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On the Essential Spectrum of a Class of Singular Matrix Differential Operators. I: Quasiregularity Conditions and Essential Self-adjointness

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Abstract. The essential spectrum of singular matrix differential operator determined by the operator matrix

$$\begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x}\rho(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x) & \frac{\mathrm{d}}{\mathrm{d}x}\frac{\beta}{x} \\ -\frac{\beta}{x}\frac{\mathrm{d}}{\mathrm{d}x} & \frac{m(x)}{x^2} \end{pmatrix}$$

is studied. It is proven that the essential spectrum of any self-adjoint operator associated with this expression consists of two branches. One of these branches (called regularity spectrum) can be obtained by approximating the operator by regular operators (with coefficients which are bounded near the origin), the second branch (called singularity spectrum) appears due to singularity of the coefficients.

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1. Introduction

Systems of ordinary and partial differential and pseudodifferential equations is a subject of interest for many mathematicians (see [19] and numerous references therein). Matrix ordinary differential operators of mixed order appear in many problems of theoretical physics: hydrodynamics, plasma physics, quantum field theory, and others. Mathematically rigorous treatment of such problems has been carried out by several authors: J. A. Adam, V. Adamyan, J. Descluox, G. Geymonat, G. Grubb, T. Kako, H. Langer, A. E. Lifchitz, R. Mennicken, M. Möller, G. D. Raikov, A. Shkalikov, and others [1, 2, 4, 5, 8–11, 15, 17, 20–23, 28, 29, 31, 37]. Matrix differential operators with singular coefficients are of special interest in plasma physics, for example so-called force operators describing equilibrium state of plasma in toroidal region are exactly of this kind [20]. A more general class of 3×3 matrix differential operators with singularities was considered by V. Hardt, R. Mennicken, and S. Naboko [17], where a new branch of the essential spectrum determined by the singularity was observed and described. This new branch had been predicted by J. Descloux and G. Geymonat [5]. To study the essential spectrum of the operator, so-called quasiregularity conditions were introduced ([17]). These conditions are necessary and sufficient for the boundedness of the essential spectrum of the singular operator. A different approach to this class of matrix operators satisfying the quasiregularity conditions was developed by M. Faierman, R. Mennicken, and M. Möller [10]. Recently, R. Mennicken, S. Naboko, and Ch. Tretter suggested clarifying approach to study this class of singular operators ([30]). It was discovered that the new branch of the essential spectrum can be characterized as the zero set for the symbol of the asymptotic Hain–Lüst operator introduced in [30]. It should be mentioned that in the new approach, the authors used Proposition A.1 from the current paper.

Investigation of the essential spectrum of differential and partial differential operators attracts attention of many scientists ([40, 41, 44]). For example the spectrum of pseudodifferential operators with piecewise continuous symbols has been investigated by S. C. Power [35, 36]. In [14] (Chapter 3), it is shown how to calculate the essential spectrum for pseudodifferential boundary value problems using the principal interior and boundary symbol operators.

A new class of matrix differential operators with singular coefficients is introduced and investigated in this paper. This class consists of 2×2 matrices instead of the 3×3 operator matrices studied in [30], which is a formal simplification. (The method elaborated in the paper can be applied to $m \times m$ operator matrices.) But all essential features of the problem are still present. Additionally the singularities of the matrix elements are distributed in a different way. We decided to study this class of singular operators in order to illustrate the mechanism of the appearance of the additional branch of essential spectrum using the most explicit example. This helps us to avoid tedious calculations and at the same time preserves the main features of the original problem. For this reason we tried to develop a proper Calkin calculus (see Appendix B), which allows one to justify calculations from [17, 20–22] being incomplete. On the other hand, employment of Calkin calculus makes all calculations transparent and easier. For example, the authors of [4], investigating nonsingular operator matrices, used subtle results from operator theory due to P. E. Sobolevskii [26]. Developing these methods, some new results on Banach space operators were obtained. These results on the spectrum of the sum of three operators are of an abstract nature and can be used in other problems as well. Investigating this problem, we tried to elaborate a new general approach to singular matrix differential operators. We hope to be able to apply this method to most general singular matrix differential operators including partial differential operators.

The operator under investigation is determined formally by the following expression:

$$\begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x}\rho(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x) & \frac{\mathrm{d}}{\mathrm{d}x}\frac{\beta}{x} \\ -\frac{\beta}{x}\frac{\mathrm{d}}{\mathrm{d}x} & \frac{m(x)}{x^2} \end{pmatrix}. \tag{1}$$

We use this form of the matrix differential operator in order to display explicitly the singularities of three matrix elements at the origin. The most interesting (and complicated) case is when the functions β and m do not vanish at the origin. Therefore, the operator defined by the functions β and m having zeroes at the origin of order 1 and 2 respectively, will be called *regular*. In this case, all singularities are artificial. The essential spectrum of the corresponding operator can easily be investigated using the methods of [4]. All other operators from the described class will be called *singular* and we are going to concentrate our attention on the case of singular operators only. It is clear that the matrix symbol does not determine the self-adjoint operator uniquely even in the regular case. The extension theory, of the minimal operator in the regular case has been developed by H. de Snoo [43] and in the case of nonsingular leading matrix coefficient in [38, 39].

Our interest in singular problem is motivated by the new spectral phenomenon which can be observed in this case: the essential spectrum of any selfadjoint operator corresponding to the symbol (1) in $L_2[0, 1] \oplus L_2[0, 1]$ cannot be described as a limit of the essential spectra of the operators determined by the same symbol in $L_2[\epsilon, 1] \oplus L_2[\epsilon, 1]$ as $\epsilon \to +0$. Such limit determines only a certain part of the essential spectrum of the operator in $L_2[0, 1] \oplus L_2[0, 1]$. An additional branch of the essential spectrum appears due to the singularity of the coefficients at the origin. Trivial counterpart of this phenomena is well known for infinite intervals, since for example the essential spectrum of $-(\mathrm{d}^2/\mathrm{d}x^2)$ on a finite interval $[-a_n, b_n]$ is empty and therefore does not give the essential spectrum of $-(\mathrm{d}^2/\mathrm{d}x^2)$ on the whole line when $a_n, b_n \to \infty$. The phenomenon described in the current note is more sophisticated and is due to rather complicated interplay between the components of the matrix differential operator. On the other hand, the coefficient of the matrix determining the operator have singularities at the boundary points. This new branch of essential spectrum is absent in the case of regular operators, since the limit procedure for the essential spectrum described above gives the correct answer in the regular case. Spectral analysis in the regular case is well known and can be carried out using methods developed in [4, 15]. In what follows, the two branches of the essential spectrum will be called the regularity spectrum and singularity spectrum, respectively. We introduce quasiregularity conditions for the singular operator which guarantees boundedness of the regularity spectrum. The quasiregularity conditions determine a special class of singular matrix differential operators for which we are able to calculate the essential spectra. Note that in many physical applications, i.e. in plasma physics ([20]), these conditions are fulfilled.

The singularities of the operator coefficients at the origin play an important role even at the stage of the definition of the self-adjoint operator corresponding to the formal expression (1). The indices of the minimal differential operator produced by the singular point are investigated by considering the extension of the minimal operator to the set of functions satisfying certain symmetric boundary condition at the regular point. (In this way the singular $x = 0$ and regular $x = 1$ endpoints are treated separately and in different ways.) It is proven that this extended operator has trivial deficiency indices (is essentially self-adjoint) if and only if the quasiregularity conditions are satisfied and $\beta(0) \neq 0$. (The condition $\beta(0) = 0$ together with the quasiregularity condition (8) imply for smooth coefficients that $m(0) = m'(0) = 0$ and therefore that the operator L is not singular.) If at least one of the quasiregularity conditions is not satisfied or the function β vanishes at the origin then the deficiency indices of the described extended operator are nontrivial like it is in the regular case. We would like to note that the quasiregularity conditions introduced originally to guarantee boundedness of the regular branch of the essential spectrum play an important role in the investigation of the deficiency indices. (Note that the name quasiregularity conditions has nothing to do with the regularity of the extension problem for the operator. It refers to the essential spectrum only.)

After the family of self-adjoint operators corresponding to the formal expression (1) is determined, we discuss the transformation of the operator using the exponential map of the interval $[0, 1]$ onto the half-infinite interval $[0, \infty)$. This map transforms the singular point at the origin to a point at ∞ and enables us to use the standard Fourier transform in $L_2(\mathbf{R})$. So the reason to use this exponential map is pure technical.

Since we are interested in the essential spectrum of the corresponding self-adjoint operators, the choice of the boundary conditions in the limit circle case is not important. The difference between the resolvents of any two operators from this family is a finite rank operator. To calculate the essential spectrum of any such selfadjoint operator we use the even stronger fact that the essential spectra of any two self-adjoint operators coincide if the difference between their resolvents is compact (Weyl theorem). We develop a so-called cleaning procedure which enables one to reduce the calculation of the essential spectrum of the complicated matrix differential operator given by (1) to the calculation of the essential spectrum of a certain asymptotic singular operator with real coefficients. The singular coefficients of the asymptotic operator are chosen to have the same singularities as those of the original operator. In other words the asymptotic operator is chosen so that the difference between the resolvents of the original and asymptotic operators is compact. The Hain–Lüst operator can be considered as a regularized determinant of the 2×2 matrix differential operator (1), and it plays a very important rôle in the cleaning procedure. In the considered case the Hain–Lüst operator is an ordinary (scalar) second order differential operator in $L_2(\mathbf{R}_+)$. We hope that the approach developed in the current paper can be applied to more general operators including arbitrary dimension matrix differential operators and matrix partial differential operators. The method of cleaning of the resolvent modulo compact operators is of general nature. Several abstract lemmas proven in the present paper can be applied without even minor changes.

To calculate the essential spectrum of the asymptotic operator we use the fact that its resolvent is equal to the separable sum of two pseudodifferential operators. We call the sum of two pseudodifferential operators separable if the symbol of one of these two operators depends only on the space variable, and the symbol of the other operator depends only on the momentum variable. Calculation of the essential spectrum of such operators is based on Proposition A.1 from Appendix A.

We observe that the essential spectrum of the model operator under consideration coincides with the set of zeroes of the symbol of the asymptotic Hain–Lüst operator. That operator is a modified version of the original Hain–Lüst operator which preserves information on the behavior of the coefficients at the singular point only. This operator has a more simple expression: it is a second-order differential operator with constant coefficients. Unfortunately all information concerning the regularity spectrum disappears during this rectification. This probably general relation between the symbol of the asymptotic Hain–Lüst operator and the singularity spectrum will be investigated in one of the forthcoming publications.

The methods developed in this article can easily be extended to include differential operators determined by operator matrices of higher dimension. For example, the case when the coefficient m appearing in (1) is a matrix can easily be investigated. The developed methods can help to study matrix partial differential operators as well. These subjects will be discussed in a future publication.

2. The Minimal Operator

Let us consider the linear operator defined by the following operator valued 2×2 matrix

$$L := \begin{pmatrix} -\dfrac{\mathrm{d}}{\mathrm{d}x}\rho(x)\dfrac{\mathrm{d}}{\mathrm{d}x} + q(x) & \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\beta}{x} \\ -\dfrac{\beta}{x}\dfrac{\mathrm{d}}{\mathrm{d}x} & \dfrac{m(x)}{x^2} \end{pmatrix}, \tag{2}$$

where the real-valued functions $\rho(x), q(x), \beta(x),$ and $m(x)$ are continuously differentiable in the closed interval [0, 1]

$$\rho, q, \beta, m \in C^2[0, 1]. \tag{3}$$

In addition we suppose that the density function ρ is positive (definite)

$$\rho(x) \geqslant \rho_0 > 0. \tag{4}$$

Certainly these conditions on the coefficients are far from being necessary for our analysis, but we assume these conditions in order to avoid unnecessary complications. In this way we are able to present certain new ideas explicitly without getting the most optimal result.

The operator matrix (2) determines rather complicated matrix differential operator. Indeed in its formal determinant which controls the spectrum of the whole operator the differential order of the formal product of the diagonal elements

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}+q(x)\right)\frac{m(x)}{x^2}$$

coincides with that of the formal product of the antidiagonal elements

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\beta}{x}\left(-\frac{\beta}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

The same holds true for the orders of the singularities at the origin. These relations can be expressed by the diagrams $2+0 = 1+1$ for the order of differential operators and $0+2 = 1+1$ for the orders of the power-like singularities at the origin. These conditions imply that the nondiagonal coupling cannot be considered as a weak perturbation of the diagonal part of the operator and therefore no existing perturbation theory can be applied to the study of the operator. The aim of this article is to describe new spectral phenomena appearing due to this interplay between the singularities.

The operator matrix given by (2) does not determine unique self-adjoint operator in the Hilbert space $\mathcal{H} = L_2[0, 1] \oplus L_2[0, 1]$. To describe the family of self-adjoint operators corresponding to (2) let us consider the minimal operator $\mathbf{L}_{\min}$ with the domain $C_0^\infty(0, 1) \oplus C_0^\infty(0, 1)$. The operator $\mathbf{L}_{\min}$ is symmetric but is not self-adjoint. Let us keep the same notation for the closure of the operator.

Any self-adjoint operator corresponding to the operator matrix (2) is an extension of the minimal operator $\mathbf{L}_{\min}$. It will be shown in Section 4 that the deficiency indices of $\mathbf{L}_{\min}$ are finite and all self-adjoint extensions of the operator can be described by certain boundary conditions at the end points of the interval [0, 1]. In what follows we are going to consider local boundary conditions only. Such boundary conditions do not connect the boundary values of functions at different end point of the interval. As usual each self-adjoint extension of the operator $\mathbf{L}_{\min}$ is a restriction of the adjoint operator $\mathbf{L}_{\min}^* \equiv \mathbf{L}_{\max}$, which is defined by the same operator matrix (2) on the domain of functions from $W_2^2[0, 1] \oplus W_2^1[0, 1] \subset \mathcal{H}$ satisfying the following two additional conditions ([33])

$$-\frac{\mathrm{d}}{\mathrm{d}x}\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}u_1 + qu_1 + \frac{\mathrm{d}}{\mathrm{d}x}\frac{\beta}{x}u_2 \in L_2[0, 1];$$

$$-\frac{\beta}{x}\frac{\mathrm{d}}{\mathrm{d}x}u_1 + \frac{m}{x^2}u_2 \in L_2[0, 1].$$

Since the original operator $\mathbf{L}_{\min}$ has finite deficiency indices, the difference between the resolvents of any two self-adjoint extensions of $\mathbf{L}_{\min}$ is a finite rank operator. Therefore all these self-adjoint operators have just the same essential spectrum by the Weyl theorem [24].

