Nonphysical Sheet and Schroedinger Evolution.*

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Abstract

The modification of the Lax-Phillips scattering theory to the case of Schroedinger equation is investigated. Formal asymptotic representation is constructed. Decay operator for the Schroedinger evolution is separated from nonexponential evolution. The developed method is applicated to exactly solvable model, constructed with the help of restriction-extension procedure.

1.INTRODUCTION.

Asymptotic behaviour of Schroedinger evolution is a very interesting object to study, especially from the physical point of view due to the contradiction with the idea of an exponential decay [1−9]. An exponential decay is a typical evolution only for badly localized quantum objects. Evolution of localized objects is nonexponential. Corresponding analysis for two-body sector of the Lie model has been carried out in [7−12]. P.Exner and J.Dittrich analysed corresponding mathematical exactly solvable model with the help of scattering theory methods in [6−9]. All these calculations contain an analytical description of the investigated phenomenon. But operator treatment of the nonphysical sheet is absent. It will be very interesting to calculate dissipative operator which eigenvalues coincide with resonances of quantum system. Such an operator was constructed for the acoustic scattering [1,13]. We shall name it the "decay operator". A spectral analysis of this operator seems to be very usefull because it provides an universal model of a dissipative operator and a contractive evolution. The main operator object of self-adjoint theory - scattering matrix - is a characteristic function of the "decay operator" [14,15]. Singularities of the S-matrix define the spectrum of this operator (discrete and continuous) [13−15].

One can conclude that such an operator is absent for Schroedinger evolution. In the spectral representation quantum-mechanical asymptotic $t \to \infty$ is very simple, but it contains nonexponential terms. The "decay operator" can be separated from the nonexponential evolution. In the second part of the paper we shall construct exactly solvable model with the help of the extensions theory in order to make all the peculiarities of the theory clear.

2.SCHROEDINGER EVOLUTION IN LAX-PHILLIPS REPRESENTATION.

Let us study the simplest scattering problem in Hilbert space $\mathcal{H} = L_2(\mathcal{R})$ for the shift group $\{e^{ikt}\}$. We fix incoming and outgoing spaces $\mathcal{D}_+$ and $\mathcal{D}_-$ by introducing some inner function $\mathcal{S}$ as a Bluashke product with simple zeroes $\{k_\ell\}$, which are situated at the upper half-plane $\Im k_\ell > 0$.

$$\mathcal{S}(k) = \prod_\ell \frac{k - k_\ell}{k - \overline{k_\ell}} \left(\frac{\imath - k_\ell}{\imath - \overline{k_\ell}}\right)^{-1} \left| \frac{\imath - k_\ell}{\imath - \overline{k_\ell}} \right| \tag{1}$$

$$\mathcal{D}_- = \mathcal{H}^2_-, \mathcal{D}_+ = \mathcal{S}\mathcal{H}^2_-.$$

Supposing the product (1) to be finite, $1 \leq \ell \leq n.$ we have Carleson condition satisfied automatically :

$$\prod_{\ell \neq m} \left| \frac{k_\ell - k_m}{k_\ell - \overline{k_m}} \right| = |\mathcal{S}'(k_m)| \, 2\Im k_m \geq \delta > 0$$

We confine ourselves with this simplest case just to share more attention to the crucial fact of nonexponential decay, caused by spectral threshold at $\lambda = 0$. By restriction of the semigroup of left shifts onto the translation-invariant subspace $\mathcal{K} = \mathcal{H}^2_+ - \mathcal{S}\mathcal{H}^2_+$ we get the family of contractions which forms the semigroup with the dissipative generator $\mathcal{B}$

$$\mathcal{Z}(t) = \mathbf{P}_\mathcal{K} \mathcal{U}_t|_\mathcal{K} = e^{\imath \mathcal{B} t}, \Im \mathcal{B} \geq 0,$$

