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VON NEUMANN–WIGNER THEOREM: LEVEL REPULSION AND DEGENERATE EIGENVALUES

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We investigate the spectral properties of Schrödinger operators with point interactions, focusing attention on the interplay between level repulsion (von Neumann–Wigner theorem) and the symmetry of the configuration of point interactions. The explicit solution of the problem allows observing level repulsion for two centers. For a large number of centers, we investigate the families of point interactions leading to the maximum degeneracy.

Keywords: von Neumann–Wigner theorem, zero-range potential, extension theory, inverse spectral problem

1. Introduction

The method of operator extensions in mathematics and its special case called the zero-range potential method in physics have been rapidly developing recently because their universality and applicability to many physical problems lead to essential simplifications (usually algebraizing the problem). Physical applications were considered in [1], and the mathematical theory was presented in [2]. In contrast to other approximation methods, the zero-range potential method contains the continuous spectrum from the very beginning and preserves important properties of the original problem such as unitarity (in contrast to the Born approximation). In many problems in physics, short-range objects are separated by large distances (galaxies, stars, planets, atoms and molecules, nuclei and elementary particles). Apart from very exotic cases, only solid state problems do not have such properties, but even in condensed matter, excitations can be regarded as quasiparticles, applying the same method.

Using the zero-range potential method, we obtain a unique possibility to solve quantum problems in an explicit form. The method is a generalization of the separable potential method, where the interaction is given by nonlocal projection operators, when δ-functions are chosen as the projection functions, and this allows preserving the locality of the operator.

Although the theory of zero-range potentials is well developed, the case of an infinite number of centers or even a few centers is still not completely studied. In this paper, we discuss properties of these models from the standpoint of the inverse spectral problem and the von Neumann–Wigner theorem.

The celebrated von Neumann–Wigner theorem [3] describes the probability that a finite-dimensional matrix has a degenerate eigenvalue. This probability is lower than might be expected: the codimension of the set of matrices with a double eigenvalue is always greater than one. It follows that for a time-dependent Hamiltonian, the probability that two energy curves intersect is extremely low, and this phenomenon is called level repulsion. Usually, two levels intersect only if the corresponding eigenfunctions have different symmetries. Hence, we can expect to observe degenerate eigenvalues primarily for Hamiltonians with

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symmetries. In this paper, we study this phenomenon using Hamiltonians with point interactions in $\mathbb{R}^1$ and $\mathbb{R}^3$. Such operators are widely used in quantum mechanics and atomic physics to model different physical processes (see [1], [2], [4]–[6], and the numerous references therein).

Every Hamiltonian with N point interactions can have at most N eigenvalues. A straightforward analysis shows that no eigenvalue of multiplicity N can appear. Our main goal here is to study the possibility for operators with point interactions to have eigenvalues of the maximum multiplicity $N-1$.

This paper is organized as follows. We rigorously define Schrödinger operators with point interactions in Sec. 2 (mainly following [1], [2]). Because two local point interactions cannot produce any degenerate eigenvalue (even of multiplicity two), we study level repulsion for two centers in Sec. 3. We study the multiplicity of levels for the cases of three, four, and five centers in Sec. 4. We show that the symmetry of the configuration of centers plays an important role in the appearance of strongly degenerate states.

2. Hamiltonians with δ-interactions

A Schrödinger operator with N local δ-interactions at the points $\{y^n\}_{n=1}^N$ is formally defined by

$$L_\alpha = -\Delta + \sum_{n=1}^{N} \alpha_n \delta(\,\cdot - y^n), \tag{1}$$

where Δ is the Laplace operator and $\delta(\,\cdot - y^n)$ is the delta function with support at the point y^n. Without loss of generality, we assume that all points y^n are different.

In what follows, we precisely define the operator L_α. We note that the operator corresponding to formal expression (1) is uniquely defined in $\mathbb{R}^1$, but to define this operator in $\mathbb{R}^3$, we must take extra assumptions into account.

We consider the Laplace operators

$$\begin{aligned} -\Delta^1 &= -\frac{d^2}{dx^2} && \text{in } L_2(\mathbb{R}^1),\\ -\Delta^3 &= -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} && \text{in } L_2(\mathbb{R}^3), \end{aligned} \tag{2}$$

which are self-adjoint when defined on the Sobolev spaces $W_2^2(\mathbb{R}^j)$. Here, x and $\vec{x} = (x_1, x_2, x_3)$ denote the respective coordinates in $\mathbb{R}^1$ and $\mathbb{R}^3$. Then the operator corresponding to formal expression (1) is one of the self-adjoint extensions of the symmetric restrictions $-\Delta^{j0}$ of the operator $-\Delta^j$, $j = 1, 3$, to the set of functions vanishing at the interaction points:

$$-\Delta^{j0} = -\Delta^j|_{\{\psi \in W_2^2(\mathbb{R}^j):\, \psi(y^n)=0,\, n=1,2,\ldots,N\}}. \tag{3}$$

The corresponding deficiency elements for $\lambda = -\chi^2$ are just solutions of the equations $-\Delta^{j0*} g + \chi^2 g = \delta(x - y^n)$:

$$g^1(x, y^n) = \frac{e^{-\chi|x-y^n|}}{2\chi}, \qquad g^3(x, y^n) = \frac{e^{-\chi|x-y^n|}}{4\pi|x-y^n|}, \quad n = 1, 2, \ldots, N. \tag{4}$$

Hence, the deficiency indices of the restricted operators are equal to (N, N). The domains of the adjoint operators are

$$\begin{aligned} \mathrm{Dom}(-\Delta^{10*}) &= W_2^2(\mathbb{R}^1 \setminus \{y^n\}_{n=1}^N) \cap C(\mathbb{R}^1),\\ \mathrm{Dom}(-\Delta^{30*}) &= W_2^2(\mathbb{R}^3 \setminus \{y^n\}_{n=1}^N). \end{aligned} \tag{5}$$

To describe self-adjoint extensions of $-\Delta^{j0}$, we use the boundary values for the functions from the domain of the adjoint operator:

$$\begin{aligned}\psi(x) &\sim_{x\to y^n} -\frac{1}{2}|x-y^n|\,\psi_{-n}+\psi_{0n}+o(1) \quad \text{for } \mathbb{R}^1,\\ \psi(x) &\sim_{x\to y^n} \frac{1}{4\pi|x-y^n|}\psi_{-n}+\psi_{0n}+o(1) \quad \text{for } \mathbb{R}^3.\end{aligned} \tag{6}$$

Then the boundary forms of the adjoint operators are given by

$$\langle(-\Delta^{j0*})u,v\rangle-\langle u,(-\Delta^{j0*})v\rangle=\sum_{n=1}^{N}(u_{0n}\bar{v}_{-n}-u_{-n}\bar{v}_{0n}). \tag{7}$$

We introduce the diagonal matrix $\alpha=\operatorname{diag}\{\alpha_1,\alpha_2,\dots,\alpha_N\}$.

