ON APPROXIMATIONS OF RANK ONE $\mathcal{H}_{-2}$-PERTURBATIONS

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ABSTRACT. Approximations of rank one $\mathcal{H}_{-2}$-perturbations of self-adjoint operators by operators with regular rank one perturbations are discussed. It is proven that in the case of arbitrary not semibounded operators such approximations in the norm resolvent sense can be constructed without any renormalization of the coupling constant. Approximations of semibounded operators are constructed using rank one non-symmetric regular perturbations.

1. INTRODUCTION

The theory of finite rank perturbations of self-adjoint operators attracted much attention of physicists and mathematicians, in particular it leads to exactly solvable models [1, 4, 5, 17, 18, 19, 20, 21, 22, 29]. This theory has been presented in several review papers and summarized in research monographs [3, 9, 14, 22, 31, 32, 35]. The aim of the current paper is to clarify a few additional properties of such perturbations, namely the possibility to construct approximations by operators with regular perturbations. Such approximations can be constructed in many different ways, but approximations in the norm resolvent sense attract special attention, since they can give much information concerning the spectrum of the operator to be approximated. It has been discovered already in 1961 by F.Berezin and L.Faddeev [12] that such approximations of highly singular (of order $\mathcal{H}_{-2}$) perturbations of positive operators cannot be constructed without any renormalization of the coupling constant. To construct approximations the authors used ideas from field theory, where renormalization became a standard tool. It has been shown in [7] that approximations of rank one $\mathcal{H}_{-2}$-perturbations of not semibounded operators can be constructed if a certain additional condition on the perturbation is satisfied. This approximation in the strong resolvent sense was constructed without any use of the renormalization technique. In the current paper we show that if the original operator is not semibounded, then any rank one $\mathcal{H}_{-2}$-perturbation can be approximated in the norm resolvent sense. Approximations of semibounded operators by non-self-adjoint rank one regular perturbations are discussed as well.

The paper is organized as follows. In Section 2 we give an introduction into the theory of rank one singular perturbations of self-adjoint operators. The main results are presented in Section 3.

2. RANK ONE PERTURBATIONS AND THE EXTENSION THEORY

In this section we follow closely the presentation given in [9], where one can find all details and proofs. Rank one perturbations of a self-adjoint operator A acting in the Hilbert space $\mathcal{H}$ are formally given by the expression

$$(2.1) \qquad A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi,$$

where the coupling parameter α is real and the vector φ is from the scale of Hilbert spaces associated with the operator A

$$(2.2) \qquad \ldots \subset \mathcal{H}_3 \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \mathcal{H}_{-3} \subset \ldots.$$

The norm in each space $\mathcal{H}_s$ is given by $\| u \|_s^2 = \langle u, (|A| + 1)^s u \rangle$. The spaces $\mathcal{H}_2$ and $\mathcal{H}_1$ coincide with the domains of the operator and its quadratic form respectively. See [9, 35], where the geometry of this scale of Hilbert spaces is described in detail.

Rank one perturbations can be characterized by their **order**. If the vector φ belongs to the space $\mathcal{H}_{-s}(A)$, $s \geq 1$, and does not belong to $\mathcal{H}_{-s+1}(A)$, then we say that the perturbation has order $-s$. The vector φ has order zero if and only if $\varphi \in \mathcal{H}$. Let us consider perturbations of order $0, -1, -2, -3$, respectively.

Order 0. If $\varphi \in \mathcal{H}$, then the perturbation operator $\alpha \langle \varphi, \cdot \rangle \varphi$ is bounded and the operator A_α is self-adjoint on the domain of the operator A. The resolvent of the perturbed operator contains the complete information on its spectrum and is given by

$$(2.3) \qquad \frac{1}{A_\alpha - \lambda} = \frac{1}{A - \lambda} - \frac{1}{1/\alpha + \langle \varphi, \frac{1}{A-\lambda}\varphi \rangle} \left\langle \frac{1}{A - \bar{\lambda}}\varphi, \cdot \right\rangle \frac{1}{A - \lambda}\varphi.$$