$\mathcal{S}$ being the characteristic function of $\mathcal{B}$. It's zeroes k_ℓ are eigenvalues of the generator. The corresponding eigenfunctions are:

$$\psi_\ell(k) = \mathcal{S}(k)(k - k_\ell)^{-1}.$$

It is easy to proof that eigenfunctions of the generator $\mathcal{B}^*$ of the conjugated semigroup $\mathbf{P}_\mathcal{K} \mathcal{U}^*_t|_\mathcal{K} = e^{-\imath \mathcal{B}^* t}, t > 0$,are:

$$\varphi_\ell(k) = -\imath (k - \overline{k_\ell})^{-1}.$$

Together with $\{\psi_\ell\}$ they form a biorthogonal system in $\mathcal{K}$:

$$\langle \psi_\ell, \varphi_m \rangle = \frac{1}{2\pi} \int \psi_\ell \overline{\varphi_m} dk = \mathcal{S}'(k_\ell) \delta_{\ell m}.$$

In terms of this system there can be written the spectral representation of $\mathcal{B}$:

$$f=\sum_{\ell}\langle f,\varphi_\ell\rangle\psi_\ell\frac{1}{\mathcal{S}'(k_\ell)}$$

and the restricted evolution:

$$e^{\imath\mathcal{B}t}f=\sum_{\ell}\frac{e^{\imath k_\ell t}}{\mathcal{S}'(k_\ell)}\langle f,\varphi_\ell\rangle\psi_\ell \tag{2}$$

This decreasing evolution is of an exponential type. The corresponding decrement of the decay is equal to the smallest imaginary part of an eigenvalue $min\Im k_\ell$.

Now let us study the Schroedinger evolution $\mathcal{V}(t)$, connected with the shift group. It's generator is equal to the square of the generator of the shift group :

$$\mathcal{L}*=k^2*,$$
$$\mathcal{V}(t)*=e^{\imath k^2 t}\,*\,.$$

We shall investigate the evolution on the translational invariant subspace $\mathcal{K}$ for large times, $t\to\infty$.

Theorem 1. *The Schroedinger evolution, restricted on $\mathcal{K}$, has the following asymptotic behaviour for large t :*

$$\mathbf{P}_{\mathcal{K}}\mathcal{V}(t)f=\frac{f(0)}{2p\sqrt{t\pi}}\frac{-\mathcal{S}(0)+\mathcal{S}(p)}{\mathcal{S}(0)}e^{-\imath\frac{\pi}{4}}+O(\frac{1}{t^{\frac{3}{2}}}).$$

Proof. In our representation the restricted Schroedinger evolution is presented as an operator with an unitary symbol: $e^{\imath k^2 t}>0,t\geq 0$.

$$\mathbf{P}_{\mathcal{K}}\mathcal{V}(t)|_{\mathcal{K}}=\mathbf{P}_{\mathcal{K}}e^{\imath k^2 t}|_{\mathcal{K}} \tag{3}$$

An integral representation for Hilbert projector should be used:

$$\mathbf{P}_+*=\frac{1}{2\pi\imath}\int\frac{1}{k-p}*\,dk,\Im p>0$$

$$\mathbf{P}_{\mathcal{K}}=\mathbf{P}_+-\mathbf{P}_{\mathcal{S}\mathcal{H}^2_+}=\mathbf{P}_+-\mathcal{S}\mathbf{P}_+\mathcal{S}^*.$$

One can calculate the Schroedinger evolution on $\mathcal{K}$:

$$\mathbf{P}_{\mathcal{K}}\mathcal{V}_t f=\mathbf{P}_+\mathcal{V}_t f-\mathcal{S}\mathbf{P}_+\mathcal{S}^*\mathcal{V}_t f$$

The integrate function in the second term

$$\frac{S(p)}{2\pi\imath}\int\frac{1}{k-p}e^{\imath k^2 t}\frac{f(k)}{S(k)}dk, \Im p>0$$

has singularities in the upper half-plane in the point $k=p$ and at the zeroes of the S-function:

$$\frac{f(k)}{S(k)}=\sum_{\ell}\frac{1}{k-k_\ell}f(k_\ell)\frac{1}{S'(k_\ell)}.$$

In the lower half-plane the function fS^{-1} is an analytic one. On the contrary the integrate function in the first term

$$\frac{1}{2\pi\imath}\int\frac{e^{\imath k^2 t}}{k-p}f(k)dk$$

is an analytic function in the upper half-plane due to $f\in\mathcal{K}\subset\mathcal{H}_+^2$ and has poles in the lower one at points $\overline{k_\ell}$:

$$f(k)=\sum_{\ell}\frac{\imath}{k-\overline{k_\ell}}\langle f,\psi_\ell\rangle\frac{1}{\overline{S'(k_\ell)}}.$$

