda)\mathcal{U} = \mathcal{V}.$$

The last equation can be written as

$$\begin{cases} (A - \lambda) u_r - (a_2 + \lambda) u_2 g_2 & = & v; \\ \\ u_2 - (a_1 + \lambda) u_1 & = & v_1. \end{cases}$$

The first of these equations can be rewritten as

$$u_r - (a_2 + \lambda) \frac{1}{A - \lambda} g_2 u_2 = \frac{1}{A - \lambda} v,$$

which implies

$$\langle \varphi, u_r \rangle - (a_2 + \lambda) \langle \varphi, \frac{1}{A - \lambda} g_2 \rangle u_2 = \langle \varphi, \frac{1}{A - \lambda} v \rangle.$$

The vector $\mathcal{U} = (u_r + u_2 g_2, u_1)$ should satisfy the boundary condition (3.5). Hence the vector $(\langle \varphi, u_r \rangle, u_1, u_2) \in \mathbf{C}^3$ solves the system of linear equations

$$\begin{pmatrix} 1 & 0 & -(a_2 + \lambda)\langle \varphi, \frac{1}{A-\lambda} g_2 \rangle \\ 0 & -(a_1 + \lambda) & 1 \\ b \sin\theta & -a \sin\theta - c \cos\theta & b \cos\theta \end{pmatrix} \begin{pmatrix} \langle \varphi, u_r \rangle \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \langle \varphi, \frac{1}{A-\lambda} v \rangle \\ v_1 \\ 0 \end{pmatrix}. \tag{4.1}$$

The determinant of this system is given by

$$-(a_1+\lambda)b\cos\theta - b\sin\theta(a_1+\lambda)(a_2+\lambda)\langle\varphi, \frac{1}{A-\lambda}g_2\rangle + a\sin\theta + c\cos\theta. \tag{4.2}$$

To prove the theorem we have to show that the determinant is not equal to zero for some λ.

Suppose first that $\theta \neq 0$. Then the second term is dominant for very small negative λ, $\lambda < 0$. Let us denote by σ_φ the spectral measure associated with the positive self–adjoint operator A and the element $\varphi \in H_{-3} \setminus H_{-2}$. Then we have:

$$\int_0^\infty \frac{1}{(\mu+1)^3} d\sigma_\varphi(\mu) = \langle\varphi, \frac{1}{(A+1)^3}\varphi\rangle; \quad \lim_{N\to\infty}\int_0^N \frac{1}{(\mu+1)^2} d\sigma_\varphi(\mu) = \infty. \tag{4.3}$$

Actually the estimate

$$\begin{aligned} \left|\left\langle\varphi, \tfrac{\lambda}{A-\lambda}\tfrac{1}{A+a_1}\tfrac{1}{A+a_2}\varphi\right\rangle\right| &= \textstyle\int_0^\infty \frac{|\lambda|}{\mu+|\lambda|}\frac{1}{\mu+a_1}\frac{1}{\mu+a_2} d\sigma_\varphi(\mu) \\ &\geq \textstyle\int_0^N \frac{|\lambda|}{\mu+|\lambda|}\frac{1}{\mu+a_1}\frac{1}{\mu+a_2} d\sigma_\varphi(\mu) \\ &\geq \textstyle\frac{|\lambda|}{N+|\lambda|}\int_0^N \frac{1}{\mu+a_1}\frac{1}{\mu+a_2} d\sigma_\varphi(\mu), \end{aligned}$$

which is uniform in $N > 0$, implies that

$$\lim_{\lambda\to-\infty}\left|\left\langle\varphi, \frac{\lambda}{A-\lambda}\frac{1}{A+a_1}\frac{1}{A+a_2}\varphi\right\rangle\right| = \infty,$$

if one takes into account (4.3). It follows that the asymptotics of the determinant for small negative λ is given by

$$-b\sin\theta\lambda\langle\varphi, \frac{\lambda}{A-\lambda}g_2\rangle$$

and it follows that the determinant is different from zero in a neighborhood of $-\infty$.

Consider now the case $\theta = 0$. The determinant is then given by

$$-a_1 b + c - b\lambda$$

and is different from zero for small negative λ, since $b \neq 0$ according to (2.4).

