On the Inverse Scattering Problem for Rational Reflection Coefficients

P. B. Kurasov

Department of Mathematical and Computational Physics, Institute for Physics, St. Petersburg University, St.Petersburg, 198904 Russia and Manne Siegbahn Institute, Freskativägen 24, 10405 Stockholm, Sweden

Abstract *The inverse scattering problem on the half-axis for long range potentials is studied. It is shown that the solution of the inverse problem contains arbitrary real parameters iven if no bound states are present. Connections with the inverse problem on the whole axis are discussed.*

1 Introduction

The present paper is devoted to the inverse scattering problem on the half line [1],[2]. New exact solutions of this inverse problem are constructed. Such exact solutions play an important role in the theory of nonlinear equations. Nonuniqueness of the solution of the inverse problem in the presence of bound states allows to construct soliton solutions of nonlinear equations. Several examples of the ambiguity potentials without any bound states for the case of the whole line scattering [3] were discovered during the last years [4, 5, 6, 7, 8, 9]. The most general description of such potentials is given in [2]. This class of ambiguities is connected with a slow decrease of the potentials at infinity (like const./x^2). But such potentials do not produce any ambiguities for half line scattering. In order to obtain ambiguities in that case the class of admissible potentials must be extended. In the present paper we shall investigate the solution of the inverse problem on the half line for potentials with the following behaviour at infinity

$$V(x) \sim \frac{\sum_{j=1}^{M} A_j \sin k_j x}{x}.$$

The corresponding scattering matrix does not satisfy the Levinson condition. Similar scattering matrices were obtained for delta-functional potentials and selfadjoint perturbations constructed with the help of the extension-restriction procedure [10, 11, 12]. We shall restrict our consideration to the case of rational reflection coefficients only in order to avoid additional unessential difficulties.

The direct scattering problem for oscillating potentials is discussed in the second part of the paper. The third part is devoted to the solution of the inverse problem for rational reflection coefficients corresponding to the short range potentials. We recall simple analytical formulas for the solution, obtained from the Gelfand-Levitan-Marchenko equation. In the fourth part we discuss the solution of the inverse problem for the rational reflection coefficients violating the Levinson theorem. It is shown that the resulting oscillating potential is not defined uniquely by the reflection coefficient even if no bound state is present. The family of ambiguity potentials contains a finite number of real parameters.

2 The direct scattering problem

We shall discuss the scattering problem for the Schroedinger operator

$$\mathcal{A} = -\frac{d^2}{dx^2} + V(x) \tag{1}$$

on the half axis $[0, \infty)$ with Dirichlet boundary condition at the zero point: $Dom(\mathcal{A}) = \{u \in W_2^2(\mathbb{R}_+), u(0) = 0\}$. The unperturbed operator is

$$\mathcal{A}_0 = -\frac{d^2}{dx^2}.$$

Usually the scattering problem is investigated in the case of real potentials satisfying the condition

$$\int_0^\infty |V(x)| dx < \infty. \tag{2}$$

Under this condition the Jost solutions exist for all real nonzero values of the spectral parameter k and the asymptotics of the regular solution

$$\varphi(k, x) : -\frac{d^2}{dx^2}\varphi(k, x) + V(x)\varphi(k, x) = k^2\varphi(k, x), \varphi(k, 0) = 0$$

for large x is a combination of plane waves

$$\varphi(k, x) \sim A_+(k)e^{ikx} + A_-(k)e^{-ikx} + o(\frac{1}{\sqrt{x}}), \quad k \in [0, \infty). \tag{3}$$

The scattering operator coincides with the reflection coefficient $S(k) = -A_-(k)/A_+(k)$. Under the condition of the finitness of the first moment:

$$\int_0^\infty x|V(x)| dx < \infty. \tag{4}$$

the Jost solution $f(k, x), -f''(k, x) + V(x)f(k, x) = k^2 f(k, x), f(k, x) = e^{ikx} + o(1/\sqrt{x}), x \to \infty$ and the Jost function $F(k) = f(k, 0)$ are analytic functions of the spectral parameter k in the upper half plane, continuous up to the real axis. The scattering matrix $S(k)$ coincides with the ratio of two Jost functions $S(k) = F(-k)/F(k)$.

The Jost function for potentials violating the condition (4) can have singularities on the real axis. This phenomenon was studied for the first time in connection with the zero energy bound state, when potential decreases at infinity like $1/|x|^2$ and the Jost function has singularity at point zero. Condition (2) is sufficient but not necessary condition for the wave operators to be complete.