3. Quasiregularity Conditions

Consider an arbitrary self-adjoint extension $\mathbf{L}$ of the operator $\mathbf{L}_{\min}$. The essential spectrum of the operator $\mathbf{L}$ will be denoted by $\sigma_{\text{ess}}(\mathbf{L})$ in what follows. One part of $\sigma_{\text{ess}}(\mathbf{L})$ can be calculated using the Glazman splitting method (see [3]) already at this stage. Indeed consider the operator $\mathbf{L}^0(\epsilon)$ being the restriction of the operator $\mathbf{L}$ to the domain

$$\text{Dom}(\mathbf{L}^0(\epsilon)) = \left\{ F = (f_1, f_2) \in \text{Dom}(\mathbf{L}) : f_1(\epsilon) = \frac{\mathrm{d}}{\mathrm{d}x} f_1(\epsilon) = f_2(\epsilon) = 0 \right\}.$$

Consider the following decomposition of the Hilbert space

$$L_2[0,1] = L_2[0,\epsilon] \oplus L_2[\epsilon, 1].$$

The corresponding decomposition of the Hilbert space $\mathcal{H}$ is defined as follows

$$\mathcal{H} = \mathcal{H}_\epsilon \oplus \mathcal{H}^\epsilon = (L_2[0,\epsilon] \oplus L_2[0,\epsilon]) \oplus (L_2[\epsilon,1] \oplus L_2[\epsilon,1]).$$

Using this decomposition the operator $\mathbf{L}^0(\epsilon)$ can be represented as an orthogonal sum of two symmetric operators acting in $\mathcal{H}_\epsilon$ and $\mathcal{H}^\epsilon$ respectively. The point $x = \epsilon$ is regular for the operator matrix (2) and one of the self-adjoint extensions of the operator $\mathbf{L}^0(\epsilon)$ is defined by Dirichlet boundary conditions at $x = \epsilon^\pm$. (The fact that the Dirichlet boundary condition at any regular point determines a self-adjoint extension is not trivial for matrix differential operators and has been proven rigorously in [43].) Let us denote this extension by $\mathbf{L}(\epsilon)$.

The difference between the resolvents of the operators $\mathbf{L}(\epsilon)$ and $\mathbf{L}$ is at most a rank 2 operator. Therefore the essential spectra of these two operators coincide. In particular the essential spectrum of the operator $\mathbf{L}$ contains the essential spectrum of the operator $\mathbf{L}(\epsilon)$ restricted to the subspace $\mathcal{H}^\epsilon = L_2[\epsilon,1] \oplus L_2[\epsilon,1]$

$$\sigma_{\text{ess}}(\mathbf{L}) \supset \sigma_{\text{ess}}(\mathbf{L}(\epsilon)|_{\mathcal{H}^\epsilon}), \quad \epsilon \in (0,1). \tag{5}$$

The restricted operator $\mathbf{L}(\epsilon)|_{\mathcal{H}^\epsilon}$ is a regular matrix self-adjoint operator and its essential spectrum can be calculated using the results of [4] (Theorem 4.5)

$$\sigma_{\text{ess}}(\mathbf{L}(\epsilon)|_{L_2(\epsilon,1)}) = \text{Range}_{x\in[\epsilon,1]} \left(\frac{m(x)}{x^2} - \frac{\beta(x)^2}{x^2 \rho(x)} \right). \tag{6}$$

For any $\epsilon > 0$ the essential spectrum of $\mathbf{L}(\epsilon)|_{\mathcal{H}^\epsilon}$ fills in a certain finite interval, since the functions m, β, and ρ^{-1} are finite and therefore bounded on $[\epsilon, 1]$. Since obviously

$$\sigma_{\text{ess}}(\mathbf{L}) \supset \bigcup_{\epsilon>0} \sigma_{\text{ess}}(\mathbf{L}(\epsilon)|_{\mathcal{H}^\epsilon}) = \text{Range}_{x\in(0,1]} \left(\frac{m(x)}{x^2} - \frac{\beta(x)^2}{x^2\rho(x)} \right), \tag{7}$$

the essential spectrum of $\mathbf{L}$ is bounded only if the following quasiregularity conditions hold

$$\rho m - \beta^2|_{x=0} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)|_{x=0} = 0. \tag{8}$$

The quasiregularity conditions appeared first in [17] and were also used later in [9, 10]. Note that the function $(\rho m - \beta^2)/x^2$ is related to the leading coefficient of the formal determinant of the matrix L (2).

The rôle of the quasiregularity conditions is explained by the following statement based on formula (51) to be proven in Section 8.

LEMMA 3.1. *Under the assumptions* (3) *and* (4) *on the coefficients* ρ, β, m, *and* q *the quasiregularity conditions are fulfilled if and only if the essential spectrum of at least one* (*and, hence, any*) *self-adjoint extension of* $\mathbf{L}_{\min}$ *is bounded.*

Proof. Formula (7) implies that quasiregularity conditions are fulfilled if the essential spectrum for at least one self-adjoint extension of $\mathbf{L}_{\min}$. Here we used that the coefficients satisfy (3). On the other hand, formula (51) valid for any operator matrix satisfying the quasiregularity conditions implies the boundedness of the essential spectrum for all self-adjoint extensions of $\mathbf{L}_{\min}$. The lemma is proven, provided formula (51) holds true. □

In what follows we are going to call the matrix L *quasiregular* if the quasiregularity conditions (8) on the coefficients are satisfied. Regular matrices form a subset of quasiregular operator matrices. The subfamily of regular matrices can be characterized by one of the following two additional conditions

$$m(0) = 0 \quad \vee \quad \beta(0) = 0. \tag{9}$$

Really each of these conditions together with the first quasiregularity condition imply the other one. Then the second quasiregularity condition implies that $m'(0) = 0$. Hence, the corresponding matrix is regular, since $m(0) = m'(0) = \beta(0) = 0$. Therefore we are going to concentrate our attention on the case of quasiregular matrices which are not regular, since the regular matrices have been studied earlier ([43]).

4. Deficiency Indices

Self-adjoint extensions of the minimal operator $\mathbf{L}_{\min}$ are investigated in this section. These extensions can be described by certain (generalized) boundary conditions on the functions from the domain of the extended operator. These boundary conditions relates the boundary values at the endpoints $x = 0$ and $x = 1$. We restrict our studies to local boundary conditions. The boundary conditions are called *local* if they do not join together the boundary values at different points.

Every self-adjoint extension of the operator $\mathbf{L}_{\min}$ is a certain restriction of the adjoint operator $\mathbf{L}^*_{\min}$. To calculate the adjoint operator it is enough to consider the operator $\mathbf{L}_{\min}$ restricted to the set of functions from $C_0^\infty(0, 1) \oplus C_0^\infty(0, 1)$, since the adjoint operator is invariant under closure. One concludes using standard calculations ([33]) that the adjoint operator is determined by the same operator valued matrix (2) on the set of functions satisfying the following five conditions

(1) $U = (u_1, u_2) \in L_2[0, 1] \oplus L_2[0, 1];$ (10)

(2) $u_1 \in W_2^1(\epsilon, 1)$ for any $0 < \epsilon < 1$; (11)

(3) The function

$$\omega_U(x) := -\rho(x)u_1'(x) + \frac{\beta(x)}{x}u_2(x) \tag{12}$$

is absolutely continuous on [0, 1];

(4) $$\frac{\mathrm{d}}{\mathrm{d}x}\omega_U(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}u_1 + \frac{\beta(x)}{x}u_2\right) \in L_2[0, 1]; \tag{13}$$

(5) $$-\frac{\beta(x)}{x}\frac{\mathrm{d}}{\mathrm{d}x}u_1 + \frac{m}{x^2}u_2 \in L_2[0, 1]. \tag{14}$$

The function ω_U is called *transformed derivative*⋆ and is well-defined for any function

$$U = (u_1, u_{2)}), \qquad u_1 \in W^1_{2,\mathrm{loc}}(0, 1) \cap L_2[0, 1], \qquad u_2 \in L_2[0, 1].$$

The transformed derivative appearing in the boundary conditions for the matrix differential operator L plays the same rôle as the usual derivative for the standard one-dimensional Schrödinger operator. The function ω_U corresponding to $U \in \mathrm{Dom}(L^*)$ belongs to $W_2^1(0, 1)$, since it is absolutely continuous and (13) holds.

Let us calculate the sesquilinear boundary form of the adjoint operator. This form can be used to describe all self-adjoint extensions of $\mathbf{L}_{\min}$ as restrictions of the adjoint operator to Lagrangian planes with respect to this form. Let U, $V \in \mathrm{Dom}(\mathbf{L}^*_{\min})$, then integrating by parts we get

$$\begin{aligned}
&\langle \mathbf{L}^*_{\min}U, V\rangle - \langle U, \mathbf{L}^*_{\min}V\rangle \\
&= \left\langle \frac{\mathrm{d}}{\mathrm{d}x}\left(-\rho u_1' + \frac{\beta}{x}u_2\right), v_1\right\rangle + \left\langle -\frac{\beta}{x}\frac{\mathrm{d}}{\mathrm{d}x}u_1 + \frac{m}{x^2}u_2, v_2\right\rangle - \\
&\quad - \left\langle u_1, \frac{\mathrm{d}}{\mathrm{d}x}\left(-\rho v_1' + \frac{\beta}{x}v_2\right)\right\rangle - \left\langle u_2, -\frac{\beta}{x}\frac{\mathrm{d}}{\mathrm{d}x}v_1 + \frac{m}{x^2}v_2\right\rangle \\
&= \lim_{\epsilon\searrow 0, \tau\nearrow 1}\left\{\int_\epsilon^\tau \left(\frac{\mathrm{d}}{\mathrm{d}x}\omega_U\right)\overline{v_1}\,\mathrm{d}x + \int_\epsilon^\tau\left(-\frac{\beta}{x}u_1' + \frac{m}{x^2}u_2\right)\overline{v_2}\,\mathrm{d}x - \right.\\
&\quad \left. - \int_\epsilon^\tau u_1\left(\frac{\mathrm{d}}{\mathrm{d}x}\overline{\omega_V}\right)\mathrm{d}x - \int_\epsilon^\tau u_2\left(-\frac{\beta}{x}\overline{v_1}' + \frac{m}{x^2}\overline{v_2}\right)\mathrm{d}x\right\} \\
&= \lim_{\epsilon\searrow 0, \tau\nearrow 1}\left\{\omega_U(x)\overline{v}_1'(x)|_{x=\epsilon}^\tau - \int_\epsilon^\tau \omega_U\overline{v_1}\,\mathrm{d}x - \int_\epsilon^\tau \frac{\beta}{x}u_1'\overline{v_2}\,\mathrm{d}x - \right.
\end{aligned}$$

⋆ The transformed derivative is a generalization of the quasi-derivatives described, for example, by W. N. Everitt, C. Bennewitz and L. Markus [6, 7].

$$- u_1(x)\overline{\omega_V}(x)|_{x=\epsilon}^{\tau} + \int_{\epsilon}^{\tau} u_1'\overline{\omega_V}\,\mathrm{d}x + \int_{\epsilon}^{\tau} u_2\frac{\beta}{x}\overline{v_1}'\,\mathrm{d}x\Big\}$$

$$= \lim_{\epsilon\searrow 0,\tau\nearrow 1}\{\omega_U(x)\overline{v}_1(x)|_{x=\epsilon}^{\tau} - u_1(x)\overline{\omega_V}(x)|_{x=\epsilon}^{\tau}\}. \tag{15}$$

Note that the limits in the last formula cannot be always substituted by the limit values of the functions, since the functions u_1 and v_1 are not necessarily bounded at the origin. On the other hand the limit as $\tau \nearrow 1$ can be calculated using continuity of all four functions at the regular endpoint $x = 1$. This boundary form will be used to determine the deficiency indices of the operator $\mathbf{L}_{\min}$ and describe its self-adjoint extensions. This method of using boundary forms to describe self-adjoint extensions of symmetric operators is classical and is well described for example in [3] (vol. 2) and [33].

THEOREM 4.1. *The operator $\mathbf{L}_{\min}$ is a symmetric operator in the Hilbert space $\mathcal{H}$ with finite equal deficiency indices.*

(1) *If the operator matrix L is singular quasiregular (i.e. quasiregularity conditions are satisfied and $m(0) \neq 0$), then the deficiency indices of $\mathbf{L}_{\min}$ are equal to $(1, 1)$ and all self-adjoint extensions of $\mathbf{L}_{\min}$ are described by the standard boundary condition*

$$\omega_U(1) = h_1 u_1(1), \quad h_1 \in \mathbf{R} \cup \{\infty\}. \tag{16}$$

(2) *If the operator matrix is regular or is not quasiregular then the deficiency indices of $\mathbf{L}_{\min}$ are equal to $(2, 2)$. The self-adjoint extensions of $\mathbf{L}_{\min}$ are described by pair of boundary conditions using the following alternatives covering all possibilities:*

(a) *If $\rho(0)m(0) - \beta^2(0) \neq 0$ or $\beta(0) = 0$, then the first component u_1 of any vector from the domain of the adjoint operator $\mathbf{L}^*_{\min}$ is continuous on the closed interval $[0, 1]$. All local* ⋆ *self-adjoint extensions of the operator $\mathbf{L}_{\min}$ are described by the* standard *boundary conditions* ⋆⋆

$$\omega_U(1) = h_1 u_1(1), \qquad \omega_U(0) = h_0 u_1(0), \quad h_{0,1} \in \mathbf{R} \cup \{\infty\}. \tag{17}$$

(b) *If*

$$\rho(0)m(0) - \beta^2(0) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)(0) \neq 0, \quad \textit{and} \quad \beta(0) \neq 0,$$

then the first component u_1 of any vector from the domain of the adjoint operator $\mathbf{L}^*_{\min}$ admits the asymptotic representation

$$u_1(x) = k w_U(0)\ln x + c_U + \mathrm{o}(1), \quad \textit{as } x \to 0, \tag{18}$$

⋆ The family of all self-adjoint extensions of $\mathbf{L}_{\min}$ can easily be described using our analysis. The corresponding formulas are not written here only in order to make the presentation more transparent.

⋆⋆ In the case $h_\alpha = \infty$, $\alpha = 0, 1$ the corresponding boundary condition should be written as $u_1(\alpha) = 0$ or $c_U = 0$.

where

$$k = -\frac{\beta^2(0)}{\rho(0)} \frac{1}{\frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)|_{x=0}}$$

and c_U is an arbitrary constant depending on U. Then all local self-adjoint extensions of the operator $\mathbf{L}_{\min}$ are described by the nonstandard *boundary conditions* ⋆⋆

$$\omega_U(1) = h_1 u_1(1), \qquad \omega_U(0) = h_0 c_U, \quad h_{0,1} \in \mathbf{R} \cup \{\infty\}. \tag{19}$$

Information concerning the deficiency indices of $\mathbf{L}_{\min}$ and self-adjoint local boundary conditions is collected in Table I.

Proof. In order to describe all local boundary conditions the points $x = 0$ and $x = 1$ can be considered separately. The point $x = 1$ is a regular boundary point, since the functions $\rho^{-1}, \beta/x, m/x^2$ are infinitely differentiable in a neighborhood of this point. The symmetric boundary condition at the point $x = 1$ can be written in the form

$$\omega_U(1) = h_1 u_1(1), \tag{20}$$

where $h_1 \in \mathbf{R} \cup \infty$ is a real constant parametrizing all symmetric conditions (see [43] and Case C below for details). The extension of the operator $\mathbf{L}_{\min}$ to the set of infinitely differentiable functions with support separated from the origin and satisfying condition (20) at the point $x = 1$ will be denoted by $\mathbf{L}_{h_1}$.

Let us study the deficiency indices of the operator $\mathbf{L}_{h_1}$. The operator adjoint to $\mathbf{L}_{h_1}$ is the restriction of $\mathbf{L}^*_{\min}$ to the set of functions satisfying (20). This operator is defined by the operator matrix with real coefficients, therefore the deficiency

Table I.