Definition. The operator L^j_α is the restriction of the adjoint operator $-\Delta^{j0*}$ to the set of functions from $\psi\in\operatorname{Dom}(-\Delta^{j0*})$ satisfying the boundary conditions

$$\vec{\psi}_0=-\alpha^{-1}\vec{\psi}_-, \tag{8}$$

where $\vec{\psi}_-=(\psi_{-1},\psi_{-2},\dots,\psi_{-N})^{\mathrm{T}}$ and $\vec{\psi}_0=(\psi_{01},\psi_{02},\dots,\psi_{0N})^{\mathrm{T}}$ are vectors of boundary values of the function ψ. In other words, the operator L^j_α coincides with the Laplace operator defined on the domain of functions satisfying (8).

In one dimension, we can prove that the operator corresponding to formal expression (1) is given by this definition. To show this in three dimensions, we must use certain additional assumptions such as the homogeneity of the Laplace operator and of the δ-distribution (see Sec. 1.5.1 in [5]). We do not want to dwell on this point, because our further studies are based on the definition and are independent of these assumptions.

In what follows, we consider only local point interactions, i.e., interactions corresponding to the diagonal matrix α. The question of which point interactions are local was exhaustively investigated in [7] (it can be shown that nondiagonal matrices α in condition (8) lead to nonlocal interactions). Without loss of generality, we assume that all coefficients α_n are nonzero. If this is not the case, then the set of singular points y^n, $n=1,2,\dots,N$, can simply be reduced.

The resolvent of the perturbed operator L_α can be calculated using Krein's formula [8]–[10] because each operator L^j_α is a finite-dimensional perturbation of the Laplace operator $-\Delta^j$ in the resolvent sense. Hence, the essential spectrum of L_α is purely absolutely continuous and coincides with the interval $[0,\infty)$ (it has multiplicity two in $\mathbb{R}^1$ and infinite multiplicity in $\mathbb{R}^3$). The number of negative eigenvalues cannot exceed N (the rank of the perturbation). In addition, a straightforward analysis shows that no positive eigenvalues occur. The discrete spectrum of the operator is given by the zeroes of the perturbation determinant appearing in Krein's formula.

Another way to obtain the equation for the discrete spectrum is to consider the ansatz for the eigenfunction

$$\psi=\sum_{n=1}^{N}a_n g^j(x,y^n), \tag{9}$$

where $g^j(x,y)$ are the Green's functions for the Laplace operator given by (4). The function ψ given by formula (9) satisfies the eigenfunction equation for the energy $\lambda=-\chi^2$

$$L^{j0*}\psi=-\chi^2\psi \tag{10}$$

for any value of the complex parameters a_n. It is an eigenfunction if and only if it satisfies boundary condition (8). We consider the boundary values of the Green's functions

$$g^1 = g^1(x-y^n) \Rightarrow g^1_{-m} = \begin{cases} 1, & m=n, \\ 0, & m\neq n, \end{cases} \qquad g^1_{0m} = \begin{cases} 1/(2\chi), & m=n, \\ \dfrac{e^{-\chi|y^n-y^m|}}{2\chi}, & m\neq n, \end{cases}$$
$$g^1 = g^1(x-y^n) \Rightarrow g^3_{-m} = \begin{cases} 1, & m=n, \\ 0, & m\neq n, \end{cases} \qquad g^3_{0m} = \begin{cases} -\chi/(4\pi), & m=n, \\ \dfrac{e^{-\chi|y^n-y^m|}}{4\pi|y^n-y^m|}, & m\neq n. \end{cases} \tag{11}$$

We obtain the respective dispersion equations defining the discrete spectrum in $\mathbb{R}^1$ and $\mathbb{R}^3$:

$$\det(1+G^1+2\chi\alpha^{-1}) = 0, \tag{12}$$

$$\det(-\chi+G^3+4\pi\alpha^{-1}) = 0, \tag{13}$$

where G^j are the Hermitian $N\times N$ matrices

$$G^1_{nm} = \begin{cases} e^{-\chi|y^n-y^m|}, & n\neq m, \\ 0, & n=m, \end{cases}$$
$$G^3_{nm} = \begin{cases} \dfrac{e^{-\chi|y^n-y^m|}}{|y^n-y^m|}, & n\neq m, \\ 0, & n=m. \end{cases} \tag{14}$$

In what follows, we study solutions of these dispersion equations, focusing on the degeneracy of the eigenvalues.

In addition to eigenfunctions exponentially decreasing at infinity (corresponding to negative eigenvalues), there exist solutions with a power-law decrease (corresponding to the eigenvalue zero). The decrease of these functions at infinity is related to their angular dependence. Spherically symmetric functions decrease as c/r and therefore are not normalizable. All other values of the angular momentum are admissible (for $E=0$).

3. Level repulsion for two centers in $\mathbb{R}^1$ and $\mathbb{R}^3$

Two local point interactions cannot produce a degenerate eigenvalue; therefore, we study the level repulsion in this section. The operator with two δ-potentials can be parameterized by three real parameters: the distance $d>0$ between the centers and the strengths α_j, $j=1,2$, of the point interactions.