Order -1. If $\varphi \in \mathcal{H}_{-1}(A) \setminus \mathcal{H}$, then the perturbation operator is infinitesimally form bounded with respect to the operator A and the perturbed operator is uniquely defined, but its domain is different from the domain of the original operator A. The operators A_α and A are two (different) self-adjoint extensions of the operator A_0 being the restriction of A to the domain

$$(2.4) \qquad \mathrm{Dom}\,(A_0) = \{\psi \in \mathrm{Dom}\,(A) : \langle \varphi, \psi \rangle = 0\}.$$

The operator A_0 has deficiency indices $(1, 1)$ and any function ψ from the domain of the adjoint operator A_0^* possesses the following representation:

$$\psi = \tilde{\psi} + b(\psi)\frac{A}{A^2 + 1}\varphi,$$

where $\tilde{\psi} \in \mathrm{Dom}\,(A)$ and $b(\psi)$ is a certain complex constant depending on the element ψ. Then all self-adjoint extensions of A_0 can be described as the restrictions of the adjoint operator with the domain

$$(2.5) \qquad \mathrm{Dom}\,(A_0^*) = \{\psi = \tilde{\psi} + b(\psi)\frac{A}{A^2 + 1}\varphi, \tilde{\psi} \in \mathrm{Dom}\,(A), b(\psi) \in \mathbf{C}\}$$

to the set of functions satisfying the boundary conditions

$$(2.6) \qquad \langle \varphi, \tilde{\psi} \rangle = \gamma b(\psi),$$

where $\gamma \in \mathbf{R} \cup \{\infty\}$ is the parameter describing the extension.[1] To establish the relations between the extension parameter γ and the coupling constant α one can simply apply the linear operator A_α to an arbitrary function from the domain of the adjoint operator A_0^*

$$(2.7) \quad \begin{aligned} A_\alpha \psi &= (A + \alpha \langle \varphi, \cdot \rangle \varphi) \left(\tilde{\psi} + b(\psi) \frac{A}{A^2+1} \varphi \right) \\ &= \left\{ A\tilde{\psi} - b(\psi) \frac{1}{A^2+1} \varphi \right\} + \left[\alpha \langle \varphi, \tilde{\psi} \rangle + b(\psi) + \alpha b(\psi) \langle \varphi, \frac{A}{A^2+1} \varphi \rangle \right] \varphi. \end{aligned}$$

The latter formula defines an operator in the Hilbert space $\mathcal{H}$ only if the expression $[\,\cdot\,]$ vanishes, i.e. if

$$(2.8) \qquad \langle \varphi, \tilde{\psi} \rangle = - \left(\frac{1}{\alpha} + \left\langle \varphi, \frac{A}{A^2+1} \varphi \right\rangle \right) b(\psi).$$

It follows that the parameters α and γ are related by

$$(2.9) \qquad \gamma = -\frac{1}{\alpha} - \left\langle \varphi, \frac{A}{A^2+1} \varphi \right\rangle = -\frac{1}{\alpha} - c,$$

where we introduce the parameter $c = \left\langle \varphi, \frac{A}{A^2+1} \varphi \right\rangle$, which plays a very important role for perturbations of order -2. Since the operators A_α and A are two extensions of one symmetric operator with deficiency indices $(1, 1)$, their resolvents are related by Krein's formula [2, 23, 24]

$$(2.10) \qquad \frac{1}{A_\alpha - \lambda} = \frac{1}{A - \lambda} - \frac{1}{1/\alpha + c + \langle \varphi, \frac{1+A\lambda}{A-\lambda} \frac{1}{A^2+1} \varphi \rangle}, \ \lambda \in \rho(A_\alpha) \cap \rho(A).$$

This formula describes the spectral properties of the operator A_α in terms of the coupling parameter α and Krein's Q-function $\langle \varphi, \frac{1+A\lambda}{A-\lambda} \frac{1}{A^2+1} \varphi \rangle$.

Order -2. If $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$, then the perturbed operator cannot be uniquely defined by the formal expression. This fact can be explained using different points of view. Here we follow the lines of extension theory. The restricted operator A_0 determined by the same formula (2.4) is a symmetric operator with deficiency indices $(1, 1)$, and the operator A_α should be one of its self-adjoint extensions characterized by boundary conditions of the form (2.6). But it is impossible without taking into account any additional assumptions to establish the relation between the extension parameter γ and the formal coupling constant α. The reason is that the functional φ is not a bounded linear functional on $\text{Dom}(A_0^*)$ and therefore calculation similar to (2.7) cannot be carried out. One of the possibilities to establish this relation is to extend the functional φ from $\mathcal{H}_2(A) = \text{Dom}(A)$ to $\text{Dom}(A_0^*)$ as a bounded linear functional. All such extensions φ^c are parameterized by one real parameter c and are determined by [6, 7, 9]

$$(2.11) \qquad \langle \varphi^c, \tilde{\psi} + b(\psi) \frac{A}{A^2+1} \varphi \rangle = \langle \varphi, \tilde{\psi} \rangle + cb(\psi).$$