	$\rho(0)m(0) - \beta^2(0) \neq 0$	$\rho(0)m(0) - \beta^2(0) = 0$	
		$\frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)\vert_{x=0} \neq 0$	$\frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)\vert_{x=0} = 0$
	A	B	C
$\beta(0) = 0$	*indices* (2,2)	*indices* (2,2)	*indices* (2,2)
	2 *standard b.c.* (17)	2 *standard b.c.* (17)	2 *standard b.c.* (17)
$\beta(0) \neq 0$	*indices* (2,2)	*indices* (2,2)	*indices* (1,1)
	2 *standard b.c.* (17)	2 *nonstandard b.c.* (19)	1 *standard b.c.* (16)

The letters A, B, and C refer to the three cases considered in the proof of the theorem.

indices of $\mathbf{L}_{h_1}$ are equal. Moreover, the differential equation on the deficiency element g^λ for any $\lambda \notin \mathbf{R}$ [3] is given by

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(-\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}g_1^\lambda + \frac{\beta(x)}{x}g_2^\lambda\right) + q(x)g_1^\lambda = \lambda g_1^\lambda,\\ &-\frac{\beta(x)}{x}\frac{\mathrm{d}}{\mathrm{d}x}g_1^\lambda + \frac{m(x)}{x^2}g_2^\lambda = \lambda g_2^\lambda;\end{aligned} \tag{21}$$

and it can be reduced to the following scalar differential equation for the first component

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho(x) + \frac{\beta(x)}{x}\frac{1}{\lambda - m(x)/x^2}\frac{\beta(x)}{x}\right)\frac{\mathrm{d}}{\mathrm{d}x}g_1^\lambda + q(x)g_1^\lambda = \lambda g_1^\lambda. \tag{22}$$

The component g_2^λ can be calculated from g_1^λ using the formula

$$g_2^\lambda = -\frac{1}{\lambda - m(x)/x^2}\frac{\beta(x)}{x}\frac{\mathrm{d}}{\mathrm{d}x}g_1^\lambda.$$

Equation (22) is a second-order ordinary differential equation with continuously differentiable coefficients. Since the principle coefficient in this equation for non-real λ is separated from zero on the interval $(\epsilon, 1]$, the solutions are two times continuously differentiable functions (18).

Boundary condition (20) implies that the first component satisfies the boundary condition at point $x = 1$

$$-\left(\rho(1) + \frac{\beta^2(1)}{\lambda - m(1)}\right)\frac{\mathrm{d}}{\mathrm{d}x}g_1^\lambda(1) = h_1 g_1^\lambda(1). \tag{23}$$

This condition is nondegenerate, since λ is nonreal. Therefore the subspace of solutions to Equation (21) satisfying condition (20) has dimension 1. But these solutions do not necessarily belong to the Hilbert space $\mathcal{H} = L_2[0,1] \oplus L_2[0,1]$. If the nontrivial solution is from the Hilbert space, $g^\lambda \in \mathcal{H}$, then the operator $\mathbf{L}_{h_1}$ is symmetric with deficiency indices $(1, 1)$. Otherwise the operator $\mathbf{L}_{h_1}$ is essentially self-adjoint ([42]). If the principal coefficient of Equation (22) is bounded and separated from zero on the interval $[0, 1]$, then $g^\lambda \in \mathcal{H}$ and the operator $\mathbf{L}_{h_1}$ has deficiency indices (1,1). The last condition is satisfied if for example $m(0) \neq 0$ and $\rho(0)m(0) - \beta^2(0) \neq 0$, since $\Im\lambda \neq 0$. Complete analysis of Equation (22) can be carried out using WKB method ([34]). We are going instead to analyze the boundary form.

Let us study the singular point $x = 0$ in more detail. We are going to consider the following three possible cases:

(A) The first quasiregularity condition (8) is not satisfied.
(B) The first quasiregularity condition is satisfied, but the second quasiregularity condition (8) is not satisfied.
(C) The quasiregularity conditions (8) are satisfied.

The case C includes the set of regular operator matrices.

Case A. Consider arbitrary cutting function $\varphi \in C^\infty[0, 1]$ equal to 1 in a certain neighborhood of the origin and vanishing in a neighborhood of the point $x = 1$. The function

$$W = (m(0)x\varphi(x), \beta(0)x\varphi(x))$$

obviously belongs to the domain of the adjoint operator $\mathbf{L}^*_{h_1}$, since the support of the function W is separated from the point $x = 1$ and condition (20) is therefore satisfied. The function W is not identically equal to zero, since the first quasiregularity condition (8) is not satisfied.

Consider arbitrary $U \in \mathrm{Dom}\,(\mathbf{L}^*_{\min})$. Then formula (15) implies that the limit

$$\lim_{\epsilon\searrow 0}\{-\omega_U(\epsilon)\overline{w_1}(\epsilon) + u_1(\epsilon)\overline{\omega_W}(\epsilon)\}$$

exists. Taking into account that

ω_U is absolutely continuous on the interval [0, 1];

$\lim_{\epsilon\searrow 0} w_1(\epsilon) = 0$;

$\lim_{\epsilon\searrow 0} \omega_W(\epsilon) = -\rho(0)m(0) + \beta^2(0) \neq 0$;

we conclude that the limit $u_1(0) = \lim_{\epsilon\searrow 0} u_1(\epsilon)$ exists for arbitrary function $U \in \mathrm{Dom}\,(L^*)$. Hence, the boundary form of the operator $\mathbf{L}^*_{h_1}$ is given by

$$\langle \mathbf{L}^*_{h_1} U, W\rangle - \langle U, \mathbf{L}^*_{h_1} W\rangle = -\omega_U(0)\overline{w_1}(0) + u_1(0)\overline{\omega_W}(0),$$

and is not degenerate. The operator $L(h_1)$ has deficiency indices (1,1), and all symmetric boundary conditions at the point $x = 0$ are standard

$$\omega_U(0) = h_0 u_1(0). \tag{24}$$

Case B. Let us introduce the following notation

$$c_0 = \frac{\mathrm{d}}{\mathrm{d}x}(\rho(x)m(x) - \beta^2(x))|_{x=0} \neq 0. \tag{25}$$

In addition we suppose that $\beta(0) \neq 0$. To prove that the boundary form is not degenerate (and hence the deficiency indices of $\mathbf{L}_{h_1}$ are (1, 1)) consider the two vector functions

$$F = \begin{pmatrix} 1 + \int_0^x \frac{\beta(t)}{\rho(t)}\,\mathrm{d}t \\ x \end{pmatrix}, \tag{26}$$

$$G = \begin{pmatrix} -\int_x^1 \left(\frac{c_0}{\rho(0)\beta(0)} + \frac{\beta(t)}{t\rho(t)}\right)\mathrm{d}t \\ 1 \end{pmatrix}. \tag{27}$$

Multiplying the functions F and G by the scalar function φ introduced above one gets functions from the domain of the operator $\mathbf{L}^*_{h_1}$. The fact that these functions satisfy (10), (11), (13), (14) is a result of straightforward calculations. We have

$$\omega_F(\epsilon) \equiv 0, \qquad \lim_{\epsilon\searrow 0} f_1(\epsilon) = 1,$$

and

$$\omega_G(\epsilon) = -\frac{c_0}{\beta(0)\rho(0)}\rho(\epsilon), \qquad g_1(\epsilon) = \frac{\beta(0)}{\rho(0)}(\ln\epsilon) + c_G + \mathrm{o}(1).$$

Hence the boundary form of $\mathbf{L}^*_{h_1}(h_1)$ calculated on φF and φG is given by

$$\langle \mathbf{L}^*_{h_1}\varphi G, \varphi F\rangle - \langle \varphi G, \mathbf{L}^*_{h_1}\varphi F\rangle = \frac{c_0}{\beta(0)} \neq 0.$$

Therefore the deficiency indices of $\mathbf{L}_{h_1}$ are equal to (1,1).

Let us prove that the asymptotic representation (18) holds for any function V from the domain of the operator adjoint to $\mathbf{L}_{\min}$. Consider the boundary form of the adjoint operator calculated on the function V and the above introduced function G. The following limits obviously exist

$$\begin{aligned}
&\exists \lim_{\epsilon\searrow 0}[-\omega_G(\epsilon)\bar{v}_1(\epsilon) + g_1(\epsilon)\bar{\omega}_V(\epsilon)]\\
&\quad = \lim_{\epsilon\searrow 0}\Bigg[-\left(-\frac{c_0}{\beta(0)} + \mathrm{o}(\sqrt{\epsilon})\right)\bar{v}_1(\epsilon) +\\
&\qquad\qquad + \left(\frac{\beta(0)}{\rho(0)}\ln\epsilon + c_U + \mathrm{o}(1)\right)(\bar{\omega}_V(0) + \mathrm{o}(\sqrt{\epsilon}))\Bigg]\\
&\quad \Rightarrow \exists \lim_{\epsilon\searrow 0}\left[\frac{c_0}{\beta(0)}(1 + \mathrm{o}(\sqrt{\epsilon}))\bar{v}_1(\epsilon) + \frac{\beta(0)}{\rho(0)}\bar{\omega_V}(0)\ln\epsilon\right].
\end{aligned}$$

It follows that (18) holds. The parameters $\omega_U(0)$ and c_U are independent, when U runs over $\mathrm{Dom}(\mathbf{L}^*_{h_1})$. This follows easily from the fact that the function $(u_1, u_2) = (1, 0)$ belongs to the domain of $\mathbf{L}^*_{\min}$.

Substituting the asymptotic representation (18) for arbitrary $U, V \in \mathrm{Dom}(\mathbf{L}_{h_1})$ into the boundary form

$$\begin{aligned}
\langle \mathbf{L}^*_{h_1}U, V\rangle - \langle U, \mathbf{L}^*_{h_1}V\rangle &= \lim_{\epsilon\searrow 0}(-\omega_U(\epsilon)\overline{v}_1(\epsilon) + u_1(\epsilon)\overline{\omega_V}(\epsilon))\\
&= -\omega_U(0)\overline{c_V} + c_U\overline{\omega_V}(0).
\end{aligned}$$

Hence all local self-adjoint extensions are described by nonstandard boundary conditions (19).

To complete the study of Case B, let $\beta(0) = 0$. Consider the function F given by (26) and the function $S = \begin{pmatrix} x \\ 0 \end{pmatrix}$. Then the boundary form calculated on the vectors φF and φS is nondegenerate

$$\langle \mathbf{L}^*_{h_1}\varphi S, \varphi F\rangle - \langle \varphi S, \mathbf{L}^*_{h_1}\varphi F\rangle = \rho(0) \neq 0,$$

and therefore the operator $\mathbf{L}_{h_1}$ has deficiency indices (1, 1). Let us prove that the component u_1 of any vector from the domain of the adjoint operator is continuous in the closed interval. Note that

$$\omega_S(x) = -\rho(x) \quad \text{and} \quad s_1(0) = 0.$$

Consider the boundary form of $\mathbf{L}_{h_1}^*$ calculated on φS and arbitrary $V \in \mathrm{Dom}\,(\mathbf{L}_{h_1})$

$$\begin{aligned}\langle \mathbf{L}_{h_1}\varphi S, V\rangle - \langle \varphi S, \mathbf{L}_{h_1} V\rangle &= \lim_{\epsilon\searrow 0}(\rho(\epsilon)\overline{v_1}(\epsilon) + \epsilon\overline{\omega_V}(\epsilon))\\ &= -\lim_{\epsilon\searrow 0}\rho(\epsilon)\overline{v_1}(\epsilon).\end{aligned}$$

Since $\rho(0)$ is not equal to zero, the limit $\lim_{\epsilon\searrow 0} v_1(\epsilon)$ exists and therefore self-adjoint boundary conditions can be written in the standard form (17) as in Case A. This completes investigation of Case B.

Case C. Suppose in addition that $\beta(0) \neq 0$. It follows that the matrix is singular quasiregular. Consider the vector function

$$E = \begin{pmatrix} -\displaystyle\int_x^1 \frac{\beta(t)}{t\rho(t)}\,\mathrm{d}t \\ 1 \end{pmatrix},$$

which belongs to the domain of the adjoint operator $\mathbf{L}_{\min}^*$ due to quasiregular conditions. Therefore $\varphi E \in \mathrm{Dom}(\mathbf{L}_{h_1}^*)$. Then for any function $U \in \mathrm{Dom}(\mathbf{L}_{h_1}^*)$ the boundary form is given by

$$\langle \mathbf{L}_{h_1}^* U, \varphi E\rangle - \langle U, \mathbf{L}_{h_1}^*\varphi E\rangle = -\lim_{\epsilon\searrow 0}\omega_U(\epsilon)\overline{e_1(\epsilon)},$$

since $\omega_E(\epsilon) \equiv 0$. Note that e_1 diverges to infinity due to our assumption $\beta(0) \neq 0$

$$v_1(\epsilon) \sim_{\epsilon\searrow 0} \frac{\beta(0)}{\rho(0)}\ln\epsilon \to \infty.$$

Since the limit $\lim_{\epsilon\searrow 0}\omega_U(\epsilon)$ exists it should be equal to zero

$$\omega_U(0) = 0. \tag{28}$$

Hence taking into account that $\omega_U \in W_2^1[0, 1]$ one concludes that

$$\omega_U(\epsilon) = \mathrm{o}(\sqrt{\epsilon}). \tag{29}$$

On the other hand, condition (13) implies that

$$x\frac{\mathrm{d}}{\mathrm{d}x}u_1 = \frac{\beta}{\rho}u_2 - \frac{x}{\rho}\omega_U \in L_2[0, 1]. \tag{30}$$

It follows from Cauchy inequality that

$$u_1(\epsilon) = \mathrm{O}\left(\frac{1}{\sqrt{\epsilon}}\right). \tag{31}$$

Formulas (29) and (31) imply that the boundary form is identically equal to zero. Therefore the operator $L(h_1)$ is essentially self-adjoint in this case. (Note that each function from the domain of arbitrary self-adjoint extension of $\mathbf{L}_{\min}$ automatically satisfies the boundary condition (28) at the singular point.)

To accomplish the investigation of Case C, assume $\beta(0) = 0$. The first quasiregularity condition (8) implies that $m(0) = 0$. The second quasiregularity condition (8) implies then that $(\mathrm{d}/\mathrm{d}x)m|_{x=0} = 0$. It follows that point zero is a regular point for the operator matrix L. Therefore the deficiency indices of $L(h_1)$ are equal to $(1, 1)$ and the local self-adjoint extensions are described by standard boundary conditions ([17, 43]). We have already proven this result. Indeed taking into account that $u_1 \in W_2^1(0, 1)$ and that the function $\omega(\epsilon)$ is absolutely continuous the above mentioned fact follows immediately from (15). This accomplishes the investigation of Case C. The theorem is proven. □

COROLLARY 4.1. *The theorem implies that the operator* $\mathbf{L}_{h_1}$ *is essentially self-adjoint if and only if the operator matrix is singular quasiregular. Otherwise it has deficiency indices* (1,1).

Nonstandard boundary conditions (19) at the singular point described by Theorem 4.1 are similar to the boundary conditions appearing in the studies of one-dimensional Schrödinger operator with Coulomb potential

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\gamma}{x} \quad \text{in } L_2(\mathbf{R}).$$

In what follows we are going to study the essential spectrum of the self-adjoint extensions of the operator $\mathbf{L}_{\min}$. Since the deficiency indices of this operator are always finite, the essential spectrum does not depend on the particular choice of the boundary conditions. The same holds true for nonlocal boundary conditions and therefore our restriction to the case of local boundary conditions can be waived. Therefore in the course of the paper we are going to denote by $\mathbf{L}$ some self-adjoint extension of the minimal operator.

5. Transformation of the Operator

In the current section we are going to transform the self-adjoint operator $\mathbf{L}$ to another self-adjoint operator acting in the Hilbert space $\mathbf{H} = L_2[0, \infty) \oplus L_2[0, \infty)$. The reason to carry out this transformation is pure technical – we would like to be able to use Fourier transform.