We first consider the case of two point centers in $\mathbb{R}^1$. In this case, the dispersion equation is given by the formula

$$(1+2\chi\alpha_1^{-1})(1+2\chi\alpha_2^{-1}) - e^{-2\chi d} = 0. \tag{15}$$

Using the two parameters

$$\gamma_i = -\frac{1}{2}\alpha_i, \quad i=1,2, \tag{16}$$

for convenience, we write it in the form

$$\frac{(\chi-\gamma_1)(\chi-\gamma_2)}{\gamma_1\gamma_2} - e^{-2\chi d} = 0. \tag{17}$$

Let $L^1(\gamma_1, \gamma_2)$ denote the corresponding operator.

The dispersion equation for two centers in $\mathbb{R}^3$ is given by

$$(-\chi + 4\pi\alpha_1^{-1})(-\chi + 4\pi\alpha_2^{-1}) - \frac{e^{-2\chi d}}{d^2} = 0. \tag{18}$$

We introduce the two new parameters

$$\gamma_j = \frac{4\pi}{\alpha_j}, \quad j = 1, 2, \tag{19}$$

and obtain the dispersion equation

$$(\chi - \gamma_1)(\chi - \gamma_2) - \frac{e^{-2\chi d}}{d^2} = 0. \tag{20}$$

The corresponding operator is denoted by $L^3(\gamma_1, \gamma_2)$.

The parameters γ_j just introduced for the one- and three-dimensional problems can be interpreted as the energies of the bound states associated with each of the two point centers separately.

We consider the Schrödinger operator with one δ-interaction $-\Delta + \alpha\delta(x)$. Then the corresponding operator has exactly one bound state with the energy

$$E = -\gamma^2 = -\frac{\alpha^2}{4} \tag{21}$$

under the condition $\alpha < 0$ in $\mathbb{R}^1$ and

$$E = -\gamma^2 = -\frac{(4\pi)^2}{\alpha^2} \tag{22}$$

under the condition $\alpha > 0$ in $\mathbb{R}^3$.

The energies corresponding to single interactions can be obtained from dispersion equations (17) and (20) in the limit $d \to \infty$. The exponential function tends to zero, and the two dispersion equations are transformed into the equation

$$(\chi - \gamma_1)(\chi - \gamma_2) = 0.$$

This equation has two solutions $\chi = \gamma_{1,2}$ and determines the energies of two bound states $E_1 = -\gamma_1^2$ and $E_2 = -\gamma_2^2$ under the conditions $\gamma_{1,2} > 0$.

Another way to obtain these bound states is to consider the limit where the interaction at one of the centers vanishes. We note that a vanishing interaction $\alpha_j = 0$ formally corresponds to $\gamma_j = 0$ in $\mathbb{R}^1$ and $\gamma_j = \infty$ in $\mathbb{R}^3$. The limit of the respective Eqs. (17) and (20) as $\gamma_2 \to 0$ and $\gamma_2 \to \infty$ is the equation

$$\chi - \gamma_1 = 0,$$

which determines the unique bound state with the energy $E_1 = -\gamma_1^2$ (under the condition $\gamma_1 < 0$).

It seems more convenient to use the parameters γ_j instead of α_j to define the operators with point interactions.

We study the number of eigenvalues depending on the values of the two parameters. Without loss of generality, we can assume that $\gamma_1 \geq \gamma_2$. We introduce the notation

$$P^1(\chi) = \frac{(\chi - \gamma_1)(\chi - \gamma_2)}{\gamma_1\gamma_2},$$

$$P^3(\chi) = (\chi - \gamma_1)(\chi - \gamma_2).$$

The eigenvalues of the operator $L(\gamma_1, \gamma_2)$ correspond to positive (real) solutions of the dispersion equation. We note that Eq. (17) has one "parasite" solution $\chi = 0$, which is unphysical because no eigenfunction corresponds to $E = 0$ in this case. The corresponding function does not belong to the Hilbert space.

We study the three cases covering all possibilities (the cases where γ_1 or γ_2 are equal to zero can be excluded from consideration) separately for point interactions in $\mathbb{R}^1$ and $\mathbb{R}^3$.

1. In the case $\gamma_2 \le \gamma_1 < 0$, the functions $P^1(\chi)$ and $e^{-2\chi d}$ for positive χ satisfy the inequalities

$$P^1(\chi) \ge 1 \ge e^{-2\chi d},$$

which are strict for $\chi \ne 0$. Therefore, Eq. (17) has no solution on the interval $(0, \infty)$ in this case.

The function P^3 increases to infinity for positive χ, and the function $e^{-2\chi d}/d$ decreases. Comparing the values of the functions at the origin, we can deduce that Eq. (20) has one solution if and only if

$$\gamma_1\gamma_2 < \frac{1}{d^2}. \tag{23}$$

2. In the case $\gamma_2 < 0 < \gamma_1$, the functions $P^1(\chi)$ and $e^{-2\chi d}$ are equal to 1 at $\chi = 0$; their second derivatives are respectively negative and positive. Therefore, Eq. (17) has at most one positive solution, and this solution exists if and only if

$$\frac{d}{d\chi}(e^{-2\chi d})\bigg|_{\chi=0} < \frac{d}{d\chi}P^1(\chi)\bigg|_{\chi=0},$$

i.e.,

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} < 2d.$$

The solution belongs to the interval $(0, \gamma_1)$; let χ_1 denote this solution.

Equation (20) always has one solution in this case because the function P^3 increases to infinity and has a positive zero and the function $e^{-2\chi d}/d^2$ is positive and decreases to zero. The solution belongs to the interval (γ_1, ∞).

3. In the case $0 < \gamma_2 \le \gamma_1$, the function P^1 is equal to zero at the points $\chi = \gamma_1$ and $\chi = \gamma_2$ and increases to $+\infty$ on the interval (γ_1, ∞). Hence there exists a solution of Eq. (17) in the interval (γ_1, ∞). This solution is denoted by χ_1. The second solution may be located in the interval $(0, \gamma_2)$, and it exists if and only if the condition

$$\frac{d}{d\chi}(e^{-2\chi d})\bigg|_{\chi=0} < \frac{d}{d\chi}P^1(\chi)\bigg|_{\chi=0}$$

is satisfied, i.e.,

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} < 2d.$$

Let χ_2 denote this solution.