The parameter c can be chosen arbitrary real $c \in \mathbf{R}$. Taking into account additional properties of the functional one can sometimes determine the parameter c in a unique way if one requires that the extended functional φ^c has the same properties as φ. In [6, 7, 9] this question was discussed in detail supposing that the functional

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[1] This approach to the extension theory for symmetric (not necessarily semibounded) operators is similar to Birman-Krein-Vishik theory applicable to semibounded operators only (see [11] for a recent review and references).

φ and the operator A are homogeneous with respect to a certain group of unitary transformations. In what follows we are always going to suppose that the real parameter c is fixed and therefore the perturbed operator is defined uniquely by the following expression:

$$(2.12) \qquad A_\alpha = A + \alpha \langle \varphi^c, \cdot \rangle \varphi.$$

Then the resolvent of the operator A_α is given by (2.10), since the regularized resolvent

$$Q(\lambda) = \langle \varphi, \frac{1 + A\lambda}{A - \lambda} \frac{1}{A^2 + 1} \varphi \rangle$$

is well defined for any $\varphi \in \mathcal{H}_{-2}(A)$. This function is a Nevanlinna function, i.e. it is analytic and has positive imaginary part in $\Im \lambda > 0$.

Order -3. If $\varphi \in \mathcal{H}_{-3}(A) \setminus \mathcal{H}_{-2}(A)$, then the restricted operator A_0 is densely defined but no self-adjoint operator in $\mathcal{H}$ (different from A) can be associated with the formal expression (2.1). One can construct an operator corresponding to this rank one perturbation using a certain extended Hilbert or Pontryagin space. The approach involving spaces with indefinite metrics was developed in [15, 16, 34, 33]. A realization of the operator A_α using Hilbert spaces is given in [27]. Generalizations of the latter approach to general perturbations of order $-n, n \geq 4$, leads to non-self-adjoint operators with real spectrum [26, 28]. Discussions of such high order perturbations are outside the scope of the present article.

In what follows we are going to study rank one perturbations of order -1 and -2 only. Such perturbations are called singular, since the domain of the operator is not invariant under them. The perturbations which are determined by zero order vectors are called regular. The characteristic property of such perturbations is that the domain of the perturbed and original operators coincide. In particular approximations of singular perturbations by regular ones will be discussed. The question of approximating singular perturbations is of particular interest for perturbations of order -2, since the perturbed operator is not uniquely defined in this case.

3. APPROXIMATIONS OF RANK ONE SINGULAR PERTURBATIONS

In this section we are going to discuss approximations of rank one perturbations in the norm resolvent sense. Consider a sequence of operators with bounded rank one perturbations

$$(3.1) \qquad A_\alpha^n = A + \alpha \langle \varphi_n, \cdot \rangle \varphi_n, \quad \varphi_n \in \mathcal{H}.$$

We say that the operator sequence approximates the operator A_α in the norm resolvent sense if and only if

$$(3.2) \qquad \left\| \frac{1}{A_\alpha^n - \lambda} - \frac{1}{A_\alpha - \lambda} \right\| \to 0 \text{ as } n \to \infty,$$

for any $\lambda, \Im \lambda \neq 0$.[2] If $\varphi \in \mathcal{H}_{-1}$, then such approximations can be constructed choosing any sequence $\varphi_n \in \mathcal{H}$ converging to φ in the norm of the Hilbert space $\mathcal{H}_{-1}$. Such sequences always exist, since the space $\mathcal{H}$ is dense in $\mathcal{H}_{-1}$ (in the norm $\| \cdot \|_{-1}$) [9, 35]. Therefore we are going to discuss the question of how to construct such

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[2]Note that we are not going to consider the approximations with the renormalization of the coupling constant, i.e. in what follows the real parameter α is always independent of n. For approximations with such renormalizations see [12, 20].

approximations only in the case $\varphi \in \mathcal{H}_{-2}$. Approximations of such perturbations in the strong resolvent sense were discussed in detail in [9].