We have proven that the linear system (4.1) has a unique solution. Then the element $\mathcal{U}$ can be calculated to be

$$\mathcal{U} = \left(\frac{1}{A-\lambda}v + \frac{1}{A-\lambda}g_1 u_2, u_1\right).$$

The theorem is proven.

We have actually proven that the operator $\mathcal{A}_\theta$ is not only self–adjoint, but semibounded from below. Moreover the resolvent of the operator can easily be calculated. The solution of the linear system (4.1) is given by

$$\begin{aligned} \langle\varphi, u_r\rangle &= \frac{(a\sin\theta+(c-b(a_1+\lambda))\cos\theta)\langle\varphi,\frac{1}{A-\lambda}v\rangle+(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle(a\sin\theta+c\cos\theta)v_1}{\cos\theta(c-b(a_1+\lambda))+\sin\theta\big(a-b(a_1+\lambda)(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\big)}; \\ u_1 &= \frac{b\sin\theta\langle\varphi,\frac{1}{A-\lambda}v\rangle+b\big(\cos\theta+(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\sin\theta\big)v_1}{\cos\theta(c-b(a_1+\lambda))+\sin\theta\big(a-b(a_1+\lambda)(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\big)}; \\ u_2 &= \frac{b\sin\theta(a_1+\lambda)\langle\varphi,\frac{1}{A-\lambda}v\rangle+(a\sin\theta+c\cos\theta)v_1}{\cos\theta(c-b(a_1+\lambda))+\sin\theta\big(a-b(a_1+\lambda)(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\big)}. \end{aligned}$$

Then the resolvent can be calculated as

$$\frac{1}{\mathcal{A}_\theta-\lambda}(v,v_1)=\left(\frac{1}{A-\lambda}v+\left(\frac{1}{A-\lambda}g_1\right)u_2,u_1\right). \tag{4.4}$$

The resolvent restricted to the subspace $H_1\subset\mathcal{H}$ of functions $\mathcal{V}=(v,v_1)\in\mathcal{H}$ with zero component $v_1=0$ is given by

$$\rho_{\frac{1}{\mathcal{A}_\theta-\lambda}}|_{H_1}v=\tfrac{1}{A-\lambda}v+ \tag{4.5}$$

$$+\frac{b\sin\theta}{\cos\theta(c-b(a_1+\lambda))+\sin\theta\left(a-b(a_1+\lambda)(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\right)}\left(\tfrac{1}{A-\lambda}\varphi\right)\left\langle\varphi,\tfrac{1}{A-\lambda}v\right\rangle.$$

Consider the special case $\theta=0$. In this case the resolvent and the restricted resolvent are given by

$$\frac{1}{\mathcal{A}_0-\lambda}\mathcal{V}=\left(\frac{1}{A-\lambda}v+\left(\frac{1}{A-\lambda}g_1\right)\frac{1}{\frac{c}{b}-a_1-\lambda}v_1,\frac{1}{\frac{c}{b}-a_1-\lambda}v_1\right) \tag{4.6}$$

and

$$\frac{1}{A-\lambda}|_{H_1}=\frac{1}{\mathcal{A}_0-\lambda}|_{H_1}, \tag{4.7}$$

respectively. The range of the restricted resolvent in this case is a subset of H_1 again. Moreover the restricted resolvent coincides with the resolvent of the original operator A, and this property is characteristic of the operator $\mathcal{A}_0$. In other words, the domain of the operator $\mathcal{A}_0$ contains the domain of the original operator A, and the action of the operators $\mathcal{A}_0$ and A restricted to this domain coincide,

$$\mathcal{A}_0|_{\mathrm{Dom}(A)}=A.$$

Therefore the operator $\mathcal{A}_0$ should be considered as an **unperturbed operator**, since this is the unique operator possessing the properties described above. All of the other operators $\mathcal{A}_\theta$ corresponding to $\theta\neq0$ are perturbations of $\mathcal{A}_0$.

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Dept. of Math., Stockholm University, 10691 Stockholm, Sweden
E-mail address: pak@matematik.su.se

Dept. of Mathematics, Gakushuin Univ., 1-5-1 Mejiro, Toshima-ku, Tokyo, 171-8588 Japan
E-mail address: kazuo.watanabe@gakushuin.ac.jp