Similarly, Eq. (20) has two solutions χ_1 and χ_2 in the intervals $(0, \gamma_2)$ and (γ_1, ∞) if and only if the condition

$$\gamma_1\gamma_2 > \frac{1}{d^2} \tag{24}$$

is satisfied. Otherwise, the equation has a unique solution in the interval (γ_1, ∞).

We have thus again proved that the discrete spectrum of the two considered problems consists of at most two distinct eigenvalues located on the negative half-axis. We now study the inverse spectral problem for these operators: *reconstruct the coupling constants γ_1 and γ_2 for the operators $L^1(\gamma_1, \gamma_2)$ and $L^3(\gamma_2, \gamma_3)$*

from the bound-state energies $-\chi_1^2 = E_1 < E_2 = -\chi_2^2 < 0$ *for a fixed distance* d *between the centers.* We can set $d = 1$ without loss of generality.

We first consider the dispersion equation corresponding to the one-dimensional problem. We suppose that χ_1 and χ_2 are solutions of (17). Then the parameters $\gamma_{1,2}$ satisfy the system of equations

$$\begin{aligned}(\chi_1 - \gamma_1)(\chi_1 - \gamma_2) &= \gamma_1\gamma_2 e^{-2\chi_1},\\ (\chi_2 - \gamma_1)(\chi_2 - \gamma_2) &= \gamma_1\gamma_2 e^{-2\chi_2},\end{aligned} \tag{25}$$

which can be rewritten in the form

$$\begin{aligned}\gamma_1\gamma_2 &= \frac{\chi_1\chi_2(\chi_1 - \chi_2)}{\chi_1(1 - e^{-2\chi_2}) - \chi_2(1 - e^{-2\chi_1})} \equiv A(\chi_1, \chi_2),\\ \gamma_1 + \gamma_2 &= \frac{\chi_1^2(1 - e^{-2\chi_2}) - \chi_2^2(1 - e^{-2\chi_1})}{\chi_1(1 - e^{-2\chi_2}) - \chi_2(1 - e^{-2\chi_1})} \equiv B(\chi_1, \chi_2).\end{aligned} \tag{26}$$

Each of these two equations is linear in γ_i. Therefore, this system of equations can be solved, for example, by expressing γ_1 from the first equation and substituting it in the second equation. We obtain the quadratic equation

$$\gamma^2 - B\gamma + A = 0, \tag{27}$$

which can always be solved. But the solution of the equations can be complex, while only real parameters γ_j define a self-adjoint operator. Hence, the bound-state energies are not arbitrary but satisfy the inequality

$$D(\chi_1, \chi_2) \equiv B^2(\chi_1, \chi_2) - 4A(\chi_1, \chi_2) \geq 0. \tag{28}$$

Similarly, for the three-dimensional problem, we have the system of equations

$$\begin{aligned}(\chi_1 - \gamma_1)(\chi_1 - \gamma_2) &= e^{-2\chi_1},\\ (\chi_2 - \gamma_1)(\chi_2 - \gamma_2) &= e^{-2\chi_2}\end{aligned} \tag{29}$$

or

$$\begin{aligned}\gamma_1\gamma_2 &= \chi_1\chi_2 + \frac{\chi_1 e^{-2\chi_2} - \chi_2 e^{-2\chi_1}}{\chi_1 - \chi_2} \equiv A(\chi_1, \chi_2),\\ \gamma_1 + \gamma_2 &= \chi_1 + \chi_2 - \frac{e^{-2\chi_1} - e^{-2\chi_2}}{\chi_1 - \chi_2} \equiv B(\chi_1, \chi_2),\end{aligned} \tag{30}$$

and again there exists a self-adjoint operator if and only if discriminant (28) is nonnegative.

Theorem 1. *Let* $E_1 = -\chi_1^2 < -\chi_2^2 = E_2$ *be eigenvalues of the operator* $L^j(\gamma_1, \gamma_2)$, $j = 1, 3$. *Let the energy* $E_1 = -\chi_1^2$ *be fixed. Then all possible values of* χ_2 *fill the interval* $[0, \chi_2^{\max}]$, *where* $\chi_2^{\max} = \chi_2^{\max}(\chi_1)$ *is the value of* χ_2 *corresponding to the symmetric interaction* $\gamma_1 = \gamma_2$. *This value of* $\chi_2^{\max}$ *is the unique solution of the respective equations*

$$\frac{1 - e^{-\chi_2^{\max}}}{\chi_2^{\max}} = \frac{1 + e^{-\chi_1}}{\chi_1}, \qquad \chi_2^{\max} + e^{-\chi_2^{\max}} = \chi_1 - e^{-\chi_1} \tag{31}$$

for $\mathbb{R}^1$ *and* $\mathbb{R}^3$.

Proof. We prove the theorem for the one and three dimensions separately.

1. In one dimension, the energies χ_1 and $\chi_2^{\max}$ corresponding to the symmetric case $\gamma_1 = \gamma_2 \equiv \gamma$ can be calculated from the equation

$$(\chi - \gamma)^2 = \gamma^2 e^{-2\chi} \Rightarrow \chi - \gamma = \pm\gamma e^{-\chi} \Rightarrow$$

$$\Rightarrow \frac{1 + e^{-\chi_1}}{\chi_1} = \frac{1 - e^{-\chi_2^{\max}}}{\chi_2^{\max}}.$$

To prove the theorem, it suffices to show that the discriminant of system (27) is positive for $\chi_2 < \chi_2^{\max}$ and negative for $\chi_2^{\max} < \chi_2$. Taking into account that the discriminant is equal to zero for $\chi_2 = \chi_2^{\max}$ ($\gamma_1 = \gamma_2$ in this case), it suffices to show that $\partial D(\chi_1, \chi_2)/\partial\chi_2$ is negative. We consider the function $f(x) = (1 - e^{-2x})/x$. Direct calculations show that

$$0 \le f(x) \le 2, \qquad -2 \le f'(x) \le 0, \qquad 0 \le f''(x)$$

for $x > 0$. It is easy to show that

$$A = \frac{\chi_1 - \chi_2}{f(\chi_2) - f(\chi_1)}.$$

Then the mean value theorem implies that $0 \le A(\chi_1, \chi_2) \le 1/2$. Taking into account that $B = \chi_1 + A(\chi_1, \chi_2)f(\chi_1)$, we can evaluate the derivative of the discriminant:

$$\frac{\partial D(\chi_1, \chi_2)}{\partial\chi_2} = 2((\chi_1 + A(\chi_1, \chi_2)f(\chi_1))f(\chi_1) - 2)\frac{\partial A(\chi_1, \chi_2)}{\partial\chi_2}.$$