For example the following theorem has been proven in [9] (Theorem 1.4.4).

Theorem 3.1. *Let A be a self-adjoint operator and φ be a functional from $\mathcal{H}_{-2}(A)$, $\| (1/(A-i))\varphi \| = 1$. Let φ_n be any sequence from the Hilbert space converging to φ in $\mathcal{H}_{-2}(A)$ and let $\lim_{n\to\infty}\langle \varphi_n, (A/(A^2+1))\varphi_n \rangle = c$. Then the sequence of self-adjoint operators*

$$A_\alpha^n = A + \alpha\langle \varphi_n, \cdot \rangle \varphi_n$$

converges to A_α in the strong resolvent sense. If

$$\lim_{n\to\infty} \left| \left\langle \varphi_n, \frac{A}{A^2+1}\varphi_n \right\rangle \right| = \infty,$$

the operators A_α^n converge to the original operator in the strong resolvent sense.

Using this theorem it was shown that if the operator A is not semibounded, then it is possible to construct approximations without any renormalization of the coupling constant, but only if the vector φ satisfies a certain additional condition [9] (Theorem 1.4.5) (see also [7]).

Theorem 3.2. *Let $E_{\mathbf{R}_\pm}\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$. Then there exist two sequences $c_n, d_n \to \infty$ such that $\varphi_n = E_{(-c_n,d_n)}\varphi$ determines the sequence of self-adjoint operators $A_\alpha^n = A + \alpha\langle \varphi_n, \cdot \rangle \varphi_n$ involving bounded perturbations of A converging to the perturbed operator $A_\alpha = A + \alpha\langle \varphi, \cdot \rangle \varphi$ in the strong resolvent sense.*

By $E_{(a,b)}$ we denote the spectral projector onto the interval (a, b) associated with the self-adjoint operator A. The advantage of Theorem 3.2 is that the approximating sequence of regular vectors φ_n is explicitly constructed using the spectral projector of A and the vector φ only. If we drop this restriction, then any singular perturbation of any self-adjoint operator which is not semibounded can be approximated even in the norm resolvent sense.

Theorem 3.3. *Let the self-adjoint operator A not be semibounded and let $\varphi \in \mathcal{H}_{-2}(A)$. Then there exists a sequence of regular vectors $\varphi_n \in \mathcal{H}$, $\varphi_n \overset{\mathcal{H}_{-2}}{\to} \varphi$, such that the sequence of self-adjoint operators $A_\alpha^n = A + \alpha\langle \varphi_n, \cdot \rangle \varphi_n$ involving bounded perturbations of A converges to the perturbed operator $A_\alpha = A + \alpha\langle \varphi, \cdot \rangle \varphi$ in the norm resolvent sense.*

Proof. Let φ_n be a sequence of regular vectors (from the original Hilbert space) converging to $\varphi \in \mathcal{H}_{-2}$ in the norm of $\mathcal{H}_{-2}$. Such a sequence always exists, since the original Hilbert space is dense in $\mathcal{H}_{-2}$ [9, 35]. Then the resolvents of the operators A_α^n and A_α for any $\lambda, \Im\lambda \neq 0$ are given by

$$(3.3) \qquad \frac{1}{A_\alpha^n - \lambda} = \frac{1}{A - \lambda} - \frac{\alpha}{1 + \alpha\langle \varphi_n, \frac{1}{A-\lambda}\varphi_n \rangle} \left\langle \frac{1}{A - \bar\lambda}\varphi_n, \cdot \right\rangle \frac{1}{A - \lambda}\varphi_n,$$

$$(3.4) \qquad \frac{1}{A_\alpha - \lambda} = \frac{1}{A - \lambda} - \frac{\alpha}{1 + \alpha(c + \langle \varphi, \frac{1+\lambda A}{A-\lambda}\frac{1}{A^2+1}\varphi \rangle)} \left\langle \frac{1}{A - \bar\lambda}\varphi, \cdot \right\rangle \frac{1}{A - \lambda}\varphi,$$

where c is the real parameter parameterizing all possible definitions of the singularly perturbed operator in the case $\varphi \in \mathcal{H}_{-2}(A)\setminus\mathcal{H}_{-1}(A)$ via (2.11) and $c = \left\langle \varphi, \frac{A}{A^2+1}\varphi \right\rangle$