Consider the following change of variables

$$x = \mathrm{e}^{-y}, \qquad \mathrm{d}x = -\mathrm{e}^{-y}\,\mathrm{d}y = -x\,\mathrm{d}y, \tag{32}$$

mapping the interval $[0, \infty)$ onto the interval $[0, 1]$ and the corresponding unitary transformation between the spaces $L_2[0, 1]$ and $L_2[0, \infty)$

$$\Phi\colon\ \psi(x) \mapsto \tilde{\psi}(y) = \psi(\mathrm{e}^{-y})\mathrm{e}^{-y/2}. \tag{33}$$

The points 0 and ∞ are mapped to 1 and 0, respectively, and the following formula holds

$$\int_0^1 \| \psi(x) \|^2 \, \mathrm{d}x = \int_0^\infty \| \psi(\mathrm{e}^{-x}) \|^2 \, \mathrm{e}^{-y} \, \mathrm{d}y.$$

The inverse transform is given by

$$\Phi^{-1}\colon \tilde{\psi}(y) \mapsto \psi(x) = \frac{1}{\sqrt{x}} \tilde{\psi}(-\ln x). \tag{34}$$

To determine the transformed operator denoted by $\mathbf{K}$ let us calculate the transformed operator matrix first componentwise

K_{11}:

$$\begin{aligned}
&\sqrt{x}\left(\left[-\frac{\mathrm{d}}{\mathrm{d}x}\rho\frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right]\frac{1}{\sqrt{x}}\tilde{\psi}(-\ln x)\right)\\
&\quad = \sqrt{x}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\rho\left[\frac{1}{2x^{3/2}}\tilde{\psi}(-\ln x) + \frac{1}{x^{3/2}}\tilde{\psi}'(-\ln x)\right]\right) + q(x)\tilde{\psi}(-\ln x)\\
&\quad = \sqrt{x}\left(\rho'_x\left(\frac{1}{2x^{3/2}}\tilde{\psi}(-\ln x) + \frac{1}{x^{3/2}}\tilde{\psi}'(-\ln x)\right) + \right.\\
&\qquad\qquad + \rho\left[-\frac{3}{4x^{5/2}}\tilde{\psi}(-\ln x) + \frac{1}{2x^{3/2}}\tilde{\psi}'(-\ln x)\frac{-1}{x} + \frac{-3}{2x^{5/2}}\tilde{\psi}'(-\ln x) + \right.\\
&\qquad\qquad\quad \left.\left. + \frac{1}{x^{3/2}}\tilde{\psi}''(-\ln x)\frac{-1}{x}\right]\right) + \\
&\qquad + q(x)\tilde{\psi}(-\ln x)\\
&\quad = -\frac{\rho}{x^2}\tilde{\psi}''(-\ln x) + \left(\frac{\rho'_x}{x} - 2\frac{\rho}{x^2}\right)\tilde{\psi}'(-\ln x) + \left(\frac{\rho'_x}{2x} - \frac{3}{4}\frac{\rho}{x^2}\right)\tilde{\psi}(-\ln x) + \\
&\qquad + q(x)\tilde{\psi}(-\ln x)\\
&\quad = -\frac{\mathrm{d}}{\mathrm{d}y}\frac{\rho}{x^2}\frac{\mathrm{d}}{\mathrm{d}y}\tilde{\psi}(-\ln x) + \left(q(x) + \frac{\rho'_x}{2x} - \frac{3\rho}{4x^2}\right)\tilde{\psi}(-\ln x).
\end{aligned}$$

K_{12}:

$$\begin{aligned}
&\sqrt{x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\beta}{x}\frac{1}{\sqrt{x}}\tilde{\psi}(-\ln x)\right)\\
&\quad = \sqrt{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta}{x^{3/2}}\tilde{\psi}(-\ln x)\right)\\
&\quad = -\frac{\beta}{x^2}\tilde{\psi}'(-\ln x) + \tilde{\psi}(-\ln x)\left(\frac{\beta'_x}{x} - \frac{3\beta}{2x^2}\right)
\end{aligned}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\beta}{x^2}\tilde{\psi}(-\ln x)\right) - x\left(\frac{\beta'_x}{x^2} - \frac{2\beta}{x^3}\right)\tilde{\psi}(-\ln x) + \left(\frac{\beta'_x}{x} - \frac{3\beta}{2x^2}\right)\tilde{\psi}(-\ln x)$$
$$= -\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\beta}{x^2}\tilde{\psi}\right) + \frac{\beta}{2x^2}.$$

K_{21} is the conjugated expression to K_{12}

$$\frac{\beta}{x^2}\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{2}\frac{\beta}{x^2}.$$

K_{22}: m/x^2.

Finally the transformed operator matrix will be denoted by K and it is given by

$$K = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}y}\frac{\rho}{x^2}\frac{\mathrm{d}}{\mathrm{d}y} + \left(q(x) + \frac{\rho'_x}{2x} - \frac{3\rho}{4x^2}\right) & -\frac{\mathrm{d}}{\mathrm{d}y}\frac{\beta}{x^2} + \frac{\beta}{2x^2} \\ \frac{\beta}{x^2}\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{2}\frac{\beta}{x^2} & \frac{m}{x^2} \end{pmatrix} := \begin{pmatrix} A & C^* \\ C & D \end{pmatrix}. \quad (35)$$

To define a self-adjoint operator corresponding to this operator matrix one has to consider first the minimal operator $\mathbf{K}_{\min}$ being the closure of the differential operator given by (35) on the domain of functions from $C_0^\infty[0,\infty) \oplus C_0^\infty[0,\infty)$. Then one has to study the deficiency indices of this operator and describe all its self-adjoint extensions. This analysis is equivalent to the one carried out in the previous section for the operator $\mathbf{L}_{\min}$. The self-adjoint extensions of the operators $\mathbf{L}_{\min}$ and $\mathbf{K}_{\min}$ are in one-to-one correspondence given by the unitary equivalence (33), (34). Therefore we conclude that the deficiency indices of the operator $\mathbf{K}_{\min}$ are equal and finite ($(1,1)$ or $(2,2)$ depending on the properties of the coefficients). Let us denote by $\mathbf{K}$ one of the self-adjoint extensions of the minimal operator. The essential spectrum of the operator will be studied. The analysis does not depend on the choice of self-adjoint extension, since the deficiency indices of the minimal operator are finite.

It is easier to study pseudodifferential operators on the whole axis instead of the half axis. The reason is that the manifold $[0,\infty)$ has nontrivial boundary and therefore even the momentum operator cannot be defined as a self-adjoint operator in $L_2[0,\infty)$. It appears more convenient for us to study the corresponding problem on the whole real line in order to avoid these nonessential difficulties related to the boundary point $y = 0$. In this way the problem of studies of the matrix differential operator can be reduced to a certain pure algebraic problem.

Consider the Hilbert space $\mathbb{H} = L_2(\mathbf{R}) \oplus L_2(\mathbf{R})$. The operator $\mathbb{K}$ acting in $\mathbb{H}$ can be chosen in such a way that its essential spectrum coincides with the essential spectrum of the operator $\mathbf{K}$.

In order to simplify the discussion of the essential spectrum we have to chose special continuation of the operator. However, this program applied to the operator $\mathbf{K}_{\min}$ itself meets some difficulties and it appears more convenient for us to perform this program on a later stage of the investigation of the operator, namely during the studies of the cleaned resolvent of the operator.

6. Resolvent Matrix and the Hain–Lüst Operator

The resolvent of the operator $\mathbf{K}$ will be used to study its essential spectrum. The difference between the resolvents of any two self-adjoint extensions of the minimal operator $\mathbf{K}_{\min}$ is a finite rank operator and it follows that the essential spectrum is independent of the chosen self-adjoint extension. In fact it is enough to calculate the resolvent of the operator $\mathbf{K}$ on any subspace of finite codimension, for example on the range of the minimal operator $\mathbf{K}_{\min}$. We are going to consider the resolvent equation

$$(\mathbf{K}_{\min} - \mu)^{-1} F = U,$$

for μ satisfying one of the following two conditions

(i) $\Re\mu \neq 0$;
(ii) $\mu \in \mathbf{R}$, $|\mu| \gg 1$.

Formula (36) below shows that resolvent's denominator $T(\mu)$ has no additional singularities outside $x = 0$ for all nonreal values of the parameter μ. For sufficiently large real μ the same holds true if either $m(0) \neq 0$, or $m(0) = 0$, the quasiregularity conditions (8) hold and

$$\operatorname{sign} \mu \operatorname{sign} m(0^+) = -1.$$

If the quasiregularity conditions hold then $m(0^+) \geqslant 0$ and the parameter μ can always be chosen to be small negative, $\mu \ll -1$.

For $F \in \mathcal{R}(\mathbf{K}_{\min})$ and $U \in C_0^\infty[0, \infty) \oplus C_0^\infty[0, \infty)$ the resolvent equation can be written as follows

$$f_1 = (A - \mu)u_1 + C^* u_2, \qquad f_2 = C u_1 + (D - \mu) u_2.$$

Using the fact that the operator $(D-\mu)$ is invertible for nonreal μ one can calculate u_2 from the second equation

$$u_2 = (D - \mu)^{-1} f_2 - (D - \mu)^{-1} C u_1$$

and substitute it into the first equation to get

$$f_1 = ((A - \mu) - C^*(D - \mu)^{-1} C) u_1 + C^*(D - \mu)^{-1} f_2.$$

The last equation can easily be resolved using *Hain–Lüst operator*, which is analogous to the regularized determinant of the matrix K

$$\begin{aligned} T(\mu) &= (A - \mu I) - C^*(D - \mu I)^{-1} C \\ &= -\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m - \mu x^2)}\right)\frac{\mathrm{d}}{\mathrm{d}y} - \mu + \\ &\quad + \left\{ q(x) + \frac{\rho'_x}{2x} - \frac{3\rho}{4x^2} - \frac{\beta^2}{4x^2(m - \mu x^2)} - x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2}{2x^2(m - \mu x^2)}\right)\right\}. \end{aligned} \quad (36)$$

Elementary calculations show that under quasiregular conditions (8) both coefficients in the expression above are smooth and bounded. The principle coefficient

$$\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}$$

is uniformly separated from zero. We consider this operator for $\mu \ll -1$ on the set $C_0^\infty[0,\infty)$ and use the same notation for its Friedrichs extension described by the Dirichlet boundary condition at the origin. This operator has been introduced in a special case by K. Hain and R. Lüst during the investigation of problems of magnetohydrodynamics. In what follows we are going to show that Hain–Lüst operator plays the key rôle in the investigation of the essential spectrum.

The rôle of the quasiregularity conditions for the Hain–Lüst operator is explained by the following lemma.

LEMMA 6.1. *Let $\mu \notin \mathrm{Range}_{x\in[0,1]}((m(x))/x^2)$, then the coefficients of the Hain–Lüst operator* (36)

$$f(x) = \frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)},$$

and

$$g(x) = q(x) + \frac{\rho_x'}{2x} - \frac{3\rho}{4x^2} - \frac{\beta^2}{4x^2(m-\mu x^2)} - x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2}{2x^2(m-\mu x^2)}\right) - \mu,$$

are uniformly bounded functions if and only if the quasiregularity conditions (8) *hold.*

Comment. The condition $\mu \notin \mathrm{Range}_{x\in[0,1]}((m(x))/x^2)$ holds, for example, if the parameter μ either nonreal or $\mu \in \mathbf{R}$, $\mu \ll -1$.

Proof. Let the quasiregularity conditions (8) be satisfied. Then the coefficient

$$f(x) = \frac{\rho m - \beta^2 - \mu x^2}{x^2(m-\mu x^2)}$$

is uniformly bounded, since by (8)

$$\rho(x)m(x) - \beta^2(x) \sim_{x\to 0} cx^2$$

and the factor $m - \mu x^2$ is uniformly separated from 0. The function

$$\begin{aligned} &g(x) - q(x) + \mu + \frac{f(x)}{4} \\ &\quad = \frac{\rho_x'}{2x} - \frac{\rho}{x^2} - x\left(\frac{\beta^2}{2x^2(m-\mu x^2)}\right)_x' \end{aligned}$$

$$= \frac{\rho'_x}{2x} - \frac{\rho}{x^2} - x\left(\frac{\beta^2 - \rho\mu}{2x^2(m - \mu x^2)}\right)'_x - x\left(\frac{\rho\mu}{2x^2(m - \mu x^2)}\right)'_x$$

$$= -x\left(\frac{\beta^2 - \rho\mu}{2x^2(m - \mu x^2)}\right)'_x + \mu x\left(\frac{\rho x^2}{2x^2(m - \mu x^2)}\right)'_x$$

is also uniformly bounded.

On the other hand, the boundedness of the leading coefficient

$$f(x) = \frac{\rho m - \beta^2 - \mu x^2}{x^2(m - \mu x^2)}$$

implies conditions (8) under the assumptions of the lemma. The lemma is proven. □

Similar result has been proven for magnetohydrodynamic operator in [17].

The resolvent matrix can be presented by

$$\mathbf{M}(\mu) \equiv (\mathbf{K}_{\min} - \mu)^{-1}$$
$$= \begin{pmatrix} T^{-1}(\mu) & -T^{-1}(\mu)[C^*(D - \mu I)^{-1}] \\ -[(D - \mu I)^{-1}C]T^{-1}(\mu) & (D - \mu I)^{-1} + [(D - \mu I)^{-1}C]T^{-1}(\mu)[C^*(D - \mu I)^{-1}] \end{pmatrix}. \quad (37)$$

The last expression determines the resolvent of any self-adjoint extension $\mathbf{K}$ of the minimal operator $\mathbf{K}_{\min}$ on the subspace $\mathcal{R}(\mathbf{K}_{\min})$ which has finite codimension. Therefore this resolvent matrix determines the essential spectrum of any self-adjoint extension $\mathbf{K}$. In order to calculate the essential spectrum we are going to consider perturbations of the calculated resolvent by compact operators. This is discussed in the following section.

7. The Asymptotic Hain–Lüst Operator

The essential spectra of two operators coincide if the difference between their resolvents is a compact operator. This idea of relatively compactness was used in applications to magnetohydrodynamics by T. Kako [22]. Even if the expression for the resolvent is much more complicated than the one for operator itself we prefer to handle with the resolvent. We are going to simplify the expression for the resolvent step by step using Weyl theorem. We call this procedure *cleaning of the resolvent*. Therefore we are going to perturb the resolvent operator $\mathbf{M}(\mu)$ by compact operators in order to simplify it. Our aim is to factorize the pseudodifferential operator $\mathbf{M}(\mu)$ into a sum of two pseudodifferential operators with symbols depend on the coordinate and momentum, respectively. In our calculations we are going to use the Calkin calculus [13]. We say that any two operators A and B are equal in Calkin algebra if their difference is a compact operator. The following notation for the equivalence relation in Calkin algebra will be used throughout the paper: $A \doteq B$. Since all operators appearing in the decomposition (37) are in fact pseudodifferential the following notation for the momentum operator will be used

$$p = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}y}. \tag{38}$$

This symbol will denote the differential expression in the first half of this section. The same notation will be used for the symbol of the pseudodifferential operator on the real line in the rest of the paper.

Let us introduce the *asymptotic Hain–Lüst operator* for the generic case $m(0) \neq 0$

$$T_{\mathrm{as}}(\mu) = a(\mu)\left(-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + c(\mu)\right) \equiv a(\mu)(p^2 + c(\mu)), \tag{39}$$

where

$$\begin{aligned} a(\mu) &= \lim_{x\to 0}\left(\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}\right) = l_0 - \mu\frac{\rho(0)}{m(0)}, \\ l_0 &= \lim_{x\to 0}\left(\frac{\rho - \frac{\beta^2}{m}}{x^2}\right), \\ c(\mu) &= \frac{1}{4} - \frac{\mu}{a(\mu)}. \end{aligned} \tag{40}$$

The domain of the asymptotic Hain–Lüst coincides with the set of functions from the Sobolev space W_2^2 satisfying the Dirichlet boundary condition at the origin: $\{\psi \in W_2^2([0,\infty)), \psi(0) = 0\}$. We obtain the asymptotic Hain–Lüst operator by substitution the coefficients of the second-order differential Hain–Lüst operator by their limit values at the singular point. It will be shown that the additional branch of essential spectrum of $\mathbf{L}$ is determined exactly by the symbol of asymptotic Hain–Lüst operator.