The bisector of the 1-3 quadrants is the steepest descent line for the entire function $e^{\imath k^2 t}, t>0$,

$$e^{\imath k^2 t}|_{k=exp(\imath\frac{\pi}{4})\chi}=e^{-\chi^2 t}.$$

One can deforme the contour in the formula (3). It is only necessary to calculate the residues at points $k_\ell,\overline{k_\ell}$, which lie in the sectors $0<argk<\frac{\pi}{4}$ and $\pi<arg\overline{k}<\pi+\frac{\pi}{4}$. So the restricted Schroedinger evolution (3) can be presented in the following form for $p=q+\imath 0, q<0$:

$$\mathcal{P}_{\mathcal{K}}\mathcal{V}(t)f=$$

$$\frac{e^{\imath\frac{\pi}{4}}}{2\pi\imath}\int_{-\infty}^{+\infty}\frac{e^{-\chi^2 t}f(\chi e^{\imath\frac{\pi}{4}})}{\chi e^{\imath\frac{\pi}{4}}-p}d\chi-\frac{S(p)}{2\pi\imath}e^{\imath\frac{\pi}{4}}\int_{-\infty}^{+\infty}\frac{e^{-\chi^2 t}}{\chi e^{\imath\frac{pi}{4}}-p}\frac{f(\chi e^{\imath\frac{\pi}{4}})}{S(\chi e^{\imath\frac{\pi}{4}})}d\chi-$$

$$-\sum_{\ell_-}e^{\imath\overline{k_\ell^2}t}\frac{\langle f,\psi_\ell\rangle}{\overline{k_\ell}-p}\frac{\imath}{\overline{S'(k_\ell)}}-\sum_{\ell_+}e^{\imath k_\ell^2 t}\frac{S(p)}{k_\ell-p}\frac{f(k_\ell)}{S'(k_\ell)}. \tag{4}$$

The first (second) sum $\sum_{\ell_-}(\sum_{\ell_+})$ is a sum over the "physical" resonances k_ℓ lying in the sector $\pi-\frac{\pi}{4}<argk_\ell<\pi$ $(0<argk_\ell<\frac{\pi}{4})$. We include all resonances lying on the bisector into the integrals. The residues for $p = q + \imath 0, q > 0$ in the first and second terms compensate one another. So the formula (4) is valid for an arbitrary real q.

These two sums provide exponentialy decreasing terms:

$$\left|e^{\imath k_\ell^2 t}\right| = e^{-2\Im k_\ell t} = \left|e^{\imath \overline{k_\ell^2} t}\right|.$$

Amplitudes of the corresponding exponential modes can be calculated up to $S'(k_\ell)$ as a scalar product of the initial state f and eigenfunctions ψ_ℓ, φ_ℓ of the generator $\mathcal{B}$ and it's adjoint $\mathcal{B}^*$:

$$f_\ell = \langle f, \varphi_\ell \rangle.$$

The integrals in the formula (4) can be calculated with the help of the steepest descent path method due to the analyticity of the integrated functions at zero:

$$\frac{e^{\imath \frac{\pi}{4}}}{2\pi\imath}\int_{-\infty}^{+\infty} e^{-\chi^2 t}\frac{f(\chi e^{\imath\frac{\pi}{4}})}{\chi e^{\imath\frac{\pi}{4}} - q}d\chi = \frac{f(0)}{q\sqrt{t}}\frac{e^{3\imath\frac{\pi}{4}}}{2\sqrt{\pi}} + O(\frac{1}{t^{\frac{3}{2}}}) \tag{5}$$

$$\frac{-S(q)e^{\imath\frac{\pi}{4}}}{2\pi\imath}\int_{-\infty}^{\infty}\frac{e^{-\chi^2 t}}{\chi e^{\imath\frac{\pi}{4}} - q}\frac{f(\chi e^{\imath\frac{\pi}{4}})}{S(\chi e^{\imath\frac{\pi}{4}})}d\chi = \frac{f(0)}{S(0)}\frac{S(q)}{q\sqrt{t}}\frac{e^{-\imath\frac{\pi}{4}}}{2\sqrt{\pi}} + O(\frac{1}{t^{\frac{3}{2}}}) \tag{6}$$