To prove that the derivative $\partial A(\chi_1, \chi_2)/\partial\chi_2$ is positive, we again use the mean value theorem,

$$\frac{\partial A(\chi_1, \chi_2)}{\partial\chi_2} = \frac{f(\chi_1) - (f(\chi_2) + (\chi_1 - \chi_2)f'(\chi_2))}{(f(\chi_1) - f(\chi_2))^2},$$

and take into account that the second derivative of f is positive. The expression in the parentheses is negative:

$$\chi_1 f(\chi_1) + A(\chi_1, \chi_2)f^2(\chi_1) - 2 \le \chi_1 f(\chi_1) + \frac{f^2(\chi_1)}{2} - 2 \le 0.$$

The last inequality follows directly from the properties of the function $f(x)$. Hence, $D(\chi_1, \chi_2)$ is positive for $\chi_2 < \chi_2^{\max}$ and negative for $\chi_2^{\max} < \chi_2$.

2. In three dimensions, we introduce the new parameters

$$\xi = \frac{1}{2}(\gamma_1 + \gamma_2), \qquad \eta = \frac{1}{2}(\gamma_1 - \gamma_2), \tag{32}$$

i.e.,

$$\gamma_1 = \xi + \eta, \qquad \gamma_2 = \xi - \eta.$$

Dispersion equation (20) becomes

$$(\chi - \xi)^2 - \eta^2 - e^{-2\chi} = 0. \tag{33}$$

Because χ_1 is a solution of the last equation, we obtain

$$Q^3(\xi, \chi) \equiv (\chi - \xi)^2 - (\chi_1 - \xi)^2 - e^{-2\chi} + e^{-2\chi_1} = 0. \tag{34}$$

We can now estimate the partial derivative $\partial\chi_2/\partial\eta$ as

$$\left.\frac{\partial\chi}{\partial\eta}\right|_{\chi=\chi_2} = \frac{\eta(\chi_1-\chi_2)}{(\chi_1-\xi)(\chi_2-\xi+e^{-2\chi_2})} < 0,$$

taking into account that

$$\chi_2 - \xi + e^{-\chi} = -\sqrt{e^{-2\chi_2}+\eta^2} + e^{-2\chi_2} \le e^{-\chi_2}(e^{-\chi_2}-1) < 0.$$

It follows that χ_2 takes its maximum value at $\eta = 0$, i.e., in the symmetric case $\gamma_1 = \gamma_2 \equiv \gamma$. This case corresponds to χ_1 and $\chi_2 = \chi_2^{\max}$ satisfying the equation

$$(\chi-\gamma)^2 = e^{-2\chi} \Rightarrow \chi_1 - \chi_2^{\max} = e^{-\chi_1} + e^{-\chi_2^{\max}}.$$

Hence, the estimate

$$\chi_1 - \chi_2 \ge e^{-\chi_1} + e^{-\chi_2} \tag{35}$$

holds for all $\chi_2 \colon 0 \le \chi_2 \le \chi_2^{\max}$.

We show that the discriminant is positive for all $\chi_2 \le \chi_2^{\max}$. The discriminant of quadratic equation (30) for γ_1 and γ_2 is

$$D(\chi_1,\chi_2) = (\chi_1-\chi_2)^2 - 2(e^{-2\chi_1}+e^{-2\chi_2}) + \frac{(e^{-2\chi_1}-e^{-2\chi_2})^2}{(\chi_1-\chi_2)^2}.$$

We can estimate the sum of the first and third terms by

$$(e^{-\chi_1}+e^{-\chi_2})^2 + \frac{(e^{-2\chi_1}-e^{-2\chi_2})^2}{(e^{-\chi_1}+e^{-\chi_2})^2},$$

taking into account that $(\chi_1-\chi_2)^2 \ge e^{-2\chi_1} - e^{-2\chi_2}$ in accordance with estimate (35). Hence, the discriminant can be estimated from below:

$$D(\chi_1,\chi_2) \ge (e^{-\chi_1}+e^{-\chi_2})^2 - 2(e^{-2\chi_1}+e^{-2\chi_2}) + \frac{(e^{-2\chi_1}-e^{-2\chi_2})^2}{(e^{-\chi_1}+e^{-\chi_2})^2} = 0.$$

It follows that system (30) has real solutions for all $0 \le \chi_2 \le \chi_2^{\max}(\chi_1)$. The theorem is proved.

The theorem states that the system of two local point interactions never has a multiple eigenvalue. The distance between the eigenenergies is minimum in the symmetric case $\gamma_1 = \gamma_2$. This is illustrated in Fig. 1 (Figs. 1a and 1b respectively correspond to $\mathbb{R}^1$ and $\mathbb{R}^3$). The domains between the curves are forbidden, i.e., it is impossible to find two point interactions at the distance $d = 1$ such that the eigenvalues are located in these domains. In the limit as $\chi_1, \chi_2 \to \infty$, these curves approach the line $\chi_1 = \chi_2$. This means that two deep eigenvalues may be located rather close to each other. Both curves intersect the corresponding coordinate axes at the points that are a unique solution of the equation $x = 1 + e^{-x}$.