To prove that the difference between the inverse Hain–Lüst and inverse asymptotic Hain–Lüst operators is compact we are going to use Lemma B.4. We decided to devote a separate appendix to this lemma which is of special interest in the theory of pseudodifferential operators (see Appendix B, where the proof of this lemma can be found). This lemma implies that the difference of the inverse Hain–Lüst operators is compact

$$T^{-1}(\mu) - T_{\mathrm{as}}^{-1}(\mu) \in S_\infty \tag{41}$$

for sufficiently large $|\mu|$ to guarantee the invertibility of the both operators. Note that both operator functions $-T^{-1}(\mu)$ and $-T_{\mathrm{as}}^{-1}(\mu)$ are operator valued Herglotz functions ([32]).

8. Cleaning of the Resolvent

This section is devoted to the cleaning of the resolvent, which is based on formula (41). The main algebraic tool is Calkin calculus ([13]) and Appendix B.

Using Calkin algebra and Lemma B.1 formula (41) can be almost rigorously written as follows

$$p\,T^{-1}(\mu)p \doteq \frac{1}{\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}}. \tag{42}$$

In fact to apply Lemma B.1 one needs extra regularizator h – any bounded vanishing at infinity function (see formula (80)). The operator $p\,T^{-1}(\mu)p$ here is the closure of the bounded operator defined originally on $W_2^1[0, \infty)$. Let us introduce the function

$$b(x, \mu) = \frac{\beta}{m - \mu x^2}. \tag{43}$$

Our aim is to find a matrix differential operator equivalent in Calkin algebra to the operator $\mathbf{M}(\mu)$ given by (37). Using (41) and the fact (the result of straightforward calculations) that the operators $C^*(D - \mu I)^{-1}$ and $(D - \mu I)^{-1}C$ *under quasiregular conditions* are first order differential operators with bounded smooth coefficients we obtain ⋆

$$\mathbf{M}(\mu) \doteq \begin{pmatrix} \frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)} & -\frac{b(0,\mu)}{a(\mu)}\frac{ip+1/2}{p^2+c(\mu)} \\ -\frac{b(0,\mu)}{a(\mu)}\frac{-ip+1/2}{p^2+c(\mu)} & \frac{x^2}{m-\mu x^2} + [(D-\mu I)^{-1}C]T^{-1}(\mu)[C^*(D-\mu I)^{-1}] \end{pmatrix}. \tag{44}$$

The expressions $(\pm ip + 1/2)/(p^2 + c(\mu))$ are considered as bounded operators defined on $L_2[0, \infty)$ by

$$(\pm ip + 1/2)(p^2 + c(\mu))^{-1},$$

where $(p^2 + c(\mu))^{-1}$ is the resolvent of the Laplace operator p^2 with the Dirichlet boundary condition at the origin. Substituting expressions for the operators C and D from (35) we get

$$\mathbf{M}(\mu) \doteq \begin{pmatrix} \frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)} & -\frac{b(0,\mu)}{a(\mu)}\frac{ip+1/2}{p^2+c(\mu)} \\ -\frac{b(0,\mu)}{a(\mu)}\frac{-ip+1/2}{p^2+c(\mu)} & \frac{x^2}{m-\mu x^2} + b(x,\mu)(-ip+1/2)T^{-1}(\mu)(ip+1/2)b(x,\mu) \end{pmatrix}.$$

Let us concentrate our attention to the element (22). We consider this differential operator on the set $W_2^1[0, \infty)$.

$$\begin{aligned} &b(x,\mu)(-ip+1/2)T^{-1}(\mu)(ip+1/2)b(x,\mu) \\ &\quad = b(x,\mu)(-ip+1/2)T_{\text{as}}^{-1}(\mu)(ip+1/2)b(x,\mu) + \\ &\qquad + b(x,\mu)(-ip+1/2)T^{-1}(\mu)(T_{\text{as}}(\mu) - T(\mu))T_{\text{as}}^{-1}(\mu)(ip+1/2)b(x,\mu) \\ &\quad \doteq b(x,\mu)\frac{p^2+1/4}{a(\mu)(p^2+c(\mu))}b(x,\mu) + \end{aligned}$$

⋆ In fact only the condition $m(0) \neq 0$ is used here. This relation follows from the first quasiregularity condition (8).

$$+b(x,\mu)(-ip+1/2)T^{-1}(\mu)\left[\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\rho}{x^2}-\frac{\beta^2}{x^2(m-\mu x^2)}-a(\mu)\right)\frac{\mathrm{d}}{\mathrm{d}y}\right]\times$$
$$\times\frac{ip+1/2}{a(\mu)(p^2+c(\mu))}b(x,\mu).$$

The last equality in Calkin algebra holds due to the following observations:

(1) The operator $T_{\mathrm{as}}^{-1}(\mu)(ip+1/2)$ is bounded.

(2) Since the minor terms in both $T(\mu)$ and $T_{\mathrm{as}}(\mu)$ are bounded functions, Lemma B.3 and (1) imply that the following operator is compact

$$\begin{aligned}&(-ip+1/2)T^{-1}(\mu)\ \{\text{bounded function tending to 0 at infinity}\}\\ &\doteq(-ip+1/2)T_{\mathrm{as}}^{-1}(\mu)\ \{\text{bounded function tending to 0 at infinity}\}\\ &\doteq 0.\end{aligned}$$

To transform the first term the following equality has been used

$$(-ip+1/2)T_{\mathrm{as}}^{-1}(\mu)(ip+1/2)\doteq\frac{p^2+1/4}{p^2+c(\mu)}.$$

Using

$$b(x,\mu)\frac{p^2+1/4}{p^2+c(\mu)}b(x,\mu)\doteq b(0,\mu)\frac{p^2+1/4}{p^2+c(\mu)}b(0,\mu)$$

($b\in L_\infty[0,\infty)$ and has limit at ∞, Lemma 6.1 from [17]), we get

$$\begin{aligned}&b(x,\mu)(-ip+1/2)T^{-1}(\mu)(ip+1/2)b(x,\mu)\\ &\doteq\frac{b^2(x,\mu)}{a(\mu)}+\frac{b^2(0,\mu)}{a(\mu)}\frac{1/4-c(\mu)}{p^2+c(\mu)}+\\ &\quad+b(x,\mu)(-ip+1/2)T^{-1}(\mu)\left[-p\left(\frac{\rho}{x^2}-\frac{\beta^2}{x^2(m-\mu x^2)}-a(\mu)\right)\right]\times\\ &\quad\times\frac{ip^2+p/2}{a(\mu)(p^2+c(\mu))}b(x,\mu).\end{aligned}$$

The operator

$$\frac{ip^2+p/2}{a(\mu)(p^2+c(\mu))}b(x,\mu)\equiv(ip^2+p/2)T_{\mathrm{as}}^{-1}(\mu)b/x,\mu)$$

is bounded. Consider the operator

$$b(x,\mu)(-ip+1/2)T^{-1}(\mu)\left[-p\left(\frac{\rho}{x^2}-\frac{\beta^2}{x^2(m-\mu x^2)}-a(\mu)\right)\right]$$
$$\doteq b(x,\mu)\frac{1}{\frac{\rho}{x^2}-\frac{\beta^2}{x^2(m-\mu x^2)}}\left(\frac{\rho}{x^2}-\frac{\beta^2}{x^2(m-\mu x^2)}-a(\mu)\right)$$

due to Lemma B.1 and the equality following from (8)

$$\left(\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)} - a(\mu)\right)|_{x=0} = 0. \tag{45}$$

Lemma B.1 could be applied here, since one can easily that the operator

$$b(x,\mu)(1/2)T^{-1}(\mu)\left[-p\left(\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)} - a(\mu)\right)\right]$$

is compact.

Therefore the element (22) is equivalent in Calkin algebra to the following operator

$$\frac{x^2}{m-\mu x^2} + \frac{b^2(x,\mu)}{a(\mu)} + \frac{b^2(0,\mu)}{a(\mu)}\frac{1/4 - c(\mu)}{p^2 + c(\mu)} -$$
$$- b(x,\mu)\frac{1}{\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}}\left(\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)} - a(\mu)\right)\frac{1}{a(\mu)}b(x,\mu).$$

The following formula for the cleaned resolvent matrix has been obtained

$$\mathbf{M}(\mu) \doteq \begin{pmatrix} \frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)} & -\frac{b(0,\mu)}{a(\mu)}\frac{ip+1/2}{p^2+c(\mu)} \\ -\frac{b(0,\mu)}{a(\mu)}\frac{-ip+1/2}{p^2+c(\mu)} & \frac{x^2}{m-\mu x^2} + \frac{b^2(0,\mu)}{a(\mu)}\frac{1/4-c(\mu)}{p^2+c(\mu)} + \frac{b^2(x,\mu)}{\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}} \end{pmatrix}. \tag{46}$$

Let us remind that the formal expression

$$\frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)}$$

in all four matrix entries denotes the resolvent of the asymptotic Hain–Lüst operator.

The last matrix can be written (at least formally) as a sum of two matrices depending on x and p only: $M(\mu) \doteq X(x) + P(p)$, where

$$X(x) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{x^2}{m-\mu x^2} + \frac{b^2(x,\mu)}{\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}} \end{pmatrix},$$

$$P(p) = \begin{pmatrix} \frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)} & -\frac{b(0,\mu)}{a(\mu)}\frac{ip+1/2}{p^2+c(\mu)} \\ -\frac{b(0,\mu)}{a(\mu)}\frac{-ip+1/2}{p^2+c(\mu)} & \frac{b^2(0,\mu)}{a(\mu)}\frac{1/4-c(\mu)}{p^2+c(\mu)} \end{pmatrix}.$$

In Section 4, to handle pseudodifferential operators, we discussed the extension of all operators to certain operators acting in the Hilbert space $\mathbb{H} = L_2(\mathbf{R}) \oplus L_2(\mathbf{R}) \supset L_2[0,\infty) \oplus L_2[0,\infty)$. This procedure can easily be carried out for the cleaned resolvent. Let us continue all involved functions $b(x(y),\mu)$, $\rho(x(y))$ and $m(x(y))$ to the whole real line as even functions of y. Consider the operator generated by the continued matrix symbol $X(x(y)) + P(p)$. This operator is bounded operator defined on the whole Hilbert space $\mathbb{H}$. The essential spectrum of the new operator coincides (without counting multiplicity) with the essential spectrum of the original operator $\mathbf{M}(\mu)$. Really Glazman's splitting procedure ([3]) and Weyl theorem on compact perturbations ([24]) imply that the essential spectrum of the new operator coincides with the union of the essential spectra of the two operators generated by the operator matrix on the two half-axes:

$$\frac{1}{p^2+c(\mu)}|_{L_2(\mathbf{R})} \doteq \frac{1}{p^2+c(\mu)}|_{L_2(-\infty,0]} \oplus \frac{1}{p^2+c(\mu)}|_{L_2[0,\infty)},$$

where

$$\frac{1}{p^2+c(\mu)}|_{L_2(-\infty,0]} \quad \text{and} \quad \frac{1}{p^2+c(\mu)}|_{L_2[0,\infty)}$$

denote the resolvents of the Laplace operator p^2 on the corresponding semiaxis with the Dirichlet boundary condition at the origin. In the last formula p denotes the momentum operator in the left-hand side and the differential expression in the right one.

One can easily prove that the unitary transformation

$$\begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} \mapsto \begin{pmatrix} f_1(-y) \\ -f_2(-y) \end{pmatrix}$$

relates the matrix operators generated in the orthogonal decomposition of the Hilbert space

$$\mathbb{H} = (L_2(-\infty,0] \oplus L_2(-\infty,0]) \oplus (L_2[0,\infty) \oplus L_2[0,\infty)).$$

Hence, the two operators appearing in this orthogonal decomposition are unitary equivalent and therefore have the same essential spectrum.

The problem of calculation of the essential spectrum has been transformed to a pure algebraic problem.

9. Calculation of the Essential Spectrum

In order to apply Proposition A.1 from Appendix A let us introduce two matrix operator functions

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{\rho(0)}{m(0)a(\mu)} \end{pmatrix} \tag{47}$$

and

$$Y(y) = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{x^2}{m-\mu x^2} + \dfrac{b^2(x,\mu)}{\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}} - \dfrac{\rho(0)}{m(0)a(\mu)} \end{pmatrix}. \tag{48}$$

Let us remind the reader that everywhere in the paper x is considered as a function of the variable y, $x = e^{|y|}$, where we have taken into account the even continuation of all parameters of the matrix for negative values of y. The matrices Q, $Y(y)$ and $P(p)$ satisfy the conditions of Proposition A.1. In addition, the matrix functions $Y(y)$ and $P(p)$ are continuous on the real line and have zero limits at infinity. All matrix functions are depending on the parameter μ. Therefore the essential spectrum of the resolvent operator $\mathbf{M}(\mu)$ is given by (72)

$$\sigma_{\text{ess}}(\mathbf{M}(\mu)) = \sigma_{\text{ess}}(\mathbf{Q}+\mathbf{P}) \cup \sigma_{\text{ess}}(\mathbf{Q}+\mathbf{Y}).$$

To calculate the essential spectra of the operators $\mathbf{Q}+\mathbf{P}$ and $\mathbf{Q}+\mathbf{Y}$ we use the fact that the determinants of the corresponding matrices $Q+P(p)$ and $Q+Y(y)$ are equal to zero identically. It follows that one of the two eigenvalues of the each matrix is identically zero. Therefore the essential spectra of the operators coincides with the range of the second (nontrivial) eigenvalues when y resp. p runs over the whole real axis. This simple fact is a result of straightforward calculations. The nontrivial eigenvalues coincide with the traces of the corresponding 2×2 matrices $Q+P(p)$ and $Q+Y(y)$. The trace of the matrix $M(\mu)$ is given by

$$\begin{aligned}\operatorname{Tr}(M(\mu)) &= \operatorname{Tr}(Y(y)) + \operatorname{Tr}(P(p)) - \operatorname{Tr}(Q) \\ &= \frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)} + \frac{x^2}{m-\mu x^2} + \\ &\quad + \frac{b^2(0,\mu)}{a(\mu)}\frac{1/4-c(\mu)}{p^2+c(\mu)} + \frac{b^2(x,\mu)}{\frac{\rho}{x^2}-\frac{\beta^2}{x^2(m-\mu x^2)}}.\end{aligned}$$

The last expression can be factorized into the sum of three factors

$$\operatorname{Tr}(M(\mu)) = \varphi(x(y)) + \psi(p) - \operatorname{Tr}(Q),$$

$$\operatorname{Tr} Q = \frac{\rho(0)}{m(0)a(\mu)},$$

where the functions $\varphi(x(y))$ and $\psi(p)$ tend to zero as y resp. p tend to ∞. The factorization is unique and obvious

$$\begin{aligned}\varphi(x) &= \frac{x^2}{m-\mu x^2} + \frac{b^2(x,\mu)}{\frac{\rho}{x^2}-\frac{\beta^2}{x^2(m-\mu x^2)}}; \\ \psi(p) &= \frac{1}{a(\mu)}\frac{1}{p^2+c(\mu)} + \frac{b^2(0,\mu)}{a(\mu)}\frac{1/4-c(\mu)}{p^2+c(\mu)} + \frac{\rho(0)}{m(0)a(\mu)}.\end{aligned} \tag{49}$$

Proposition A.1 implies that the essential of the resolvent operator is given by

$$\sigma_{\mathrm{ess}}(\mathbf{M}(\mu)) = (\mathrm{Range}(\varphi(x)) \cup \mathrm{Range}(\psi(x)) + \varphi(0)). \tag{50}$$

Straightforward calculations imply

$$\sigma_{\mathrm{ess}}(\mathbf{L}) = \mathrm{Range}_{x\in[0,1]}\left\{\frac{m-\frac{\beta^2}{\rho}}{x^2}\right\} \cup \left[\frac{l_0}{4+\frac{\rho(0)}{m(0)}}, \frac{l_0}{\frac{\rho(0)}{m(0)}}\right], \tag{51}$$

where l_0 is given by (40). The parameter μ disappears eventually as one can expect. This parameter is pure axillary.