The usual scattering operator exists also in the case when the Jost solution is well defined for almost all k on the real axis. An additional condition on the difference between the regular and free solutions at infinity, $x \to \infty$ is necessary [13] . Hence the class of admissible potentials can be extended. For example, the wave operators are complete for potentials with the following asymptotic behaviour at infinity

$$V(x) = \frac{\sum_{j=1}^{M} A_j \sin(k_j x)}{x} + V_0(x); \tag{5}$$

$$|V_0(x)| \leq C(1+x^2)^{-1/2-\epsilon}$$

for some $C, \epsilon > 1/4$.This class of potentials was investigated in connection with the phenomenon of bound states imbedded into the continuous spectrum [14, 15, 16] . The Jost solution for a potential from the class (5) exists for $k \neq 0, k \neq k_j/2$. The reflection coefficient is defined for almost every real k.

3 Inverse scattering problem for short-range potentials

The inverse problem is the problem to restore the potential $V(x)$ from the known reflection coefficient. For the class of potentials defined by the conditions (2, 4) this problem can be solved by Gelfand-Levitan-Marchenko procedure [1, 2].

An important class of analytically solvable inverse problems is formed by the rational Jost functions. All rational Jost functions can be presented in the following form

$$F(k) = \prod_{j=1}^{M} \frac{k + \imath a_j}{k + \imath b_j} \tag{6}$$

with $\Re b_j > 0$. The sets $\{a_j\}, \{b_j\}$ are symmetric over the real axis. The constants a_j with negative real part can be only real and correspond to bound states. Such Jost functions define scattering matrices of the form

$$S(k) = \prod_{j=1}^{M} \frac{k - \imath a_j}{k - \imath b_j} \frac{k + \imath b_j}{k + \imath a_j} \tag{7}$$

corresponding to potential exponentially decreasing at infinity. Simple analytical formulas for the potential and regular solution can be obtained by solution of the Gelfand-Levitan-Marchenko equation [1, 2]

$$V(x) = -2\frac{d^2}{dx^2} \ln \det W(x),$$

$$\varphi(k,x) = \frac{\det \begin{vmatrix} W(x) & f(x) \\ \beta(k,x) & \dfrac{\sin kx}{k} \end{vmatrix}}{\det W(x)}, \tag{8}$$

where

$$W_{nm}(x) = \frac{e^{-a_n x}}{2b_m}\left(\frac{e^{b_m x}}{a_n - b_m} - \frac{e^{-b_m x}}{a_n + b_m}\right),$$

$$f_n(x) = e^{-a_n x}$$

$$\beta_n(k,x) = \int_0^x \frac{\sin \imath b_n t}{\imath b_n}\frac{\sin kt}{k}dt.$$

4 Inverse problem for oscillating potentials

The scattering matrix for potentials satisfying condition (5) is a generalized function. We shall consider the case when this function is equivalent to some rational function of the form

$$S(k) = \prod_{j=1}^{M} \frac{k - \imath a_j}{k + \imath a_j} \tag{9}$$

$\Re a_j > 0$. As the phase shift on the real axis

$$\delta(\infty) - \delta(-\infty) = \frac{1}{2\imath}\int_{-\infty}^{+\infty} \frac{S'(k)}{S(k)}dk = \pi M$$

is positive the Levinson theorem can not be fulfilled for such reflection coefficients. Then the inverse problem does not have a solution in the class of short range potentials even if we assume that some bound states are present.

The Gelfand-Levitan method is based on the representation of the scattering matrix as a ratio of the Jost functions analytical in the upper half plane $\Pi_+ = \{k, \Im k > 0\}$ continuous in the closed upper half plane $\bar{\Pi}_+ = \{k, \Im k \geq 0\}$ with the unit limit at infinity. Similar representation can be introduced in the investigated case also

$$S(k) = \frac{F_0(-k)}{F_0(k)},\ F_0(k) = \prod_{j=1}^{M} \frac{k + \imath a_j}{k + \imath b_j},\ \Re b_j = 0. \tag{10}$$

The formally introduced Jost function $F_0(k)$ has singularities on the real axis. The singularities $k = -\imath b_j$ are situated symmetrically with respect to the origin and do not give any contribution to the scattering matrix. The Jost function $F_0(k)$ contains $[M/2]$ arbitrary real parameters. To solve the inverse problem for this class of the reflection coefficients the limit of the Gelfand-Levitan procedure can be used. In the first step we approximate the Jost functions $F_0(k)$ by the functions with singularities at the lower half plane, which are Jost functions for some short-range potentials

$$F_\epsilon = \prod_{j=1}^{M} \frac{k + \imath a_j}{k + \imath(b_j + \epsilon)}, \epsilon > 0. \tag{11}$$

The potential $V_\epsilon(x)$ and the regular solution $\varphi_\epsilon(x)$ corresponding to the Jost function $F_\epsilon(k)$ can be calculated with the help of the formulas (8). Then the pointwise limits of the potential and regular solution when $\epsilon \to 0$ are considered. The potentials V_ϵ exponentially decrease at infinity, but the limit potential V_0 is from the class (5).