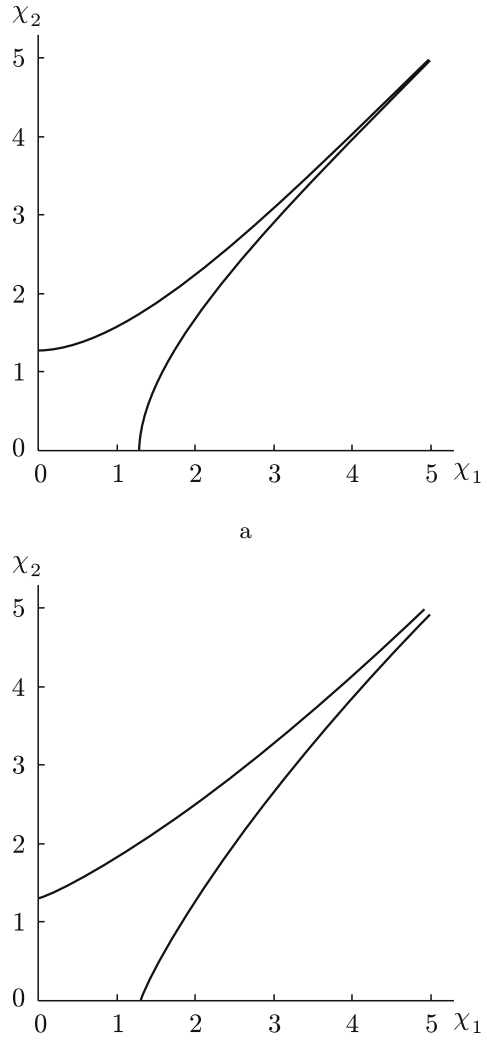

Fig. 1

We have proved that the Schrödinger operator with two local point interactions cannot have a degenerate eigenvalue. Moreover, the two eigenvalues cannot be arbitrarily close to each other. This is a certain generalization of the von Neumann–Wigner theorem [3]. We note that the last statement holds because of the special form of the boundary conditions described by diagonal matrices. Considering nonlocal point interactions, we can obtain operators with two negative eigenvalues located arbitrarily.

4. Degenerate eigenvalues in the case of several centers

The one-dimensional Schrödinger operator cannot have degenerate eigenvalues (except in the case where the operator can be represented by an orthogonal sum of operators on two intervals). Therefore, we restrict our consideration to the case of the three-dimensional Schrödinger operator with point interaction described in Sec. 2. Moreover, we study the maximum possible degeneracy for simplicity. Lower-order degeneracies can be studied by considering clusters consisting of a smaller number of centers.

We consider the possibility of the maximum degeneracy $N-1$ for the system of N point potentials.

The equation determining the eigenvalues in this system is

$$\det\begin{pmatrix} -\chi+\gamma_1 & \dfrac{e^{-\chi d_{12}}}{d_{12}} & \dfrac{e^{-\chi d_{13}}}{d_{13}} & \cdots & \dfrac{e^{-\chi d_{1N}}}{d_{1N}} \\ \dfrac{e^{-\chi d_{21}}}{d_{21}} & -\chi+\gamma_2 & \dfrac{e^{-\chi d_{23}}}{d_{23}} & \cdots & \dfrac{e^{-\chi d_{2N}}}{d_{2N}} \\ \dfrac{e^{-\chi d_{31}}}{d_{31}} & \dfrac{e^{-\chi d_{32}}}{d_{32}} & -\chi+\gamma_3 & \cdots & \dfrac{e^{-\chi d_{3N}}}{d_{3N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{e^{-\chi d_{N1}}}{d_{N1}} & \dfrac{e^{-\chi d_{N2}}}{d_{N2}} & \dfrac{e^{-\chi d_{N3}}}{d_{N3}} & \cdots & -\chi+\gamma_N \end{pmatrix} = 0, \tag{36}$$

where $\gamma_j = 4\pi\alpha_j^{-1}$. This equation determines at most N negative eigenvalues. We are interested in the case where one of these eigenvalues has the maximum possible multiplicity $N-1$. This occurs in the case where all rows in the matrix are parallel, i.e., the determinants of all 2×2 minors are zero. We estimate for which number of centers this is possible. For sufficiently large N ($N \geq 3$), the configuration of the centers is determined by $N_{\text{geom}} = 3(N-2)$ distances. In addition, there are $N_{\text{ext}} = N$ parameters determining the extensions. Hence, the matrix contains $N_{\text{par}} = N_{\text{geom}} + N_{\text{ext}} + 1 = 4N - 5$ parameters if we additionally include the spectral parameter χ. The total number of 2×2 minors is equal to $(n(n-1)/2)^2$, but the number of independent equations reduces to $n(n-1)/2$ because of the symmetry and special form of the matrix. Hence, there are $N_{\text{con}} = n(n-1)/2$ constraints in general on the parameters that guarantee the maximum degeneracy of the eigenvalue. Because the number of constraints increrases quadratically but the number of parameters increases just linearly, it is impossible to find a configuration leading to the maximum degeneracy for sufficiently large N. In Table 1, we show the calculated maximum expected dimension D of the parameter set that guarantees the maximum degeneracy.

Table 1

N	N_{geom}	N_{ext}	N_{par}	N_{con}	D
1	0	1	2	1	1
2	1	2	4	1	3
3	3	3	7	3	4
4	6	4	11	6	5
5	9	5	15	10	5
6	12	6	19	15	4
7	15	7	23	21	2
8	18	8	27	28	∅

We see that for low N, our "naive" calculations give the correct result. Thus, in the case of one point interaction, there is a one-parameter family of extensions having an eigenvalue. For two centers, the family of operators having an eigenvalue is described by three parameters: two extension parameters and one distance.

For large N ($N \geq 8$), the table indicates that no eigenvalue of maximum multiplicity is possible. In this section, we study the case of intermediate values of N.

Three centers. There are nine minors. Because the matrix is symmetric, only six minors are independent. Because of the special structure of the matrix, the number of equations reduces to three:

$$\begin{aligned} -\chi+\gamma_1 &= \frac{d_{23}}{d_{12}d_{13}}e^{\chi(d_{23}-d_{13}-d_{12})},\\ -\chi+\gamma_2 &= \frac{d_{13}}{d_{12}d_{23}}e^{\chi(d_{13}-d_{12}-d_{23})},\\ -\chi+\gamma_3 &= \frac{d_{12}}{d_{13}d_{23}}e^{\chi(d_{12}-d_{13}-d_{23})}. \end{aligned} \tag{37}$$

Theorem 2. *For an arbitrary configuration $\{y^1, y^2, y^3\}$ of three points in $\mathbb{R}^3$ and an arbitrary negative number $E=-\chi^2$, there exists a unique set of parameters α_1, α_2, and α_3 such that the Schrödinger operator with three δ-interactions of the strengths α_1, α_2, and α_3 respectively concentrated at y^1, y^2, and y^3 has a degenerate eigenvalue with the energy $E=-\chi^2$.*

Proof. We consider system of equations (37) for an arbitrary set of positive parameters d_{12}, d_{13}, d_{23}, and χ. These equations allow directly calculating three positive real numbers $\gamma_j = 4\pi/\alpha_j$ and hence reconstructing the parameters determining the δ-interactions at the points y^j. The theorem is proved.