Remark. Formulae (4,5,6) show that the exponential terms are important for the small t and for the initial data satisfying the condition of delocalization $|f(0)| \ll 1$. In this case a wave packet decay exponentialy. We shall see a nonexponential evolution for "large" t only. On the contrary, "well localized" wave packets develop nonexponentially. This conclusion is approved by the papers [1-9]. In addition to these papers we see that nonexponential terms have an operator interpretation. One can introduce dissipative "decay operators". They commute with $\mathcal{B}, \mathcal{B}^*$ correspondingly:

$$\mathcal{A}_+ = \sum_{\ell_+} k_\ell^2 \psi_\ell \langle *, \varphi_\ell\rangle (S'(k_\ell))^{-1},$$

$$\mathcal{A}_- = \sum_{\ell_-} \overline{k_\ell^2} \varphi_\ell \langle *, \psi_\ell\rangle (S'(k_\ell))^{-1}$$

So the exponential part of the evolution is:

$$e^{\imath\mathcal{A}_+ t} f + e^{\imath\mathcal{A}_- t} f$$

These terms are important for the delocalized initial data and the "small" time only. The formula (4) can be written as following:

$$\mathbf{P}_{\mathcal{K}} e^{\imath k^2 t} f \approx e^{\imath \mathcal{A}_+ t} f + e^{\imath \mathcal{A}_- t} f + \frac{f(0)[S(q) - S(0)]}{2S(0)\sqrt{t\pi}q} e^{-\imath \frac{\pi}{4}}. \tag{7}$$

Thus we see, that no "decay operator" for Schroedinger evolution exist, but there exist two mutually adjoint operators $\mathcal{A}_{\pm}$, which are associated with corresponding shift group. These operators play important role for small time and special initial data.

3. DECAY OPERATOR FOR THE POINT INTERACTION.

Let us investigate connections between wave and Schroedinger equations in the case of the simplest model, constructed with the help of the restriction-extension procedure. The unperturbed Schroedinger equation

$$-\triangle u = \frac{1}{\imath}\frac{\delta u}{\delta t} \tag{8}$$

defines the unperturbed evolution. The generator $-\triangle$ is a selfadjoint operator in the space $L_2(\mathcal{R}^3)$. By restriction of this operator on the subset of the domain of all functions, vanishing at origin we receive the symmetric operator with defficiency indices (1,1) and defficiency element for $\lambda = k_0^2, \Im k_0 > 0$:

$$g_0 = \frac{e^{\imath k_0 |x|}}{4\pi |x|}.$$

The domain of the adjoint operator consists of all Caushy data with the following singularity at the origin:

$$u = \frac{\xi_-(u)}{4\pi|x|} + \xi_0(u) + o(1)$$

$$|x| \to 0.$$

The boundary form of the adjoint operator $(-\triangle_0)^*$ is:

$$\langle (-\triangle_0)^* u, v\rangle - \langle u, (-\triangle_0)^* v\rangle = \xi_0(u)\overline{\xi_-(v)} - \xi_-(u)\overline{\xi_0(v)}.$$

We shall "switch on" a non-trivial point interaction by adding internal structure. Let us choose some self-adjoint operator $\mathcal{L}_{in}$, acting in the Hilbert space H_{in}. Let us suppose H_{in} to be finite-dimensional just for the same reason as in part 2. We shall consider the orthogonal sum of the external ($\mathcal{L}_{ex} = -\triangle$) and internal $\mathcal{L}_{in}$ operators $\mathcal{A}_0 = \mathcal{L}_{ex} \oplus \mathcal{L}_{in}$ as a background operator. It acts in the space $\mathcal{H} = L_2(\mathcal{R}^3) \oplus H_{in}$. Let us consider the perturbed operator

$$\mathcal{A}\begin{pmatrix} u_{ex} \\ u_{in}\end{pmatrix} = \begin{pmatrix} \mathcal{L}_0^* u_{ex} \\ \mathcal{L}_{in} u_{in} + (a\xi_0(u_{ex}) + b\xi_-(u_{ex}))\,\theta \end{pmatrix} \tag{9}$$

in order to "switch on" an interaction between internal and external components. Here a and b are arbitrary complex constants. Operator $\mathcal{A}$, defined on the domain $Dom(\mathcal{L}_0^*) \oplus H_{in}$ is not self-adjoint. The corresponding boundary form is:

$$[u,v] \equiv \langle \mathcal{AU}, \mathcal{V}\rangle_{H_{ex}\oplus H_{in}} - \langle \mathcal{U}, \mathcal{AV}\rangle_{H_{ex}\oplus H_{in}} =$$
$$= \left(\xi_0(u_{ex})\overline{\xi_-(v_{ex})} - \xi_-(u_{ex})\overline{\xi_0(v_{ex})}\right) +$$
$$(a\xi_0(u_{ex}) + b\xi_-(u_{ex}))\,\langle \theta, v_{in}\rangle_{H_{in}} - \langle u_{in}, \theta\rangle\overline{(a\xi_0(v_{ex}) + b\xi_-(v_{ex}))}.$$

If we shall restrict $\mathcal{A}$ on the lineal determined by the boundary conditions:

$$\langle u_{in}, \theta\rangle = c\xi_0(u_{ex}) + d\xi_-(u_{ex}) \tag{10}$$

we shall receive the following boundary form:

$$[u,v] = (1 + a\bar{d} - c\bar{b})\xi_0(u)\overline{\xi_-(v)} +$$
$$+(-1 + b\bar{c} - d\bar{a})\xi_-(u)\overline{\xi_0(v)} + (a\bar{c} - c\bar{a})\xi_0(u)\overline{\xi_0(v)} + (B\bar{d} - d\bar{b})\xi_-(u)\overline{\xi_-(v)},$$

which vanishes in the case:

$$det\begin{pmatrix} a & \bar{b} \\ c & \bar{d} \end{pmatrix} = -1 \tag{11}$$

$$a\bar{c} \in \mathcal{R},$$
$$b\bar{d} \in \mathcal{R}$$

This restriction defines a self-adjoint operator. It has branch of continuous spectrum $[0, +\infty)$. Corresponding eigenfunctions are solutions of the equation:

$$\mathcal{A}\Psi = \lambda\Psi$$

in the distributional sense. The external component is:

$$\Psi_{ex} = e^{-\imath k\langle x,\nu\rangle} + f(k)\frac{e^{\imath k|x|}}{4\pi|x|}, k^2 = \lambda \tag{12}$$

with the scattering amplitude, calculated by excluding the internal component:

$$f(k) = \frac{\mathcal{D}}{1 - \frac{\imath k}{4\pi}\mathcal{D}} \tag{13}$$

where $\mathcal{D}(k)$ is a meromorphic function with the positive imaginary part in the upper half-plane due to the conditions (11):

$$\mathcal{D}(k) = -\frac{c + a\mathbf{R}(k)}{d + b\mathbf{R}(k)}, \tag{14}$$

$$\mathbf{R}(k) \equiv \langle (\mathcal{L}_{in} - k)^{-1}\theta, \theta \rangle$$

We shall restrict our consideration by the case of absence of bound states. It means that unperturbed internal operator has not negative eigenvalues.

Our goal is a description of decay operator for constructed Schroedinger evolution:

$$\begin{aligned} \frac{1}{\imath}\frac{\delta U}{\delta t} &= \mathcal{A}U \\ U &\in \mathcal{H}. \end{aligned} \tag{15}$$

Following chapter 2, we shall write it in the spectral representation of corresponding wave-equation :

$$\frac{\delta^2 U}{\delta t^2} = \mathcal{A}U, \tag{16}$$

which is to be used in Schroedinder form :

$$\frac{\delta}{\delta t}\begin{pmatrix} U_0 \\ U_1 \end{pmatrix} = \imath \begin{pmatrix} 0 & -1 \\ \mathcal{A} & 0 \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \end{pmatrix}. \tag{17}$$