in the case $\varphi \in \mathcal{H}_{-1}(A)$. Then the difference between the resolvents can be calculated

$$\frac{1}{A^n_\alpha - \lambda} - \frac{1}{A_\alpha - \lambda} = -\frac{\alpha}{1 + \alpha\langle\varphi_n, \frac{1}{A-\lambda}\varphi_n\rangle}\left\langle\frac{1}{A-\bar{\lambda}}\varphi_n, \cdot\right\rangle\frac{1}{A-\lambda}\varphi_n$$

$$+\frac{\alpha}{1 + \alpha(c + \langle\varphi, \frac{1+\lambda A}{A-\lambda}\frac{1}{A^2+1}\varphi\rangle)}\left\langle\frac{1}{A-\bar{\lambda}}\varphi, \cdot\right\rangle\frac{1}{A-\lambda}\varphi.$$

To prove that the resolvents of A^n_α converge to the resolvent of A_α in the operator norm it is enough to show that

(3.5) $$\|\frac{1}{A-z}\varphi_n - \frac{1}{A-z}\varphi\| \to 0, \text{ as } n \to \infty$$

and

(3.6) $$\left\langle\varphi_n, \frac{1}{A-z}\varphi_n\right\rangle = \left\langle\varphi_n, \frac{A}{A^2+1}\varphi_n\right\rangle + \left\langle\varphi_n, \frac{1+zA}{A-z}\frac{1}{A^2+1}\varphi_n\right\rangle$$

$$\to c + \left\langle\varphi, \frac{1+zA}{A-z}\frac{1}{A^2+1}\varphi\right\rangle, \text{ as } n \to \infty,$$

for any $z, \Im z \neq 0$ (and in particular for $z = \lambda, \bar{\lambda}$).

The first limit (3.5) holds for any sequence φ_n converging to φ in the norm of $\mathcal{H}_{-2}$. Similarly, for any such sequence the scalar product $\left\langle\varphi_n, \frac{1+zA}{A-z}\frac{1}{A^2+1}\varphi_n\right\rangle$ converges to $\left\langle\varphi, \frac{1+zA}{A-z}\frac{1}{A^2+1}\varphi\right\rangle$. Therefore to prove the theorem it is enough to show that there exists a sequence of regular vectors with the following two properties:

(3.7) $$\varphi_n \overset{\mathcal{H}_{-2}}{\to} \varphi \text{ and } \lim_{n\to\infty}\left\langle\varphi_n, \frac{A}{A^2+1}\varphi_n\right\rangle = c.$$

The proof is not trivial only if $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$. In fact if $\varphi \in \mathcal{H}_{-1}(A)$, then consider any sequence $\varphi_n \in \mathcal{H}$ converging to φ in the norm of $\mathcal{H}_{-1}(A)$. Such a sequence always exists since $\mathcal{H}$ is dense in $\mathcal{H}_{-1}(A)$. Every such sequence satisfies (3.7).

Now consider the case $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$. Let us pick up an arbitrary vector $\psi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$ having the additional property

(3.8) $$E_{\mathbf{R}_\pm}\psi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A),$$

where $\mathbf{R}_\pm$ denotes the open half-infinite intervals on the real axis and E is the spectral projector associated with A. Such vector ψ always exists, since the operator A is not semibounded. Really, the operators $E_{\mathbf{R}_+}A$, $E_{\mathbf{R}_-}A$ are not bounded. Therefore there exist vectors $\psi_\pm \in \mathcal{H}_{-2}(E_{\mathbf{R}_\pm}A) \setminus \mathcal{H}_{-1}(E_{\mathbf{R}_\pm}A)$. Then the vector $\psi = \psi_+ + \psi_-$ satisfies the condition (3.8). Let us introduce the sequence of real numbers $a_n = \left\langle E_{[-n,n]}\varphi, \frac{A}{A^2+1}\varphi\right\rangle$. Then there exist real numbers $c_n, d_n \geq n$ such that

$$\beta^2_n = \left\langle E_{(n,d_n)}\psi, \frac{A}{A^2+1}\psi\right\rangle > |a_n - c|,$$

$$\gamma^2_n = -\left\langle E_{(-c_n,-n)}\psi, \frac{A}{A^2+1}\psi\right\rangle > |a_n - c|.$$