Theorem 2 shows that for an arbitrary fixed configuration of points supporting δ-functions, the parameter set determining the interactions that lead to double eigenvalues can be parameterized by one real parameter: the energy of the degenerate eigenvalue. Then the parameter set leading to a degenerate eigenvalue can be described by four parameters, as predicted in Table 1.

Four centers. There are 36 minors, but because of the symmetry of the matrix, only 21 minors are independent. The special form of the matrix reduces the number of independent equations to six. It is natural to divide these equations into two systems, one of four equations and one of two:

$$\begin{aligned} -\chi+\gamma_1 &= \frac{d_{23}}{d_{13}d_{12}}e^{\chi(d_{23}-d_{13}-d_{12})},\\ -\chi+\gamma_2 &= \frac{d_{34}}{d_{24}d_{23}}e^{\chi(d_{34}-d_{24}-d_{23})},\\ -\chi+\gamma_3 &= \frac{d_{14}}{d_{34}d_{13}}e^{\chi(d_{14}-d_{34}-d_{13})},\\ -\chi+\gamma_4 &= \frac{d_{12}}{d_{14}d_{24}}e^{\chi(d_{12}-d_{14}-d_{24})} \end{aligned} \tag{38}$$

and

$$e^{\chi(d_{12}+d_{34})}d_{12}d_{34} = e^{\chi(d_{13}+d_{24})}d_{13}d_{24} = e^{\chi(d_{14}+d_{23})}d_{14}d_{23}. \tag{39}$$

The second system describes a certain relation between the distances and the energy parameter. This means that not all configurations of four centers lead to a triple eigenvalue. In general, if this configuration is admissible, then it uniquely determines the possible energy of the triple bound state (except in the special cases described by Theorem 3). Then the distances and the energy of the triple eigenvalue can be used to determine the strengths of the point interactions from the first four equations, system (38).

For the system of four centers, we introduce some perimetric coordinates: the sums of the lengths of opposite edges in the tetrahedron determined by y^1, y^2, y^3, and y^4,

$$\begin{aligned} D_{12} &= d_{12} + d_{34}, \\ D_{13} &= d_{13} + d_{24}, \\ D_{14} &= d_{14} + d_{23}. \end{aligned} \tag{40}$$

These coordinates can be used to easily calculate the perimeters of all three possible quadrangles: the perimeters are equal to the sums of the corresponding two perimetric coordinates.

Theorem 3. *We consider the Schrödinger operator in $L_2(\mathbb{R}^3)$ with four point interactions of strengths α_1, α_2, α_3, and α_4 at the points y^1, y^2, y^3, and y^4. This operator has a triple eigenvalue if and only if one of the following conditions is satisfied.*

1. *If all three perimetric coordinates are different, then the distances between the centers must satisfy one of the three equivalent conditions*

$$\frac{\log d_{12} + \log d_{34} - \log d_{13} - \log d_{24}}{d_{12} + d_{34} - d_{13} - d_{24}} = \frac{\log d_{12} + \log d_{34} - \log d_{14} - \log d_{23}}{d_{12} + d_{34} - d_{14} - d_{23}} < 0, \tag{41}$$

$$\frac{\log d_{12} + \log d_{34} - \log d_{13} - \log d_{24}}{d_{12} + d_{34} - d_{13} - d_{24}} = \frac{\log d_{13} + \log d_{24} - \log d_{14} - \log d_{23}}{d_{13} + d_{24} - d_{14} - d_{23}} < 0, \tag{42}$$

$$\frac{\log d_{12} + \log d_{34} - \log d_{14} - \log d_{23}}{d_{12} + d_{34} - d_{14} - d_{23}} = \frac{\log d_{13} + \log d_{24} - \log d_{14} - \log d_{23}}{d_{13} + d_{24} - d_{14} - d_{23}} < 0. \tag{43}$$

The energy of the triple eigenvalue is uniquely determined by the geometry of the centers:

$$E = -\left(\frac{\log d_{12} + \log d_{34} - \log d_{14} - \log d_{23}}{d_{12} + d_{34} - d_{14} - d_{23}}\right)^2. \tag{44}$$

The unique values of the constants α_j are determined by Eq. (38) (*with* (19) *taken into account*).

2. *If two of the perimetric coordinates coincide, for example, $D_{12} - D_{13} = d_{12} + d_{34} - d_{13} - d_{24} = 0$, then the lengths in these pairs must be equal, i.e.,*

$$\begin{cases} d_{12} = d_{13}, \\ d_{34} = d_{24}, \end{cases} \quad \text{or} \quad \begin{cases} d_{12} = d_{24}, \\ d_{34} = d_{13}. \end{cases} \tag{45}$$

Then the triple eigenvalue exists only if the condition

$$\frac{\log d_{12} + \log d_{34} - \log d_{14} - \log d_{23}}{d_{12} + d_{34} - d_{14} - d_{23}} < 0$$

is satisfied, and its energy is uniquely determined by the geometry and is given by formula (44). *The unique values of the parameters α_j are determined by formula* (38) (*with* (19) *taken into account*).

3. *If all three perimetric coordinates are equal, then the triple eigenvalue exists if and only if the four centers are located at the vertices of a tetrahedron with at least one side given by a regular triangle and the three other sides equal. The energy of the triple eigenvalue is arbitrary* (*negative*), *and the values of the parameters α_j* (*all equal*) *are uniquely determined by this energy.*

Proof. We consider each of the three cases separately.