The generator

$$\mathcal{Q} = \imath \begin{pmatrix} 0 & -1 \\ \mathcal{A} & 0 \end{pmatrix}$$

is a self-adjoint operator in the space:

$$E = (W_2^1(\mathcal{R}^3) \oplus H_{in}) \oplus (L_2(\mathcal{R}^3) \oplus H_{in}),$$

$$\mathcal{U} = \begin{pmatrix} U_0 \\ U_1 \end{pmatrix} = \begin{pmatrix} u_{0,ex} \\ u_{0,in} \\ u_{1,ex} \\ u_{1,in} \end{pmatrix}$$

with the energy norm:

$$|\mathcal{U}|_E^2 = \frac{1}{2}\Big(\int_{\mathcal{R}^3} (\mid \nabla u_{0,ex} \mid^2 + \mid u_{1,ex} \mid^2) dx + |\xi_-(u_{0,ex})|^2 +$$

$$+|\sqrt{\mathcal{L}_{in}}u_{0,in}|^2 + |u_{1,in}|^2\Big) .$$

Note, that the energy norm can be written in this form only for smooth data with now singularities at the origin. But if we write it for all data from $Dom(\sqrt{\mathcal{A}}) \oplus \mathcal{H}$ the component $u_{0,ex}$:

$$u_{0,ex} = \frac{a}{4\pi|x|} + \tilde{u_0}, \tilde{u_{0,ex}} < \infty$$

has to be renormalized by subtracting singularity part $\frac{a}{|x|}$, which does not give any contribution into the Dirichlet integral, since this singular part is a harmonic function. Thus we can write the energy norm in the form ( ) for all data with finite energy, having in mind the described renormalization.

The generator $\mathcal{Q}$ determines an unitary evolution. Corresponding incoming and outgoing subspaces can be constructed in accordance to the standard procedure [1]. Because of the fact that we are able to construct translation representation for the unperturbed evolution in the external space which is literally both incoming and outgoing, it follows that incoming and outgoing spaces together span the set $\mathcal{H}_0$ of all initial data for the evolution in the external space with the boundary condition

$$\xi_-(u_{0,ex}) = 0.$$

So we can define an isometrical imbedding of $\mathcal{H}_0$ into $\mathcal{H}$. Then $\mathcal{H}$ can be split into the direct sum of $\mathcal{H}_0$ and some finite-dimensional space $\mathcal{K}$. All the data from $\mathcal{K}$ are of the following type:

$$\mathcal{U} = \begin{pmatrix} \frac{c_0}{|x|} \\ u_{0,in} \\ 0 \\ u_{1,in} \end{pmatrix}$$

The energy-orthogonal projection on $\mathcal{K}$ is equivalent to the projection on the internal spaces $\mathcal{H}_{in}$ and calculation of the singularity of the zero external component at the origin. Also we can consider evolution in $\mathcal{K}$ as an evolution in the space $e \oplus H_{in} \oplus H_{in}$ where e marked the one-dimensional space .

In order to calculate the compressed evolution let us solve the Caushy problem with data from $\mathcal{K}$:

$$\begin{array}{l} u_{0,ex}(x) = \frac{c_0}{|x|} \\ u_{1,ex}(x) = 0 \\ u_{0,in} \\ u_{1,in}. \end{array}$$

Note that in the region $|x| > t$ the solution does not change with the time:

$$u_{ex} = \left(\begin{array}{c} \frac{c_0}{|x|} \\ 0 \end{array} \right).$$

Using the standard anzats for incoming and outgoing solutions in the region $|x| < t$:

$$u_{0,1} = \frac{1}{4\pi|x|} \left(a_{0,1}(t+|x|) + b_{0,1}(t-|x|)\right)$$

with an arbitrary functions $a_{0,1}$ and $b_{0,1}$ we can exclude the external channel. On the surface $|x| = t$ we have the boundary condition : $u_{0,1} = const$. It means that $a(s)$ are constant for $s > 0$ and thus may be adsorbed into $b(s)$. From the evolution equation we receive immediately the connection between b_0 and b_1:

$$b_1(s) = b_0'(s).$$

Then the boundary values of the external component are :

$$\xi_-(u_{0,ex})(t) = b_0(t)$$

$$\xi_0(u_{0,ex})(t) = -\frac{b_0'(t)}{4\pi}$$

$$\xi_-(u_{1,ex})(t) = b_1(t) = b_0'(t)$$

$$\xi_0(u_{1,ex})(t) = -\frac{b_1'(t)}{4\pi} = -\frac{b_0''(t)}{4\pi}$$

By excluding the external component (function $b_0(t)$)we receive the following connections between the boundary values:

$$\xi_0(u_{0,ex}) = -\frac{\xi_-'(u_{0,ex})}{4\pi}$$

$$\xi_-(u_{1,ex}) = \xi_-'(u_{0,ex})$$

$$\xi_0(u_{1,ex}) = -\frac{\xi''_-(u_{0,ex})}{4\pi}$$

Now the compressed evolution equation in the space $\mathcal{K}$ can be calculated:

$$\begin{pmatrix} \frac{1}{\imath}\frac{4\pi d}{c} & \imath\frac{4\pi}{c} < *, \theta > & 0 \\ 0 & 0 & -\imath \\ \imath\left(-\frac{ad}{c} + b\right)\theta & \imath\mathcal{L}_{in} + \imath\frac{a}{c}\mathbf{P}_\theta & 0 \end{pmatrix} \begin{pmatrix} s \\ u_{0,in} \\ u_{1,in} \end{pmatrix} = \frac{1}{\imath}\frac{\delta}{\delta t} \begin{pmatrix} s \\ u_{0,in} \\ u_{1,in} \end{pmatrix}. \qquad (18)$$

On the contrary to the unperturbed evolution the new generator $\mathcal{B}$:

$$\mathcal{B} = \begin{pmatrix} -\imath\frac{4\pi d}{c} & \imath\frac{4\pi}{c} < *, \theta > & 0 \\ 0 & 0 & -\imath \\ \imath(-\frac{ad}{c} + b)\theta & \imath\mathcal{L}_{in} + \imath\frac{a}{c}\mathbf{P}_\theta & 0 \end{pmatrix} \qquad (19)$$

is a dissipative operator in the space K with the norm:

$$|\mathcal{U}|^2_{\mathcal{K}} = \frac{1}{2}\left(|s|^2 + < \mathcal{L}_{in}u_0, u_0 >_{H_{in}} + < u_1, u_1 >_{H_{in}}\right).$$

The eigenvalues k_j coincide with the zeroes of the S-matrix i.e. they are situated at the points, symmetric to the poles p_j:

$$k_j = -p_j.$$

The corresponding eigenfunctions are:

$$\Psi_j = \begin{pmatrix} 1 \\ -\left(\frac{\imath p_j}{4\pi}a + b\right)(\mathcal{L}_{in} - p_j^2)^{-1}\theta \\ \imath p_j\left(\frac{\imath p_j}{4\pi}a + b\right)(\mathcal{L}_{in} - p_j^2)^{-1}\theta \end{pmatrix}.$$

The adjoint generator

$$\mathcal{B}^* = \begin{pmatrix} \imath\frac{4\pi\bar{d}}{\bar{c}} & 0 & -\imath\left(-\frac{\bar{a}\bar{d}}{\bar{c}} + \bar{b}\right) < *, \theta > \\ -\imath\frac{4\pi}{\bar{c}}\mathcal{L}_{in}^{-1}\theta & 0 & -\imath\left(1 + \frac{\bar{a}}{\bar{c}}(\mathcal{L}_{in}^{-1}\theta) < *, \theta >\right) \\ 0 & \imath\mathcal{L}_{in} & 0 \end{pmatrix}$$

has eigenvalues α_j at the conjugated points:

$$\alpha_j = -\bar{p}_j = \bar{k}_j$$

with the eigenfunctions:

$$\Phi_j = \left(\begin{array}{c} 1 \\ -\imath \bar{p}_j \frac{4\pi}{\bar{a}\mathbf{R}(\bar{p}_j)+\bar{c}} \mathcal{L}_{in}^{-1} (\mathcal{L} - \bar{p}^2)^{-1}\theta \\ -\frac{4\pi}{\bar{a}\mathbf{R}(\bar{p}_j)+\bar{c}} (\mathcal{L}_{in} - \bar{p}_j^2)^{-1}\theta \end{array} \right).$$

The eigenfunctions of these two operators form a biorthogonal system in K:

$$< \Psi_i, \Phi_j >_{\mathcal{K}} =$$

$$= \left(\frac{1}{2} + \frac{\imath p_j (\imath p_j c + 4\pi d)}{\mathbf{R}(p_j)(a\mathbf{R}(p_j) + c)} < (\mathcal{L}_{in} - p_j^2)^{-2}\theta, \theta >_{H_{in}} \right) \delta_{ij} \equiv m(p_j).$$