We conclude that the essential spectrum of $\mathbf{L}$ consists of two parts having different origin. The so-called regularity spectrum ([30])

$$\mathrm{Range}_{x\in[0,1]}\left\{\frac{m-\frac{\beta^2}{\rho}}{x^2}\right\}$$

is determined by all coefficients of the operator matrix on the whole interval [0, 1]. This part of the spectrum coincides with the limit of the essential spectra of the truncated operators $\mathbf{L}(\epsilon)$

$$\mathrm{Range}_{x\in[0,1]}\left\{\frac{m-\frac{\beta^2}{\rho}}{x^2}\right\} = \overline{\bigcup_{\epsilon>0}\sigma_{\mathrm{ess}}(\mathbf{L}(\epsilon))}.$$

On the contrary the singularity spectrum

$$\left[\frac{l_0}{4+\frac{\rho(0)}{m(0)}}, \frac{l_0}{\frac{\rho(0)}{m(0)}}\right]$$

is due to the singularity of the operator matrix at the origin and depends on the behavior of the matrix coefficients at the origin only. This part of the essential spectrum is absent for all truncated operators $\mathbf{L}(\epsilon)$ and cannot be obtained by the limit procedure $\epsilon \to 0$. This fact explains the name singularity spectrum given in [30]. The appearance of this interval of the essential spectrum generated by the singularity was predicted by J. Descloux and G. Geymonat. Note that the end point $l_0/(\rho(0)/m(0))$ of the singularity spectrum always belongs to the interval of regularity spectrum, since

$$\lim_{x\to 0}\frac{m-\frac{\beta^2}{\rho}}{x^2} = \frac{l_0}{\frac{\rho(0)}{m(0)}}.$$

Remark. Let us remind that the essential spectrum has been calculated provided $m(0) \neq 0$ and the quasiregularity conditions are satisfied. If $m(0) = 0$, the quasiregularity conditions imply that $\beta(0) = 0$ and hence $m'(0) = 0$. No singularity appears in the coefficients of the matrix L given by (2). Therefore the operator is regular and its essential spectrum equals to

$$\mathrm{Range}_{x\in[0,1]}\left\{\frac{m-\frac{\beta^2}{\rho}}{x^2}\right\} \quad (4).$$

No singularity spectrum appears in this case.

There is another way to describe the singularity spectrum using the roots of the symbol of the asymptotic Hain–Lüst operator, observed first for a different matrix differential operator in [30].

LEMMA 9.1. *The singularity spectrum*

$$\left[\frac{l_0}{4+\frac{\rho(0)}{m(0)}}, \frac{l_0}{\frac{\rho(0)}{m(0)}}\right]$$

of the operator $\mathbf{L}$ *coincides with the set of singular points* (*roots*) *of the symbol of the asymptotic Hain–Lüst operator*

$$\Phi = \{\mu \in \mathbf{R} \mid \exists p \in \mathbf{R} \cup \{\infty\} : a(\mu)(p^2 + c(\mu)) = 0\}.$$

Proof. The set of singular points of the symbol $a(\mu)(p^2 + c(\mu))$ coincides with the set

$$\Phi = \{\mu \in \mathbf{R} \mid c(\mu) \leqslant 0\}.$$

Formula (40) implies

$$\Phi = \left\{\mu \in \mathbf{R} \mid 0 \leqslant \frac{l_0 - \mu\frac{\rho(0)}{m(0)}}{\mu} \leqslant 4\right\}$$
$$= \left[\frac{l_0}{4+\frac{\rho(0)}{m(0)}}, \frac{l_0}{\frac{\rho(0)}{m(0)}}\right].$$

Note that $p = \infty$ formally corresponds to right endpoint of the last interval. The lemma is proven. □

In our opinion this connection between the singular set of the symbol of the asymptotic Hain–Lüst operator and the singularity spectrum has general character. Studies in this direction will be continued in one of our forthcoming publications.

Remark. We would like to mention that the regularity spectrum

$$\mathrm{Range}_{x\in[0,1]}\left\{\frac{m-\frac{\beta^2}{\rho}}{x^2}\right\}$$

under quasiregularity conditions can be calculated using just the symbol of the Hain–Lüst operator. Really trivial calculations show that the regularity spectrum coincides with the set of real μ for which the principle coefficient of the Hain–Lüst operator degenerates, i.e. equals zero. Roughly speaking this idea has been utilized by physicists K. Hain and R. Lüst ([16]) (see also [12]).

10. Semiboundedness of the Operator

In many applications to physics semibounded operators play very important rôle. Semiboundedness of the considered operator is related to the quasiregularity conditions.

THEOREM 10.1. *Suppose that the real valued functions q, β, ρ, m satisfy the following conditions:*

$$q \in L^\infty[0,1], \qquad \beta, m, \rho \in C^2[0,1], \qquad \rho \geqslant c_0 > 0. \tag{52}$$

Then the symmetric operator $\mathbf{L}_{\min}$ corresponding to the operator matrix (2) *is semibounded if and only if one of the following three conditions is satisfied*

(1) $(m - \beta^2/\rho)|_{x=0} > 0$,
(2) $(m - \beta^2/\rho)|_{x=0} = 0$ *and* $(m - \beta^2/\rho)'|_{x=0} > 0$,
(3) $(m - \beta^2/\rho)|_{x=0} = 0$ *and* $(m - \beta^2/\rho)'|_{x=0} = 0$ (*quasiregularity conditions*).

COROLLARY 10.1. *Under assumptions of Theorem* 10.1 *the operator $\mathbf{L}_{\min}$ admits self-adjoint extensions. Every such extension $\mathbf{L}$ is a semibounded operator if and only if one of the conditions* (1)–(3) *is satisfied.*

Proof. Since the coefficients of the matrix L are real valued functions, the deficiency indices of $\mathbf{L}_{\min}$ are equal. On the other hand the equation for the deficiency element is a system of ordinary differential equations. Therefore the set of solutions has finite dimension. Hence the operator $\mathbf{L}_{\min}$ always has finite equal deficiency indices and admits self-adjoint extensions. Theorem 10.1 implies that every such extension is semibounded if and only if one of the three conditions is satisfied (see [3]). □

Proof of Theorem 10.1. Without loss of generality one can suppose that $q = 0$, since the operator corresponding to the matrix $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ is bounded in $\mathcal{H}$ and cannot change the semiboundedness of the whole operator $\mathbf{L}_{\min}$.

The theorem will be proven by estimating the quadratic form of $\mathbf{L}_{\min}$ defined on the domain $C_0^\infty[0,1] \oplus C_0^\infty[0,1]$ by the following operator matrix

$$L = \begin{pmatrix} -\dfrac{\mathrm{d}}{\mathrm{d}x}\rho\dfrac{\mathrm{d}}{\mathrm{d}x} & \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\beta}{x} \\ -\dfrac{\beta}{x}\dfrac{\mathrm{d}}{\mathrm{d}x} & \dfrac{m}{x^2} \end{pmatrix}. \tag{53}$$

The quadratic form of this operator is

$$\langle \mathbf{L}_{\min} U, U \rangle = \langle \rho u_1', u_1' \rangle - \left\langle \frac{\beta}{x} u_2, u_1' \right\rangle - \left\langle \frac{\beta}{x} u_1', u_2 \right\rangle + \left\langle \frac{m}{x^2} u_2, u_2 \right\rangle$$

$$= \left\langle \begin{pmatrix} 1 & -\frac{\beta}{\sqrt{\rho}} \\ -\frac{\beta}{\sqrt{\rho}} & m \end{pmatrix} \begin{pmatrix} \sqrt{\rho}u_1' \\ \frac{u_2}{x} \end{pmatrix}, \begin{pmatrix} \sqrt{\rho}u_1' \\ \frac{u_2}{x} \end{pmatrix} \right\rangle. \tag{54}$$

Considering functions with zero second component $U = (u_1, 0)$ we conclude that the operator $\mathbf{L}_{\min}$ is not bounded from above, since the quadratic form coincides with the quadratic form of the operator $-(\mathrm{d}/\mathrm{d}x)\rho\mathrm{d}/\mathrm{d}x$ in this case. Therefore the operator $\mathbf{L}_{\min}$ is semibounded if and only if it is bounded from below. To get the second necessary condition for the semiboundedness of the operator consider the set of functions with zero first component $U = (0, u_2)$. The quadratic form is then given by

$$\langle \mathbf{L}_{\min} U, U \rangle = \left\langle \frac{m}{x^2} u_2, u_2 \right\rangle.$$

Hence the operator $\mathbf{L}_{\min}$ is semibounded only if

$$m(0) > 0 \quad \text{or} \quad m(0) = 0 \quad \text{and} \quad m'(0) \geqslant 0. \tag{55}$$

Case A. Suppose that the determinant of the matrix

$$\det \begin{pmatrix} 1 & -\frac{\beta}{\sqrt{\rho}} \\ -\frac{\beta}{\sqrt{\rho}} & m \end{pmatrix} = m - \frac{\beta^2}{\rho}$$

is negative at point zero (and therefore in a neighborhood of this point as well)

$$m(0) - \frac{\beta(0)^2}{\rho(0)} < 0. \tag{56}$$

It follows that the matrix has precisely one negative eigenvalue $\lambda(x) < 0$ for small enough values of x. Let us denote by (α, γ) the corresponding normalized real eigenvector depending continuously on x in a neighborhood of the origin.

Suppose that $\alpha(0) = 0$. Then the first equation for the eigenvector implies that $\beta(0) = 0$ and therefore $m(0) < 0$ due to (56). This contradicts (55) and therefore $\alpha(0) \neq 0$ in a certain neighborhood of the origin due to the continuity of α.

Consider arbitrary real function $h \in C_0^\infty[0, 1]$ such that the derivative of h is equal to 1 in the interval $(1/4, 1/2)$ and the family of scaled functions $h^\epsilon = \epsilon h(x/\epsilon)$. The corresponding family of vector functions $U^\epsilon = (h^\epsilon, \frac{\gamma}{\alpha}\sqrt{\rho} x h^{\epsilon\prime})$ is well-defined for sufficiently small ϵ. Since

$$\langle \mathbf{L}_{\min} U^\epsilon, U^\epsilon \rangle = \int_0^\epsilon \frac{\lambda(x)\rho}{\alpha^2} h^{\epsilon\prime 2}\, \mathrm{d}x,$$

and

$$\|U^\epsilon\|^2 = \int_0^\epsilon \left(h^{\epsilon 2} + \frac{\gamma^2\rho}{\alpha^2} x^2 h^{\epsilon\prime 2} \right) \mathrm{d}x,$$

the quotient $\langle \mathbf{L}U, U\rangle / \| U \|^2$ tends to $-\infty$ as $\epsilon \to 0$. Hence the operator $\mathbf{L}_{\min}$ is not semibounded in this case.

Case B. Suppose that

$$m(0) - \frac{\beta(0)^2}{\rho(0)} > 0.$$

The operator $\mathbf{L}$ is semibounded in this case. Indeed the quadratic form can be decomposed as follows

$$\langle \mathbf{L}_{\min} U, U\rangle = \left\langle \begin{pmatrix} 1 & -\frac{\beta(0)}{\sqrt{\rho(0)}} \\ -\frac{\beta(0)}{\sqrt{\rho(0)}} & m(0) \end{pmatrix} \begin{pmatrix} \sqrt{\rho} u_1' \\ \frac{u_2}{x} \end{pmatrix}, \begin{pmatrix} \sqrt{\rho} u_1' \\ \frac{u_2}{x} \end{pmatrix} \right\rangle +$$

$$+ \left\langle \begin{pmatrix} 0 & \frac{\beta(0)}{\sqrt{\rho(0)}} - \frac{\beta}{\sqrt{\rho}} \\ \frac{\beta(0)}{\sqrt{\rho(0)}} - \frac{\beta}{\sqrt{\rho}} & m - m(0) \end{pmatrix} \begin{pmatrix} \sqrt{\rho} u_1' \\ \frac{u_2}{x} \end{pmatrix}, \begin{pmatrix} \sqrt{\rho} u_1' \\ \frac{u_2}{x} \end{pmatrix} \right\rangle.$$

The first term is positive and can be estimated from below by

$$\mathrm{const}(\|u_1'\|^2 + \|u_2\|^2)$$

due to the assumption. The second term is subordinated to the first one

$$\begin{aligned} &\left| \left\langle \begin{pmatrix} 0 & \frac{\beta(0)}{\sqrt{\rho(0)}} - \frac{\beta}{\sqrt{\rho}} \\ \frac{\beta(0)}{\sqrt{\rho(0)}} - \frac{\beta}{\sqrt{\rho}} & m - m(0) \end{pmatrix} \begin{pmatrix} \sqrt{\rho} u_1' \\ \frac{u_2}{x} \end{pmatrix}, \begin{pmatrix} \sqrt{\rho} u_1' \\ \frac{u_2}{x} \end{pmatrix} \right\rangle \right| \\ &\quad \leqslant \mathrm{const} \int_0^1 \left(x {u_1'}^2 + \frac{u_2^2}{x} \right) \mathrm{d}x \\ &\quad \leqslant \mathrm{const}\, \epsilon \int_0^\epsilon \left(\rho {u_1'}^2 + \frac{u_2^2}{x^2} \right) \mathrm{d}x + \mathrm{const} \int_\epsilon^1 \left({u_1'}^2 + \frac{u_2^2}{x^2} \right) \mathrm{d}x. \end{aligned}$$

The relative bound const ϵ can be chosen less than 1 and the second term is bounded for any $\epsilon > 0$.