The following theorem was formulated in [12] as a conjecture and it was proved there for $M = 1, 2$ only.

Theorem 1. *The limit of the regular solution $\varphi_0(k,x) = \lim_{\epsilon\to 0}\varphi_\epsilon(k,x)$, $x \in [0,\infty)$ is a regular solution for the Schroedinger equation for every $k \in [0,\infty), k \neq 0, ib_j$ with the limit potential $V_0(x) = \lim_{\epsilon\to 0} V_\epsilon(x), x \in [0,\infty)$. The reflection coefficient for the limit potential is given by the formula*

$$S(k) = \prod_{j=1}^{M} \frac{k - ia_j}{k + ia_j}.$$

Proof. The Theorem 1 can be proved by an iteration procedure because every scattering matrix $S(k)$ with the symmetrically situated zeroes and poles can be presented as a product of elementary unimodular functions containing not more than two factors.

The formulas (8) can be generalized for the case when the background operator has the form

$$\mathcal{A}_0 = -\frac{d^2}{dx^2} + V_0(x) \tag{12}$$

with the potential V_0 from the class (5). The generalized formulas allow to calculate the potential $V(x)$ and the regular solution corresponding to the reflection coefficient $S(k) = \prod_{j=1}^{M}(k - ia_j)/(k + ia_j)S_0(k)$, where $S_0(k)$ is the reflection coefficient for the potential $V_0(x)$. The standard Gelfand-Levitan-Marchenko procedure can not be used directly and one needs to consider the approximation procedure similar to one discussed earlier. To prove the Theorem 1 one needs to consider this approximation procedure for the case $M = 1, 2$ only.

Lemma 2. *Let the background operator be of the form (12) and let the logarithmic derivative of the corresponding Jost function be bounded at the point $t = 0$. Then for $M = 1$ the limit of the regular solution $\varphi_{\epsilon=0}(k,x)$ is a regular solution for the Schroedinger equation for every $k \in [0,\infty), k \neq 0, ib_j$ with the limit potential $V_{\epsilon=0}(x)$. The reflection coefficient for the limit potential is given by the formula*

$$S(k) = \frac{k - ia_1}{k + ia_1} S_0(k).$$

Proof. The Jost function and approximate Jost functions are introduced as follows:

$$F(k) = \frac{k + ia_1}{k} F_0(k) \Rightarrow F_\epsilon(k) = \frac{k + ia_1}{k + i\epsilon} F_0(k) \tag{13}$$

The potential and the regular solution, corresponding to the approximate Jost function are [1, 2]:

$$V_\epsilon(x) - V_0(x) =$$
$$= -2\frac{(\epsilon^2 - a^2)(f^2(\imath a_1)\varphi'^2_x(\imath\epsilon) - f'^2_x(\imath a_1)\varphi^2(\imath\epsilon)) - (\epsilon^2 - a^2)^2 f^2(\imath a_1)\varphi^2(\imath\epsilon)}{(f(\imath a_1)\varphi'_x(\imath\epsilon) - f'_x(\imath a_1)\varphi(\imath\epsilon))^2} \tag{14}$$

$$\varphi_\epsilon(k,x) = \varphi(k) + \frac{a_1^2 - \epsilon^2}{k^2 + \epsilon^2}\frac{\mathbf{W}[\varphi(\imath\epsilon), \varphi(k)]}{\mathbf{W}[\varphi(\imath\epsilon), f(\imath a_1)]} f(\imath a_1) \tag{15}$$

where the following notations for the Jost and regular solutions corresponding to the nonperturbed operator $\mathcal{A}_0$ were used $f(k) \equiv f_0(k,x), \varphi(k) \equiv \varphi_0(k,x)$. The limits of the potential and regular solution for $\epsilon \longrightarrow 0$ are