The exponential-decay modes for compressed evolution are decreasing functions:

$$e^{-\imath p_j t} \Psi_j.$$

So the reduced evolution can be written in the form:

$$\mathcal{Z}(t) = \sum e^{-\imath p_j t} m^{-1}(p_j) < *, \Phi_j > \Psi_j. \tag{20}$$

Now let us return back to the Schroedinger evolution. It can be written in the same space as wave-equation evolution, using the connection between $\mathcal{A}$ and $\mathcal{Q}$:

$$\left(\begin{array}{cc} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{array} \right) = \mathcal{Q}^2. \tag{21}$$

The Schroedinger equation

$$\frac{1}{\imath} \frac{\delta}{\delta t} U = \mathcal{Q}^2 U \tag{22}$$

decomposes into the two systems of nonconnected equivalent equations for components $u_{0,ex}, u_{0,in}$ and $u_{1,ex}, u_{1,in}$. But we are interesting in the evolution of the second pair of components $u_{1,ex}, u_{1,in}$ only. There exist a natural isometric imbedding of the space $\mathcal{H}_1$ of all initial data with the component U_0 equal to zero into the space of all Caushy data. Really the Schroedinger evolution in $\mathcal{K}$ is evolution (22), restricted on the space $\mathcal{H}_1$. Projections on $\mathcal{K}$ and $\mathcal{H}_1$ commute. So the Schroedinger evolution in $\mathcal{K}$ is the evolution, restricted on the internal component $u_{1,in}$:

$$\mathbf{P}_1 \mathcal{V}^2(t) \mathbf{P}_1 =$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \left| \frac{bc - ad}{d + b\mathbf{R}(k) + \frac{\imath k}{4\pi}(c + a\mathbf{R}(k))} \right|^2 \tag{23}$$

$$e^{\imath k^2 t} < *, (\mathcal{L}_{in} - k^2)^{-1}\theta > (\mathcal{L}_{in} - k^2)^{-1}\theta k^2 dk.$$

Here $\mathbf{P}_1$ is projector on $u_{1,in}$. The Schroedinger evolution has not incoming and outgoing subspaces. So this evolution is not semigroup evolution. To calculate the asymptotic behaviour of this integral one can use the steepest descent line method. The steepest descent line for this integral is the bisector of the 1-3 quadrants. The asymptotics has exponential and nonexponential terms:

$$\mathbf{P}_1 \mathcal{V}(t) \mathbf{P}_1 =$$

$$= \frac{1}{\imath} \frac{\partial}{\partial t} \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \left| \frac{bc - ad}{d + b\mathbf{R}(k) + \frac{\imath k}{4\pi}(c + a\mathbf{R}(k))} \right|^2$$

$$e^{\imath k^2 t} < *, (\mathcal{L}_{in} - k^2)^{-1}\theta > (\mathcal{L}_{in} - k^2)^{-1}\theta dk \sim$$

$$\sim \frac{e^{\imath 3\pi/4}}{8(\pi t)^{3/2}} \left| \frac{bc - ad}{d + b\mathbf{R}(0)} \right|^2 < *, \mathcal{L}_{in}^{-1}\theta > \mathcal{L}_{in}^{-1}\theta -$$

$$- \sum_{\ell_+} \frac{\imath e^{\imath k^2 t} |bc - ad|^2 k^2}{2\pi (d + b\mathbf{R} + \frac{\imath k}{4\pi}(c + a\mathbf{R}) \frac{\partial}{\partial k}(\bar{d} + \bar{b}\mathbf{R} - \frac{\imath k}{4\pi}(\bar{c} + \bar{a}\mathbf{R})} \Bigg|_{k = \bar{p}_\ell} +$$

$$+ \sum_{\ell_-} \frac{\imath e^{\imath k^2 t} |bc - ad|^2 k^2}{2\pi \frac{\partial}{\partial k}(d + b\mathbf{R} + \frac{\imath k}{4\pi}(c + a\mathbf{R})(\bar{d} + \bar{b}\mathbf{R} - \frac{\imath k}{4\pi}(\bar{c} + \bar{a}\mathbf{R})} \Bigg|_{k = p_\ell} \tag{24}$$

Nonexponential term is small for initial data orthogonal to the element $\mathcal{L}_{in}^{-1}\theta$. Exponential terms are formed by the "physical" poles of the S-matrix. The exponential evolution is semigroup evolution.

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