Case C. Suppose that

$$m(0) - \frac{\beta(0)^2}{\rho(0)} = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(m - \frac{\beta^2}{\rho} \right)\Big|_{x=0} < 0.$$

The quadratic form can be decomposed as

$$\langle \mathbf{L}_{\min} U, U\rangle = \left\langle \begin{pmatrix} 1 & -\beta/\sqrt{\rho} \\ -\beta/\sqrt{\rho} & \beta^2/\rho \end{pmatrix} \begin{pmatrix} \sqrt{\rho}u_1' \\ \dfrac{u_2}{x} \end{pmatrix}, \begin{pmatrix} \sqrt{\rho}u_1' \\ \dfrac{u_2}{x} \end{pmatrix} \right\rangle + \\ + \left\langle \begin{pmatrix} 0 & 0 \\ 0 & m-\beta^2/\rho \end{pmatrix} \begin{pmatrix} \sqrt{\rho}u_1' \\ \dfrac{u_2}{x} \end{pmatrix}, \begin{pmatrix} \sqrt{\rho}u_1' \\ \dfrac{u_2}{x} \end{pmatrix} \right\rangle.$$

Consider the vector function

$$V^\epsilon = \left(\int_0^x \frac{\beta(t)}{\rho(t)} h^{\epsilon\prime}(t)\,\mathrm{d}t,\ x h^{\epsilon\prime}(x) \right),$$

where the scalar h^ϵ has been introduced investigating Case A. Calculating the the quadratic form

$$\langle \mathbf{L}_{\min} V^\epsilon, V^\epsilon \rangle = \left\langle \left(m - \frac{\beta^2}{\rho} \right) h^{\epsilon\prime}, h^{\epsilon\prime} \right\rangle$$

and estimating the norm

$$\begin{aligned} \|V^\epsilon\|^2 &= \int_0^1 \left[x^2 h^{\epsilon\prime 2} + \left(\int_0^x \frac{\beta}{\rho} h^{\epsilon\prime}\,\mathrm{d}t \right)^2 \right] \mathrm{d}x \\ &\leqslant \int_0^1 [x^2 h^{\epsilon\prime 2} + \mathrm{const}\ h^{\epsilon 2}]\,\mathrm{d}x. \end{aligned}$$

Since $(m - (\beta^2/\rho))'|_{x=0} < 0$, the quotient

$$\frac{\langle \mathbf{L} U, U\rangle}{\|U\|^2}$$

tends to $-\infty$ as $\epsilon \to 0$. The operator is not semibounded in this case.

Case D. Suppose that

$$\left(m - \frac{\beta^2}{\rho} \right)\Big|_{x=0} = 0 \quad \text{and} \quad \left(m - \frac{\beta^2}{\rho} \right)'\Big|_{x=0} \geqslant 0. \tag{57}$$

The operator $\mathbf{L}_{\min}$ is semibounded in this case due to the following estimate

$$\left\langle \begin{pmatrix} 0 & 0 \\ 0 & m-\beta^2/\rho \end{pmatrix} \begin{pmatrix} \sqrt{\rho}u_1' \\ \dfrac{u_2}{x} \end{pmatrix}, \begin{pmatrix} \sqrt{\rho}u_1' \\ \dfrac{u_2}{x} \end{pmatrix} \right\rangle$$

$$= \int_0^1 \frac{m - \frac{\beta^2}{\rho}}{x^2} |u_2|^2\,\mathrm{d}x \leqslant \mathrm{const}\|u_2\|^2,$$

which is valid since the function $(m - (\beta^2/\rho))/x^2$ from below.

The Cases **A**–**D** cover all the possibilities. The Theorem is proven. □

Appendix A. On the Essential Spectrum of the Triple Sum of Operators in Banach Space

The following simple lemma will be used to calculate the essential spectrum of the separable sum of pseudodifferential operators. It allows one to pass to the limit in formula (58) below when the point λ reaches the discrete spectrum.

LEMMA A.1. *Let* $\mathbf{T}, \mathbf{Y}, \mathbf{P}$ *be bounded operators acting in a Banach space* X. *Suppose that a certain dotted neighborhood of* $\lambda = 0$ *does not belong to the spectrum of the operator* $\mathbf{T}$ *and the point* $\lambda = 0$ *is not in the essential spectrum of the operator* $\mathbf{T}$.* *Suppose in addition that*

$$\mathbf{Y}(\mathbf{T} - \lambda)^{-1}\mathbf{P} \in S_\infty \tag{58}$$

is a compact operator in the dotted neighborhood. Let $\mathbf{R_T}$ *be the parametrix of the operator* $\mathbf{T}$ ([13])

$$\mathbf{R_T T} \doteq \mathbf{T R_T} \doteq I. \tag{59}$$

Then the operator $\mathbf{Y R_T P}$ *is compact*

$$\mathbf{Y R_T P} \in S_\infty. \tag{60}$$

Proof. The following calculations prove the lemma

$$\begin{aligned}\mathbf{Y R_T P} &:= \mathbf{Y}(\mathbf{T} - \lambda)^{-1}(\mathbf{T} - \lambda)\mathbf{R_T P} \\ &\doteq \mathbf{Y}(\mathbf{T} - \lambda)^{-1}(\mathbf{I} - \lambda\mathbf{R_T})\mathbf{P} \\ &\doteq -\lambda\mathbf{Y}(\mathbf{T} - \lambda)^{-1}\mathbf{R_T P} \\ &\doteq -\lambda\mathbf{Y R_T}(\mathbf{I} - \lambda\mathbf{R_T})^{-1}\mathbf{R_T P} \\ &\doteq 0,\end{aligned} \tag{61}$$

where the second equality from the end is valid for all λ, $0 < |\lambda| < 1/\|\mathbf{R_T}\|$ not from the spectrum of the operator $\mathbf{T}$

$$(\mathbf{T} - \lambda)\mathbf{R} \doteq \mathbf{I} - \lambda\mathbf{R} \Rightarrow (\mathbf{T} - \lambda)^{-1} \doteq \mathbf{R}(\mathbf{I} - \lambda\mathbf{R})^{-1}.$$

The lemma is proven. □

* These conditions imply that the point $\lambda = 0$ is a finite type eigenvalue of $\mathbf{T}$ ([13]) or does not belong to the spectrum of $\mathbf{T}$ at all. In the last case the proof of the lemma is trivial.

THEOREM A.1. *Let* $\mathbf{M}$ *be the operator sum of three bounded operators* $\mathbf{Q}$, $\mathbf{Y}$ *and* $\mathbf{P}$ *acting in a certain Banach space*

$$\mathbf{M} = \mathbf{Q} + \mathbf{Y} + \mathbf{P}, \tag{62}$$

such that the complement in $\mathbf{C}$ *of the essential spectrum of the operator* $\mathbf{Q}$ *is connected. Suppose that the following two operators are compact for any* λ *from the regular set of* $\mathbf{Q}$

$$\mathbf{P}\frac{1}{\mathbf{Q}-\lambda}\mathbf{Y} \in S_\infty; \qquad \mathbf{Y}\frac{1}{\mathbf{Q}-\lambda}\mathbf{P} \in S_\infty. \tag{63}$$

Then the essential spectrum of the operator $\mathbf{M}$ *can be calculated as follows*

$$\sigma_{\text{ess}}(\mathbf{M}) \setminus \sigma_{\text{ess}}(\mathbf{Q}) = [\sigma_{\text{ess}}(\mathbf{Q}+\mathbf{Y}) \cup \sigma_{\text{ess}}(\mathbf{Q}+\mathbf{P})] \setminus \sigma_{\text{ess}}(\mathbf{Q}). \tag{64}$$

Proof. It has been proven in [13] (Corollary 8.5, page 204) that if the complement in $\mathbf{C}$ of the essential spectrum of a certain bounded operator is connected, then any number λ from the spectrum of the operator, but not from the essential spectrum is a finite type eigenvalue ([13]), i.e. the pole of the resolvent with finite rank Laurent coefficients with negative indices. Lemma A.1 implies that the operators

$$\mathbf{Y}\mathbf{R}_{\mathbf{Q}}(\lambda)\mathbf{P}, \qquad \mathbf{P}\mathbf{R}_{\mathbf{Q}}(\lambda)\mathbf{Y} \tag{65}$$

are compact operators, where $\mathbf{R}_{\mathbf{Q}}(\lambda)$ is one of the parametrix of the operator $\mathbf{Q}$ at point λ

$$(\mathbf{Q}-\lambda)\mathbf{R}_{\mathbf{Q}}(\lambda) \doteq \mathbf{R}_{\mathbf{Q}}(\lambda)(\mathbf{Q}-\lambda) \doteq \mathbf{I}. \tag{66}$$

Then the following equalities can be proven

$$\begin{aligned}(\mathbf{Q}+\mathbf{Y}-\lambda)\mathbf{R}_{\mathbf{Q}}(\lambda)(\mathbf{Q}+\mathbf{P}-\lambda) &\doteq \mathbf{Q}+\mathbf{Y}+\mathbf{P}-\lambda;\\ (\mathbf{Q}+\mathbf{P}-\lambda)\mathbf{R}_{\mathbf{Q}}(\lambda)(\mathbf{Q}+\mathbf{Y}-\lambda) &\doteq \mathbf{Q}+\mathbf{Y}+\mathbf{P}-\lambda.\end{aligned} \tag{67}$$

Let us prove the first equality only, since the prove of the second equality is similar. Formulas (65) imply that

$$\begin{aligned}&(\mathbf{Q}+\mathbf{Y}-\lambda)\mathbf{R}_{\mathbf{Q}}(\lambda)(\mathbf{Q}+\mathbf{P}-\lambda)\\ &\quad\doteq (\mathbf{I}+\mathbf{Y}\mathbf{R}_{\mathbf{Q}}(\lambda))(\mathbf{Q}+\mathbf{P}-\lambda)\\ &\quad\doteq \mathbf{Q}-\lambda+\mathbf{Y}\mathbf{R}_{\mathbf{Q}}(\lambda)(\mathbf{Q}-\lambda)+\mathbf{P}+\mathbf{Y}\mathbf{R}_{\mathbf{Q}}(\lambda)\mathbf{P}\\ &\quad\doteq \mathbf{Q}+\mathbf{Y}+\mathbf{P}-\lambda.\end{aligned}$$

We are going to prove now formula (64) for the essential spectra of the operators $\mathbf{M}$, $\mathbf{Q}$, $\mathbf{Y}$, and $\mathbf{P}$ following the idea of [17], where a similar fact has been proven to the sum of two operators. Let us prove the following inclusion first

$$\sigma_{\text{ess}}(\mathbf{M}) \setminus \sigma_{\text{ess}}(\mathbf{Q}) \subset [\sigma_{\text{ess}}(\mathbf{Q}+\mathbf{Y}) \cup \sigma_{\text{ess}}(\mathbf{Q}+\mathbf{P})] \setminus \sigma_{\text{ess}}(\mathbf{Q}). \tag{68}$$

Suppose that λ does not belong to the essential spectra of the operators $\mathbf{Q}$, $\mathbf{Q}+\mathbf{Y}$, and $\mathbf{Q}+\mathbf{P}$, then the operators

$$\mathbf{Q}+\mathbf{Y}-\lambda, \qquad \mathbf{R_Q}(\lambda), \qquad \mathbf{Q}+\mathbf{P}-\lambda$$

are Fredholm operators as a product of three Fredholm operators. Then formulas (67) imply that the operator

$$\mathbf{Q}+\mathbf{Y}+\mathbf{P}-\lambda$$

is a Fredholm operator. Hence the point λ does not belong to the essential spectrum of the operator $\mathbf{Q}+\mathbf{Y}+\mathbf{P}$.

In the second step let us prove the inclusion

$$\sigma_{\text{ess}}(\mathbf{M}) \setminus \sigma_{\text{ess}}(\mathbf{Q}) \supset [\sigma_{\text{ess}}(\mathbf{Q}+\mathbf{Y}) \cup \sigma_{\text{ess}}(\mathbf{Q}+\mathbf{P})] \setminus \sigma_{\text{ess}}(\mathbf{Q}). \tag{69}$$

Suppose that λ does not belong to the essential spectra of the operators $\mathbf{M}$ and $\mathbf{Q}$, i.e. that the operators $\mathbf{M}-\lambda$ and $\mathbf{Q}-\lambda$ are Fredholm operators. We are going to use Proposition 8.2 from [17] (see also [13]) stating that if the operators $\mathbf{A}$ and $\mathbf{B}$ are two bounded operators acting in a certain Banach space and the operators $\mathbf{AB}$ and $\mathbf{BA}$ are Fredholm operators, then the operators $\mathbf{A}$ and $\mathbf{B}$ are also Fredholm operators. Formulas (67) imply that the operators

$$\mathbf{R_Q}(\lambda)(\mathbf{Q}+\mathbf{Y}-\lambda)\mathbf{R_Q}(\lambda)(\mathbf{Q}+\mathbf{P}-\lambda)$$

and

$$\mathbf{R_Q}(\lambda)(\mathbf{Q}+\mathbf{P}-\lambda)\mathbf{R_Q}(\lambda)(\mathbf{Q}+\mathbf{Y}-\lambda)$$

are Fredholm operators. Then the proposition implies that the operators $\mathbf{R_Q}(\lambda)(\mathbf{Q}+\mathbf{Y}-\lambda)$ and $\mathbf{R_Q}(\lambda)(\mathbf{Q}+\mathbf{P}-\lambda)$ are Fredholm operators. It follows from (66) that the operators

$$\mathbf{Q}+\mathbf{Y}-\lambda \doteq (\mathbf{Q}-\lambda)\mathbf{R_Q}(\lambda)(\mathbf{Q}+\mathbf{Y}-\lambda)$$

and

$$\mathbf{Q}+\mathbf{P}-\lambda \doteq (\mathbf{Q}-\lambda)\mathbf{R_Q}(\lambda)(\mathbf{Q}+\mathbf{P}-\lambda)$$

are Fredholm operators. It follows that λ does not belong to the essential spectra of the operators $\mathbf{Q}+\mathbf{Y}$ and $\mathbf{Q}+\mathbf{P}$. Inclusion (69) is proven.

Formulas (68) and (69) imply (64). The Theorem is proven. □

Remark. It is possible to get read of the condition that the complement in $\mathbf{C}$ of the essential spectrum the operator $\mathbf{Q}$ is connected. Then it is necessary to suppose that the operators $\mathbf{Y}\mathbf{R_Q}(\lambda)\mathbf{P}$, $\mathbf{P}\mathbf{R_Q}(\lambda)\mathbf{Y}$ are compact for any λ outside the essential spectrum of $\mathbf{Q}$. It is possible to construct three operators $\mathbf{Q}, \mathbf{P}, \mathbf{Y}$ satisfying all conditions of the theorem except the connectivity of $\mathbf{C} \setminus \sigma_{\text{ess}}(\mathbf{Q})$ but not satisfying formula (64). This counterexample can be prepared using bilateral shift in the Hilbert space $X = \ell^2_{\mathbf{Z}}(\ell^2_{\mathbf{N}}, H \oplus H)$, where H is a certain infinite dimensional axillary Hilbert space.

PROPOSITION A.1. *Let* $\mathbf{M}$ *be any* $n \times n$ *matrix separable pseudodifferential operator generated in the Hilbert space* $L_2(\mathbf{R}, \mathbf{C}^n)$ *by the symbol*

$$M(y, p) = Q + Y(y) + P(p), \quad p = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}y}, \tag{70}$$

where Q *is a constant diagonalizable matrix with simple spectrum, and the matrix functions* $Y(y)$ *and* $P(p)$ *are essentially bounded and satisfy the following two asymptotic conditions*

$$\lim_{x\to\infty} Y(y) = 0 \qquad \lim_{p\to\infty} P(p) = 0. \tag{71}$$

Then the essential spectrum of the operator $\mathbf{M}$ *is given by*

$$\sigma_{\mathrm{ess}}(\mathbf{M}) = \sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{P}) \cup \sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{Y}). \tag{72}$$

Proof. The essential spectra of both operators $\mathbf{Q}+\mathbf{Y}$ and $\mathbf{Q}+\mathbf{P}$ contain the essential spectrum of $\mathbf{Q}$

$$\sigma_{\mathrm{ess}}(\mathbf{Q}) = \sigma(Q),$$

where $\sigma(Q)$ is the spectrum of the matrix Q. To prove this fact one can use perturbation theory and the fact that the matrices $Y(y),\ y \to \infty$ and $P(p),\ p \to \infty$ are asymptotically small ([24]).