$$V_{\epsilon=0}(x) - V_0(x) = 2a^2 \times \tag{16}$$

$$\frac{\left(\frac{F_0'(0)}{F_0(0)} f_x'(0) - f''_{xk}(0)\right)^2 f^2(\imath a_1) + \left(\frac{F_0'(0)}{F_0(0)} f(0) - f'_k(0)\right)^2 \left(a^2 f^2(\imath a_1) - f'^2_x(\imath a_1)\right)}{\left(f(\imath a_1)(f''_{xk}(0) - \frac{F_0'(0)}{F_0(0)} f_x'(0)) + f'_x(\imath a_1)\left(\frac{F_0'(0)}{F_0(0)} f(0) - f'_k(0)\right)\right)^2}$$

$$\varphi_{\epsilon=0}(k,x) = \varphi(k) + \frac{a_1^2}{k^2} f(\imath a_1) \times \tag{17}$$

$$\frac{\frac{F_0'((0)}{F_0(0)}\mathbf{W}[f(0), F_0(k)f(-k) - F_0(-k)f(k)] - \mathbf{W}[f_k'(0), F_0(k)f(-k) - F_0(-k)f(k)]}{\left(\frac{F_0'(0)}{F_0(0)}\mathbf{W}[f(0), f(\imath a_1)] - \mathbf{W}[f_k'(0), f(\imath a_1)]\right)(F_0(k)f_{0x}'(-k,0) - F_0(-k)f_{0x}'(k,0))}$$

By direct calculation one can prove that the limit of the regular solution is a regular solution for the limit potential. The asymptotics of the potential for large x is

$$V_{\epsilon=0}(x) - V_0(x) \sim \frac{2a^2}{\left(1 + a(\imath\frac{F_0'(0)}{F_0(0)} + x)\right)^2} \sim \frac{2}{x^2}$$

We note that the Jost function $F_0(k)$ has the following property: $F_0(-\bar{k}) = \bar{F}_0(k)$. Hence the logarithmic derivative of the Jost function at point zero is purely imaginary $\imath F_0'(0)/F_0(0) \in \mathbb{R}$ and the calculated potential is real. The asymptotics of the regular solution for $x \longrightarrow \infty$ is

$$\varphi_{\epsilon=0}(k,x) \sim \frac{1}{F_0(k)f_{0x}'(-k,0) - F_0(-k)f_{0x}'(k,0)}\frac{1}{k}$$
$$\left\{e^{-\imath kx}(k + \imath a_1)F_0(k) - e^{\imath kx}(k - \imath a_1)F_0(-k)\right\} \tag{18}$$

and the reflection coefficient is

$$S(k) = \frac{k - \imath a_1}{k + \imath a_1} S_0(k), \; k \neq 0, \pm\imath b_j \tag{19}$$

The Jost function and the Jost solution have singularities on the real axis at the points $k = \pm\imath b_j, 0$. The case of the multiple singularities can be studied separately. This complets the proof of Lemma 1 . □

We note that the condition of the finitness of the logarithmic derivative of the Jost function at the point zero is fulfilled automatically if the potential V_0 was constructed by the iteration procedure.

Lemma 3. *Let the background operator be of the form (12). Then for $M = 2$ the limit of the regular solution $\varphi_{\epsilon=0}(k,x)$ is a regular solution for the Schroedinger equation for every $k \in [0,\infty), k \neq 0, \imath b_j$ with the limit potential $V_{\epsilon=0}(x)$. The reflection coefficient for the limit potential is given by the formula*

$$S(k) = \frac{k - \imath a_1}{k + \imath a_1}\frac{k - \imath a_2}{k + \imath a_2}S_0(k).$$

Proof. To avoid complicated formulas we shall discuss here the proof of this Lemma for the case of $V_0 = 0$ only. The original scattering matrix is given by the expression

$$S(k) = \frac{k - \imath a_1}{k + \imath a_1}\frac{k - \imath a_2}{k + \imath a_2}, \quad a_1 = \bar{a}_2 \;\; or \;\; a_1, a_2 \in \mathbb{R}.$$

There is arbitrariness in the definition of the Jost function in this case. The family of the possible Jost functions depends on the real parameter b_0

$$F_{b_0}(k) = \frac{k + \imath a_1}{k - b_0}\frac{k + \imath a_2}{k + b_0} \tag{20}$$

Approximate Jost functions are introduced as follows

$$F_{b_0,\epsilon}(k) = \frac{k + \imath a_1}{k - b_0 + \imath\epsilon}\frac{k + \imath a_2}{k + b_0 + \imath\epsilon}, \epsilon > 0 \tag{21}$$