The sequence of regular vectors φ_n can be chosen equal to

$$\varphi_n = E_{[-n,n]}\varphi + \varepsilon_{1,n}E_{(-c_n,-n)}\psi + \varepsilon_{2,n}E_{(n,d_n)}\psi,$$

where the numbers $\varepsilon_{1,n}, \varepsilon_{2,n} \in [0,1]$ are given by $\varepsilon_{1,n} = \frac{\sqrt{a_n-c}}{\gamma_n}$, $\varepsilon_{2,n} = 0$ and $\varepsilon_{1,n} = 0, \varepsilon_{2,n} = \frac{\sqrt{c-a_n}}{\beta_n}$ if $a_n - c \geq 0$, and $a_n - c \leq 0$, respectively. The following calculations prove that the sequence φ_n satisfies conditions (3.7)

$$\left\langle \varphi_n, \frac{A}{A^2+1}\varphi_n \right\rangle = a_n - \varepsilon_{1,n}^2\gamma_n^2 + \varepsilon_{2,n}^2\beta_n^2 = c,$$

$$\begin{aligned}
\| \varphi_n - \varphi \|_{\mathcal{H}_{-2}} &\leq & \| E_{(n,\infty)}\varphi \|_{\mathcal{H}_{-2}} + \| E_{(-\infty,-n)}\varphi \|_{\mathcal{H}_{-2}} \\
& & + \| E_{(n,\infty)}\psi \|_{\mathcal{H}_{-2}} + \| E_{(-\infty,-n)}\psi \|_{\mathcal{H}_{-2}} \\
&\rightarrow & 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

The theorem is proven. $\qquad\qquad\square$

One can easily prove that this theorem is optimal in the sense that the following lemma is valid [6] (Lemma 2.3).

Lemma 3.4. *Let $A \geq 0$ and $\varphi \in \mathcal{H}_{-m}(A)\backslash\mathcal{H}_{-1}(A)$, $m \geq 2$. Let φ_n be any sequence weakly converging to φ in $\mathcal{H}_{-m}(A)$. Then the sequence of self-adjoint operators*

$$A_\alpha^n = A + \alpha\langle\varphi_n,\cdot\rangle\varphi_n$$

converges to A in the strong resolvent sense.

Approximations of not semibounded self-adjoint operators in the strong resolvent sense involving the renormalization of the coupling constant were constructed in [12, 20]. Another possibility to construct approximations of singularly perturbed operators is to use non self-adjoint operators. Consider the following family of operators:

$$(3.9) \qquad\qquad B_\alpha^n = A + \alpha\langle\psi_n,\cdot\rangle\varphi_n,$$

where $\psi_n, \varphi_n \in \mathcal{H}, \alpha \in \mathbf{R}$. Each of these operators is a bounded perturbation of the self-adjoint operator A and their resolvents can easily be calculated

$$(3.10) \qquad \frac{1}{B_\alpha^n - \lambda} = \frac{1}{A-\lambda} - \frac{\alpha}{1+\alpha\langle\psi_n,\frac{1}{A-\lambda}\varphi_n\rangle}\left\langle\frac{1}{A-\bar\lambda}\psi_n,\cdot\right\rangle\frac{1}{A-\lambda}\varphi_n,$$

where $\lambda \in \rho(B_\alpha^n)\cap\rho(A)$. The following slight modification of Theorems 3.1 and 3.3 can be proven.

Theorem 3.5. *Let A be a self-adjoint operator and φ be a functional from $\mathcal{H}_{-2}(A)$, $\| (1/(A-i))\varphi \| = 1$. Let φ_n and ψ_n be any two sequences from the Hilbert space converging to φ in $\mathcal{H}_{-2}(A)$ and let $\lim_{n\to\infty}\langle\psi_n,(A/(A^2+1))\varphi_n\rangle = c$. Then the sequence of (non-self-adjoint) operators*

$$B_\alpha^n = A + \alpha\langle\psi_n,\cdot\rangle\varphi_n$$

converges to A_α in the norm resolvent sense. If

$$\lim_{n\to\infty}\left|\left\langle\psi_n, \frac{A}{A^2+1}\varphi_n\right\rangle\right| = \infty,$$

the operators B_α^n converge to the original operator in the norm resolvent sense.