Theorem A.1 implies that

$$\sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{Y}+\mathbf{P}) \setminus \{0\} \supset \bigcup_n \sigma_{\mathrm{ess}}(P_N(\mathbf{Q}+\mathbf{Y})P_N) \setminus \{0\} \supset \sigma(Q) \setminus \{0\},$$

(using $\mathbf{A} = \mathbf{Q}+\mathbf{Y},\ \mathbf{B} = \mathbf{P}$). It follows that

$$\sigma_{\mathrm{ess}}(\mathbf{M}) \setminus \{0\} = (\sigma_{\mathrm{ess}}\,(\mathbf{Q}+\mathbf{P}) \cup \sigma_{\mathrm{ess}}\,(\mathbf{Q}+\mathbf{Y})) \setminus \{0\}. \tag{73}$$

We are going to remove the set {0} from the last formula.

Applying the same analysis for the operator $\mathbf{M} - \epsilon\mathbf{I}$ we obtain that

$$\sigma_{\mathrm{ess}}(\mathbf{M}-\epsilon\mathbf{I}) \setminus \{0\} = (\sigma_{\mathrm{ess}}\,(\mathbf{Q}-\epsilon\mathbf{I}+\mathbf{P}) \cup \sigma_{\mathrm{ess}}\,(\mathbf{Q}-\epsilon\mathbf{I}+\mathbf{Y})) \setminus \{0\}. \tag{74}$$

This implies that

$$\sigma_{\mathrm{ess}}(\mathbf{M}(y, p)) \setminus \{\epsilon\} = (\sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{P}) \cup \sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{Y})) \setminus \{\epsilon\},$$

for arbitrary real ϵ and hence

$$\sigma_{\mathrm{ess}}(\mathbf{M}) = (\sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{P}) \cup \sigma_{\mathrm{ess}}(\mathbf{Q}+\mathbf{Y})). \tag{75}$$

The proposition is proven. □

In the special case case $n = 1$ and when the symbols $Y(y)$ and $P(p)$ are piecewise continuous the last proposition can be derived from Theorem 3 in [35] (see also [36]). The advantage of our approach is its transparency compared with the technique of C^* algebras used in [35].

Remark. The condition concerning the simplicity of the spectrum of matrix $\mathbf{Q}$ can be removed in the special case where all matrices Q, $Y(y)$, and $P(p)$ are Hermitian.

Proof. Consider the family of small Hermitian perturbations $Q_\epsilon,\ \epsilon > 0$ of the matrix $\mathbf{Q}$ such that

$$\| Q_\epsilon - Q \| \leqslant \epsilon$$

and the spectrum of Q_ϵ is simple. Such matrix Q_ϵ satisfy the conditions of the theorem and hence

$$\sigma_{\mathrm{ess}}(\mathbf{Q}_\epsilon + \mathbf{Y} + \mathbf{P}) = \sigma_{\mathrm{ess}}(\mathbf{Q}_\epsilon + \mathbf{P}) \cup \sigma_{\mathrm{ess}}(\mathbf{Q}_\epsilon + \mathbf{Y}). \tag{76}$$

Let us denote by F^δ the δ-neighborhood of any set $F \subset \mathbf{R}$

$$F^\delta := \{x \in \mathbf{R} : \mathrm{dist}(x, F) \leqslant \delta\}.$$

Let A and B be two bounded self-adjoint operators acting in a certain Hilbert space. Then the essential spectra of the operators $\mathbf{A}$ and $\mathbf{A}+\mathbf{B}$ are related by the following formula ([3])

$$\sigma_{\mathrm{ess}}(\mathbf{A} + \mathbf{B}) \subset (\sigma_{\mathrm{ess}}(\mathbf{A}))^{\|\mathbf{B}\|}.$$

From (76) we immediately obtain that

$$\sigma_{\mathrm{ess}}(\mathbf{Q} + \mathbf{Y} + \mathbf{P}(p)) \subset [\sigma_{\mathrm{ess}}(\mathbf{Q}_\epsilon + \mathbf{Y}) \cup \sigma_{\mathrm{ess}}(\mathbf{Q}_\epsilon + \mathbf{P})]^{2\epsilon};$$
$$\sigma_{\mathrm{ess}}(\mathbf{Q} + \mathbf{Y}) \cup \sigma_{\mathrm{ess}}(\mathbf{Q} + \mathbf{P}) \subset [\sigma_{\mathrm{ess}}(\mathbf{Q}_\epsilon + \mathbf{Y} + \mathbf{P})]^{2\epsilon}.$$

Since the essential spectra are closed sets and ϵ is arbitrary small, we conclude that

$$\sigma_{\mathrm{ess}}(\mathbf{Q}_\epsilon + \mathbf{Y} + \mathbf{P}) = \sigma_{\mathrm{ess}}(\mathbf{Q} + \mathbf{Y}) \cup \sigma_{\mathrm{ess}}(\mathbf{Q} + \mathbf{P}). \tag{77}$$

This completes the proof. □

Appendix B. Elementary Lemmas on Calkin Calculus

The following lemmas are necessary for the transformation of the resolvent.

LEMMA B.1. *Let the real valued function $f(y)$ be positive bounded and separated from zero*

$$0 < c \leqslant f(y) \leqslant C \tag{78}$$

for some $c, C \in \mathbf{R}_+$. Let the function $g(y)$ be bounded and the operator

$$L \equiv pf(y)p + g(y) \tag{79}$$

be self-adjoint and invertible in $L_2(\mathbf{R})$. *Suppose that the operator* $pL^{-1}p$ *be bounded.* ⋆ *Then for any bounded function* $h(y)$ *such that* $\lim_{y\to\infty} h(y) = 0$ *the following equality holds in Calkin algebra*

$$pL^{-1}ph \doteq \frac{h}{f}. \tag{80}$$

Comment 1. The rôle of the function h is to regularize the equality which does not hold in Calkin algebra $pL^{-1} \doteq 1/f$. Therefore the regularizing function h cannot be cancelled in (80). To construct a counter example let us first consider similar problem on the whole axis for which all calculations are trivial. Let the functions f and g be constant functions $f = 1$, $g = 1$. Then the operator

$$pL^{-1}p - 1 = p(p^2+1)^{-1}p - 1 = -\frac{1}{p^2+1}$$

obviously is not compact, since it is a multiplication operator in the Fourier representation.

Comment 2. In [4] similar result has been obtained in the regular case. Here an abstract proof of a generalization of the result is presented. We hope that the algebraic character of the proof will enable us to generalize these results to a wider classes of PDO and ΨDO. The advantage of our approach is that no information concerning the Green's function is used.

Proof. Consider first the case where the function $h(y)$ is a $C^\infty(\mathbf{R})$ function. We need this condition in order to avoid to consider the closure of bounded operators considered below. All these operators are well defined by their differential expressions on $W_2^1(\mathbf{R})$.

The following identity holds (at least in $W_2^1(\mathbf{R})$)

$$((p+i)L^{-1}(p-i))\left(\frac{1}{p-i}L\frac{1}{p+i}\right) = I.$$

Multiplying the latter equality by the operator of multiplication by decreasing function h one can get the operator equality valid on $W_2^1(\mathbf{R})$

$$\begin{aligned}((p+i)L^{-1}(p-i))&\frac{1}{p-i}(pfp)\frac{1}{p+i}h + \\ &+((p+i)L^{-1}(p-i))\frac{1}{p-i}g\frac{1}{p+i}h = h.\end{aligned}$$

The second term in left-hand side is a compact operator as the multiplication of the bounded operator $(p+i)L^{-1}(p-i)(1/(p-i))g$ and the compact operator

⋆ The latter condition could follow from the previous conditions for sufficiently smooth function f.

$(1/(p+i))h$ (since the functions $1/(p+i)$ and h are decreasing function of p and y respectively). Hence the following equality holds in Calkin algebra

$$((p+i)L^{-1}(p-i))\frac{1}{p-i}(pfp)\frac{1}{p+i}h \doteq h.$$

Similarly taking into account that

$$p\frac{1}{p+i}h = \left(1 - \frac{i}{p+i}\right)h \doteq h$$

we get the following equality

$$((p+i)L^{-1}(p-i))\frac{1}{p-i}pfh \doteq h.$$

Multiplying by f^{-1} the latter equality, one gets

$$(p+i)L^{-1}(p-i)\frac{1}{p-i}ph \doteq \frac{h}{f},$$

using the fact the function f is boundedly invertible. Multiplying the latter equality by factor

$$\frac{p}{p+i} = 1 - \frac{i}{p+i}$$

from the left one gets in Calkin algebra

$$\frac{h}{f} \doteq \frac{p}{p+i}(p+i)L^{-1}(p-i)\frac{p}{p-i}h\frac{p}{p-i} = pL^{-1}ph.$$

Let us consider the case of decreasing bounded but otherwise arbitrary function h. Every such function can be estimated from above by a certain positive decreasing to zero $C^\infty(\mathbf{R})$ function $\tilde{h}$, $|h(y)| \leqslant \tilde{h}$. We have already proven the Lemma for the function $\tilde{h}$

$$\frac{\tilde{h}}{f} \doteq \frac{p}{p+i}(p+i)L^{-1}(p-i)\frac{p}{p-i}h\frac{p}{p-i} = pL^{-1}p\tilde{h}.$$

Of cause the multiplication by the contraction operator of multiplication by the bounded function $h/\tilde{h}$ preserves the equality in Calkin algebra. Finally one gets (80) for arbitrary h satisfying the conditions of the Lemma. □

The following lemma is well-known. (It is a special case of problems treated systematically in [19].)

LEMMA B.2. *Let the following conditions be satisfied*

(i) $f \geqslant c > 0$,
(ii) $f, g \in C^1(\mathbf{R})$,
(iii) $(pfp + g)|_{W_2^2(\mathbf{R})}$ *is invertible, then the operator* $(p + i)(pfp + g)^{-1}(p - i)$ *defined originally on the dense set* $W_2^1(\mathbf{R})$ *is bounded in* $L_2(\mathbf{R})$.

COROLLARY. *Under conditions of the lemma the operator*

$$p(pfp + g)^{-1}p|_{W_2^1(\mathbf{R})}$$

is also bounded, since the operator $p/(p \pm i)$ *is a contraction.*

Remark. Condition (iv) can be substituted by a stronger condition (iv′) the real valued function g is positive definite $g(x) \geqslant \tilde{c}_0 > 0$. Really conditions (i), (ii), (iii) (iv′) imply (iv), since the estimate

$$\langle (pfp + g)u, u\rangle = \langle fpu, pu\rangle + \langle gu, u\rangle \geqslant c_0\|pu\|^2 + \tilde{c}_0\|u\|^2$$

implies that the operator $(pfp + g)|_{W_2^2(\mathbf{R})}$ has bounded inverse.

LEMMA B.3. *Suppose that conditions* (i)–(iii) *of Lemma* B.2 *be satisfied. Let in addition the limits*

$$\lim_{y\to\infty} f(y), \qquad \lim_{y\to\infty} g(y),$$

be finite. Then the difference between the inverse Hain–Lüst and asymptotic Hain–Lüst operators is a compact operator, moreover

$$\begin{aligned}&(p + i)[T^{-1}(\mu) - T_{\mathrm{as}}^{-1}(\mu)] \in S_\infty;\\ &[T^{-1}(\mu) - T_{\mathrm{as}}^{-1}(\mu)](p - i) \in S_\infty.\end{aligned} \tag{81}$$

Proof. Consider the following chain of equalities

$$\begin{aligned}&(p + i)[T^{-1}(\mu) - T_{\mathrm{as}}^{-1}(\mu)] \in S_\infty\\ &\quad = (p + i)T^{-1}(\mu)\{T_{\mathrm{as}}(\mu) - T(\mu)]T^{-1}(\mu)\\ &\quad = (p + i)T^{-1}(\mu)(p\tilde{f}p + \tilde{g})T^{-1}(\mu)\\ &\quad = (p + i)T^{-1}(\mu)(p - i)(p - i)^{-1}(p\tilde{f}p + \tilde{g})T^{-1}(\mu),\end{aligned}$$

where

$$\tilde{f} = f - \lim_{y\to\infty} f(y) \in C^1(\mathbf{R}),$$

$$\tilde{g} = g - \lim_{y\to\infty} g(y) \in L^\infty(\mathbf{R}) \cap C^1(\mathbf{R}),$$

$$\lim_{y\to\infty} \tilde{f}(y) = \lim_{y\to\infty} \tilde{g}(y) = 0.$$

The operator $(p+i)T^{-1}(\mu)(p-i)$ is bounded and the operator $(p-i)^{-1}(p\tilde{f}p+\tilde{g})T^{-1}(\mu)$ is compact operators, since the operators

$$(p-i)^{-1}\tilde{g} \quad \text{and} \quad \tilde{f}pT^{-1}(\mu)$$

are compact. (Here we use the fact that any pseudodifferential operator determined by the symbol $\varphi(x)\psi(p)$ is compact if $\varphi, \psi \in C(\mathbf{R})$ and $\lim_{x\to\infty}\varphi(x) = 0$, $\lim_{p\to\infty}\varphi(p) = 0$.) The lemma is proven. □

LEMMA B.4. *Let conditions* (i)-(iii) *of Lemma* B.2 *be satisfied. Suppose in addition that the continuous functions* $\alpha, \gamma \in C(\mathbf{R})$ *are continuous and have finite limits at infinity. Then the following equality holds in Calkin algebra*

$$(\alpha p+\gamma)T^{-1}(\mu) \doteq (\alpha(\infty)p+\gamma(\infty))T_{\text{as}}^{-1}(\mu). \tag{82}$$

Proof. Lemma B.3 implies that

$$\begin{aligned}
&(\alpha p+\gamma)T^{-1}(\mu)\\
&\quad \doteq (\alpha p+\gamma)T_{\text{as}}^{-1}(\mu)\\
&\quad = (\alpha(\infty)p+\gamma(\infty))T_{\text{as}}^{-1}(\mu) + ((\alpha-\alpha(\infty))p+\gamma-\gamma(\infty))T_{\text{as}}^{-1}(\mu)\\
&\quad = (\alpha(\infty)p+\gamma(\infty))T_{\text{as}}^{-1}(\mu) + (\alpha-\alpha(\infty))pT_{\text{as}}^{-1}(\mu) + (\gamma-\gamma(\infty))T_{\text{as}}^{-1}(\mu)\\
&\quad \doteq (\alpha(\infty)p+\gamma(\infty))T_{\text{as}}^{-1}(\mu).
\end{aligned}$$

The lemma is proven. □

Remark. All lemmas proven in this appendix for the operators acting in $L_2(\mathbf{R})$ are in fact valid for the corresponding operators restricted to $L_2(\mathbf{R}_+)$. To make the operators self-adjoint in $L_2(\mathbf{R}_+)$ one needs to introduce some additional symmetric boundary condition at the origin, for example the Dirichlet boundary condition discussed in the paper. Let us mention here the necessary modifications of Lemma B.1 only. The other lemmas can be treated in the same way. Let L be a self-adjoint operator in $L_2(\mathbf{R}_+)$ determined by (79) and certain boundary condition at the origin. Consider the extension of L to the operator acting in $L_2(\mathbf{R})$ determined by the same expression, where the functions $f(y)$ and $g(y)$ are continued for negative values of y as even functions. Then equality (80) holds in Calkin algebra for the extended operator. Taking into account that the resolvent of the extended operator differs from the orthogonal sum of two copies of the resolvents of the initial operator taken on the positive and negative semiaxes separately by a finite rank operator. (Note that the functions from the domain of both operators satisfy proper separating boundary conditions at the origin.) We have used here the Glazman splitting method ([3]). As a result, we obtain the necessary equality for the operators in $L_2(\mathbf{R}_+)$.

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