The limit potential when $\epsilon \to 0$ is

$$V_{a_1,a_2,b_0}(x) = 16b_0^2\frac{1 - (b_0x + B)\sin 2(b_0x + \delta(b_0)) - \cos 2(b_0x + \delta(b_0))}{(2b_0x + 2B - \sin 2(b_0x + \delta(b_0)))^2} \tag{22}$$

where we used the following notations

$$B = B(b_0, a_1, a_2) = \frac{b_0(a_1 + a_2)(a_1a_2 + b_0^2)}{(a_1^2 + b_0^2)(a_2^2 + b_0^2)}$$

$$e^{2\imath\delta(b_0)} = S(b_0) \tag{23}$$

The corresponding regular solution is

$$\varphi(k,x) = \frac{\sin kx}{k} + \tag{24}$$

$$+\frac{1}{2\imath k}\frac{e^{\imath kx}\left(-w(b_0,-\imath k,a_2) + w(b_0,-\imath k,a_1)\right) + e^{-\imath kx}\left(w(b_0,\imath k,a_2) - w(b_0,\imath k,a_1)\right)}{w(b_0,a_1,a_2)}$$

where the function w is
$w(b_0, a_1, a_2) =$

$$= \frac{a_2 - a_1}{(a_1^2 - b^2)(a_2^2 - b^2)}\left(2b_0x + 2B(b_0,a_1,a_2) - \sin 2(b_0x + \delta(b_0,a_1,a_2))\right) \tag{25}$$

The asymptotics of the solution for $x \to \infty$ is:

$$\varphi(k,x) \sim \frac{e^{ikx}}{2ik}\frac{(k-ia_1)(k-ia_2)}{k^2-b_0^2} - \frac{e^{-ikx}}{2ik}\frac{(k+ia_1)(k+ia_2)}{k^2-b_0^2} \tag{26}$$

for $k \neq \pm b_0$ and the scattering matrix can be easily calculated:

$$S(k) = \frac{k-ia_1}{k+ia_1}\frac{k-ia_2}{k+ia_2} \tag{27}$$

The calculated Jost function and Jost solution have singularities at the points $k = \pm b_0$. This proof can be generalized for the case $V_0 \neq 0$. Then the additional condition $b_0 \neq ib_j$ is necessary. We have now finished the proof of Lemma 2 and Theorem 1. □

The proof of Theorem 1 shows, that in the case $M = 1$ the unique potential was calculated. Arbitrariness of the solution of the inverse problem in the second case is connected with the arbitrariness of the definition of the Jost function.

Acknowledgements The author wishes to thank professors C.Chadan, N.Elander, B.Pavlov, B.Karlson, R.Newton and J.Sjostrand for the interest to the work and fruitful discussions. He is grateful for professor L.Päivärinta for the invitation to a very stimulating conference at Saariselka. He is indebted to Manne Siegbahn Institute for Atomic Physics and Stockholm University for the financial support and hospitality.

References

1. Faddeev, L.D., Uspekhi Mat. Nauk **14**, 57 (1959); English translation: J.Math.Phys.**4** (1963) 73.
2. Chadan K., Sabatier P.C. "Inverse problems in quantum scattering theory", 2nd edition, Springer-Verlag, Berlin, 1989.
3. Deift, P., Trubowitz, E., Comm. Pure and Appl. Math. **XXXII**, 121 (1979).
4. Abraham, P.B., De Facio, B., Moses, H.E., Phys. Rev. Let. **46**, 1657 (1981).
5. Brownstein, K.R., Phys. Rev. D **25**, 2704 (1982).
6. Moses, H.E., Phys. Rev. A **27**, 2220 (1983).
7. Aktosun, T., Newton, R.G., Inverse Problems **1**, 291 (1985).
8. Degasperis, A., Sabatier, P.C., Inverse Problems **3**, 73 (1987).
9. Aktosun, T., Inverse Problems **4**, 347 (1988).
10. Albeverio S., Gesztesy F., Hoegh-Krohn R., Holden H. "Solvable models in quantum mechanics", Springer-Verlag, Berlin, 1988.
11. Pavlov, B.S., Russian Math. Surveys **42:6**, 127 (1987).
12. Kurasov P.B., Let.Math.Phys. **25**, 287 (1992).
13. Reed M., Simon B. "Methods of Modern Mathematical Physics" III "Scattering Theory", Academic Press, New York, 1979, p.155-168.
14. Matveev V.B., Skriganov M.M. Dokl. Acad. Nauk SSSR **202** (1972), p.775-758.
15. Skriganov M.M. Zap. Nauch. Sem. LOMI **38** (1973), p.149.
16. Simon B. Comm. Pure Applied Math., vol **XXII**, 531-538 (1967).