We do not give a proof here, since it follows the same lines as the proof of Theorem 3.1. The following theorem shows that approximations by non-self-adjoint operators can be constructed even if the original operator is semibounded. It is a generalization of the results obtained in [10, 30].

Theorem 3.6. *Let A be an arbitrary self-adjoint operator and let $\varphi \in \mathcal{H}_{-2}(A)$. Then there exist two sequences of regular vectors $\varphi_n, \psi_n \in \mathcal{H}$, $\varphi_n, \psi_n \overset{\mathcal{H}_{-2}}{\to} \varphi$, such that the sequence of (non-self-adjoint) operators $B_\alpha^n = A + \alpha\langle\psi_n,\cdot\rangle\varphi_n$ involving bounded perturbations of A converges to the perturbed operator $A_\alpha = A + \alpha\langle\varphi,\cdot\rangle\varphi$ in the norm resolvent sense.*

Proof. We are going to prove the theorem in the case where the operator A is semibounded, since Theorem 3.3 covers the opposite case. Moreover if A is bounded, then $\mathcal{H}_{-2}(A) = \mathcal{H}_{-1}(A) = \mathcal{H}$ and the proof is trivial. Therefore without loss of generality we can suppose that A is positive and not bounded.

Again the proof is not trivial only if $\varphi \notin \mathcal{H}_{-1}(A)$. To use Theorem 3.5 one needs to construct two sequences of regular vectors satisfying the following conditions:

$$\psi_n, \varphi_n \overset{\mathcal{H}_{-2}}{\to} \varphi \text{ and } \lim_{n\to\infty}\left\langle \psi_n, \frac{A}{A^2+1}\varphi_n \right\rangle = c.$$

Let us introduce the real sequence $a_n = \left\langle E_{[-n,n]}\varphi, \frac{A}{A^2+1}\varphi \right\rangle$. Since φ does not belong to $\mathcal{H}_{-1}(A)$, then there exists an interval $[c_n, d_n]$ outside $[-n, n]$ such that numbers $b_n = \left\langle E_{[c_n,d_n]}\varphi, \frac{A}{A^2+1}\varphi \right\rangle$ could be arbitrarily large (small), in particular satisfying $|b_n| > |c - a_n|$. Then the sequences $\varphi_n, \psi_n \in \mathcal{H}$ can be chosen equal to

$$\varphi_n = E_{[-n,n]}\varphi + \varepsilon_n E_{[c_n,d_n]}\varphi,$$

$$\psi_n = E_{[-n,n]}\varphi + \delta_n E_{[c_n,d_n]}\varphi,$$

where the numbers ε_n and δ_n are given by $\varepsilon_n = \frac{|c-a_n|^{1/2}}{|b_n|^{1/2}}$, $\delta_n = \text{sign}\,(c-a_n)\,\text{sign}\,b_n\,\varepsilon_n$. It is obvious that $|\varepsilon_n| \leq 1$, and $|\delta_n| \leq 1$ and $\left\langle\psi_n, \frac{A}{A^2+1}\varphi_n\right\rangle = a_n + \varepsilon_n\delta_n b_n = c$. Moreover,

$$\| \varphi_n - \varphi \|_{\mathcal{H}_{-2}} \leq 2(\| E_{(n,\infty)}\varphi \|_{\mathcal{H}_{-2}} + \| E_{(-\infty,-n)}\varphi \|_{\mathcal{H}_{-2}}) \to 0, \text{ as } n \to \infty.$$

Similarly one can prove that $\| \psi_n - \varphi \|_{\mathcal{H}_{-2}} \to 0$, as $n \to \infty$. Then Theorem 3.5 implies that the operator sequence B_α^n converges to the operator A_α in the norm resolvent sense. $\square$

4. Conclusions

In this article it was shown that any rank one $\mathcal{H}_{-2}$-perturbation of a not semibounded self-adjoint operator can be approximated (in the norm resolvent sense) by a sequence of operators with regular perturbations of rank one. In the case of semibounded operators, we prove that approximations by not self-adjoint rank one regular perturbations can be constructed. The approach developed in this paper can easily be generalized to include arbitrary finite rank perturbations [8, 9, 13]. In particular the case of point interactions in $\mathbf{R}^1$ has recently been treated in [10, 25, 30]. It remains an open question of how to describe approximations of the higher order rank one perturbations described in [16, 26, 27, 28, 33, 34].

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