1. If all three perimetric coordinates are different, then the parameter χ, which determines the energy of the triple bound state, can be calculated from (39), using the equations

$$\chi = -\frac{\log d_{12} + \log d_{34} - \log d_{13} - \log d_{24}}{d_{12} + d_{34} - d_{13} - d_{24}},$$
$$\chi = -\frac{\log d_{12} + \log d_{34} - \log d_{14} - \log d_{23}}{d_{12} + d_{34} - d_{14} - d_{23}},$$
$$\chi = -\frac{\log d_{13} + \log d_{24} - \log d_{14} - \log d_{23}}{d_{13} + d_{24} - d_{14} - d_{23}}.$$

Eliminating χ from the three different equations, we obtain (41)–(43), taking into account that χ must be positive. Then the parameters α_j (or γ_j) can be calculated from Eqs. (38).

2. If any two of the three perimetric coordinates coincide, for example, $D_{12} = D_{13}$, then system of equations (39) is equivalent to

$$\frac{\log d_{12} + \log d_{34} - \log d_{14} - \log d_{23}}{d_{12} + d_{34} - d_{14} - d_{23}} < 0,$$
$$d_{12}d_{34} = d_{13}d_{24}.$$

Because the two perimetric coordinates coincide, we conclude that the last equation implies Eqs. (45) and that the energy of the triple eigenvalue is given by expression (44). The extension parameters are again determined by Eqs. (38).

3. If all three perimetric coordinates coincide, then Eqs. (39) are equivalent to

$$d_{12}d_{34} = d_{13}d_{24} = d_{14}d_{23},$$

and it follows that the four centers form a tetrahedron with at least one side given by a regular triangle. The three remaining sides are equal. In this case, Eqs. (39) are satisfied for an arbitrary value of the energy parameter $E = -\chi^2$. After choosing this parameter equal to an arbitrary negative number, we can calculate the unique values of the strength parameters from Eqs. (38). The theorem is proved.

The first family of point interactions is described by five independent parameters as expected (see Table 1). The other two families are correspondingly described by four and three parameters. The last family is the most interesting: it includes the regular tetrahedron, the most symmetric configuration of four centers.

Five centers. We study whether this system can have an eigenvalue of multiplicity four. It is not obvious that the system of equations is solvable. The table predicts that the family of solutions is described by five parameters. We can show that there exists a two-parameter family. We consider the most symmetric configuration of five points: the points y^2, y^3, y^4, and y^5 are located at the corners of a regular tetrahedron, and the point y^1 is located at the center of the tetrahedron.

We let d denote the length of the tetrahedron edges and r denote the radius of the circumscribed sphere

containing the four tetrahedron vertices. Then system (36) becomes

$$\det\begin{pmatrix} -\chi+\gamma_1 & \frac{e^{-\chi r}}{r} & \frac{e^{-\chi r}}{r} & \frac{e^{-\chi r}}{r} & \frac{e^{-\chi r}}{r} \\ \frac{e^{-\chi r}}{r} & -\chi+\gamma_2 & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} \\ \frac{e^{-\chi r}}{r} & \frac{e^{-\chi d}}{d} & -\chi+\gamma_3 & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} \\ \frac{e^{-\chi r}}{r} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & -\chi+\gamma_4 & \frac{e^{-\chi d}}{d} \\ \frac{e^{-\chi r}}{r} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & -\chi+\gamma_5 \end{pmatrix} = 0. \tag{46}$$

All rows of the matrix are linearly dependent (the eigenvalue has the multiplicity four) if and only if

$$-\chi+\gamma_2 = -\chi+\gamma_3 = -\chi+\gamma_4 = -\chi+\gamma_5 = \frac{e^{-\chi d}}{d},$$

i.e.,

$$\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \chi + \frac{e^{-\chi d}}{d}$$

and

$$-\chi+\gamma_1 = \frac{(e^{-\chi r}/r)^2}{e^{-\chi d}/d},$$

i.e.,

$$\gamma_1 = \chi + \frac{d}{r^2} e^{\chi(d-2r)}.$$

Hence, for any χ, we can find parameters α_j, $j = 1, \ldots, 5$, such that the Schrödinger operator with five point interactions has an eigenvalue of multiplicity four. We obtain a family described by two parameters: the length of the edge of the regular tetrahedron and the energy of the degenerate state.

Six and more centers. We consider the system of six centers. The table predicts that the parameter set leading to the maximum degeneracy is described by four parameters. We examine the most symmetric configuration of six centers: the points located at the vertices of an octahedron. The matrix in Eq. (36) becomes

$$\begin{pmatrix} -\chi+\gamma_1 & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi\sqrt{2}d}}{\sqrt{2}d}d & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} \\ \frac{e^{-\chi d}}{d} & -\chi+\gamma_2 & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi\sqrt{2}d}}{\sqrt{2}d}d & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} \\ \frac{e^{-\chi\sqrt{2}d}}{\sqrt{2}d} & \frac{e^{-\chi d}}{d} & -\chi+\gamma_3 & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} \\ \frac{e^{-\chi d}}{d} & \frac{e^{-\chi\sqrt{2}d}}{\sqrt{2}d} & \frac{e^{-\chi d}}{d} & -\chi+\gamma_4 & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} \\ \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & -\chi+\gamma_5 & \frac{e^{-\chi\sqrt{2}d}}{\sqrt{2}d} \\ \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi d}}{d} & \frac{e^{-\chi\sqrt{2}d}}{\sqrt{2}d} & -\chi+\gamma_6 \end{pmatrix},$$

where d is the length of the octahedron edge. The rows are parallel only if

$$\frac{e^{-\chi d}}{d} = \frac{e^{-\chi\sqrt{2}d}}{\sqrt{2}d}$$

is satisfied, i.e.,

$$\chi = -\frac{\log 2}{2d(\sqrt{2}-1)} < 0,$$

but the parameter χ must be positive. It follows that this system cannot have an eigenvalue of multiplicity five. The last equation determines a resonance instead of the eigenvalue. We conjecture that the system of six point interactions cannot have an eigenvalue of multiplicity five. Similarly, we do not expect eigenvalues of the multiplicity $N-1$ for any system of N point interaction for $N > 6$. We will return to this problem in one of our forthcoming publications.

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