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On the essential spectrum of a class of singular matrix differential operators. II. Weyl's limit circles for the Hain–Lüst operator whenever quasi-regularity conditions are not satisfied*

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The essential spectrum of the singular matrix differential operator of mixed order determined by the operator matrix

$$\begin{pmatrix} -\dfrac{\mathrm{d}}{\mathrm{d}x}\rho(x)\dfrac{\mathrm{d}}{\mathrm{d}x}+q(x) & \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\beta(x)}{x} \\ -\dfrac{\beta(x)}{x}\dfrac{\mathrm{d}}{\mathrm{d}x} & \dfrac{m(x)}{x^2} \end{pmatrix}$$

is studied. Investigation of the essential spectrum of the corresponding self-adjoint operator is continued but now without assuming that the quasi-regularity conditions are satisfied. New conditions that guarantee that the operator is semi-bounded from below are derived. It is proven that the essential spectrum of any self-adjoint operator associated with the matrix differential operator is given by the range range$((m\rho-\beta^2)/\rho x^2)$ in the case where the quasi-regularity conditions are not satisfied.

1. Introduction

Matrix differential operators of mixed order have recently attracted much attention, owing to their interesting and unexpected spectral properties. Investigating one of the problems related to magnetohydrodynamics, it has been discovered that such operators may have a so-called singularity essential spectrum: the essential spectrum connected entirely with the singular point of the operator [4, 10–12, 14, 19, 27]. Mathematically rigorous studies of matrix differential operators of mixed order have been carried out in [1–3, 7–9, 13, 18, 21–23]. In order to investigate the phenomenon of the singularity essential spectrum, further study (see [17] for a detailed description of recent developments in this area) of the following matrix

*Dedicated to Boris Pavlov on the occasion of his 70th birthday.

ordinary differential operator

$$L = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x}\rho(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x) & \frac{\mathrm{d}}{\mathrm{d}x}\frac{\beta(x)}{x} \\ -\frac{\beta(x)}{x}\frac{\mathrm{d}}{\mathrm{d}x} & \frac{m(x)}{x^2} \end{pmatrix} \tag{1.1}$$

in the Hilbert space $\mathcal{H} = L_2[0,1] \oplus L_2[0,1]$ was suggested. This operator is singular if the function β is not equal to zero or the function m does not have second-order zero at the origin. It is natural to assume that the coefficients are real twice continuously differentiable functions:

$$\rho, q, \beta, m \in C^2[0,1]. \tag{1.2}$$

In addition we assume that the density function ρ is positive definite:

$$\rho(x) \geqslant \rho_0 > 0. \tag{1.3}$$

Singular matrix differential operators with coefficients of mixed order appear in different problems related to applications in physics and engineering, in particular in magnetohydrodynamics. It appears that these operators may have an interesting spectrum structure and therefore they have attracted the attention of specialists in spectral theory. The operators appearing in realistic problems are rather complicated and their spectral analysis leads to tedious calculations which make it difficult to study the interplay between the matrix coefficients. It appears to us that the operator (1.1) is one of the simplest matrix differential operators possessing the following spectral property: its essential spectrum cannot be obtained as the limit as $\varepsilon \to 0^+$ of the essential spectrum of the same differential operator restricted to the interval $[\varepsilon, 1]$. Note that the differential order of the coefficients and the orders of the singularities cancel in the formal determinant of the operator L: the differential order of the product of the diagonal coefficients is $2 + 0$ and of the anti-diagonal is $1 + 1$. Similarly, for the orders of the singularities we have $0 + 2 = 1 + 1$. This property allows the unusual interplay between the matrix coefficients.

It has been proven that the essential spectrum of the corresponding operator $\boldsymbol{L}$ is bounded if and only if the following quasi-regularity conditions are satisfied:

$$\left.\begin{aligned} \rho m - \beta^2|_{x=0} &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)\Big|_{x=0} &= 0. \end{aligned}\right\} \tag{1.4}$$

It appears that under these conditions the essential spectrum consists of two parts: regularity and singularity spectra. The first part of the essential spectrum is determined by the behaviour of the coefficients over the whole interval $x \in [0,1]$ and is given by the range

$$\operatorname{range}\left(\frac{\rho m - \beta^2}{\rho x^2}\right). \tag{1.5}$$

This spectrum can be obtained by considering the sequence of regular matrix differential operators on the intervals $(\varepsilon, 1]$ as $\varepsilon \to 0$. The second part of the essential spectrum is called the singularity spectrum and is determined by the limits of the coefficients at the origin, i.e. exclusively by the singularity. This spectrum cannot be obtained as a limit described above. The singularity spectrum is equal to

$$\left[\frac{l_0}{4+(\rho(0)/m(0))}, \frac{l_0}{\rho(0)/m(0)}\right], \tag{1.6}$$

where $l_0 = \lim_{x\to 0}((\rho m - \beta^2)/mx^2)$. A similar operator has been studied later in [18] under quasi-regularity conditions as well.

Here we study the case in which the essential spectrum of the matrix differential operator is not necessarily bounded, but the operator is just semi-bounded from below. This assumption is natural in numerous physical applications.

The differential expression L does not determine the self-adjoint operator in $\mathcal{H}$ uniquely and, therefore, in the first step it is natural to associate with L a certain minimal operator $\boldsymbol{L}_{\min}$. Since the endpoint, $x = 1$, is regular for the matrix differential operator, we define $\boldsymbol{L}_{\min}$ on the set of functions from $C_0^\infty(0,1] \oplus C_0^\infty(0,1]$ satisfying a certain symmetric boundary condition at $x = 1$. Consider the transformed derivative

$$w_U(x) = -\rho(x)u_1'(x) + \frac{\beta(x)}{x}u_2(x). \tag{1.7}$$

Then any symmetric boundary condition at the regular point $x = 1$ can be written as [3, 17]

$$w_U(1) = h_1 u_1(1), \quad h_1 \in \mathbb{R} \cup \{\infty\}. \tag{1.8}$$

Therefore, the minimal operator $\boldsymbol{L}$ is defined by (1.1) on the domain

$$\operatorname{dom}(\boldsymbol{L}_{\min}) = \{U \in C_0^\infty(0,1] \oplus C_0^\infty(0,1], w_U(1) = h_1 u_1(1)\}. \tag{1.9}$$

Note that the domain includes functions which are infinitely many times differentiable and which vanish in a neighbourhood of the origin. In what follows, we keep the same notation for the closure of the operator $\boldsymbol{L}_{\min}$ in $\mathcal{H}$.

Only the simplest symmetric matrix differential operator is studied here. However, it is not difficult to consider additional even non-Hermitian low-order perturbations. (Here we mean both differential and algebraic order.)

In this paper we concentrate our attention on the case in which the quasi-regularity conditions are not satisfied. Note that a similar problem for a 3×3 differential operator is addressed in [23]. It is proven in the following section that the differential operator is semi-bounded from below if and only if conditions (2.1)–(2.3) are satisfied. These conditions include quasi-regularity conditions as a special case. The Friedrichs extension of the minimal operator is described in § 5. Finally, it is proven that the essential spectrum of any self-adjoint operator associated with (1.1) is given just by (1.5) in the case where conditions (2.1)–(2.3) are satisfied but the quasi-regularity conditions are not. Thus, the following striking theorem is proven: the singularity essential spectrum (1.6) for the matrix differential operator is present only if the quasi-regularity conditions are satisfied, provided that the operator is singular and semi-bounded from below. Under the same assumptions it is proven that the singularity spectrum appears if and only if the Hain–Lüst operator (see § 5), following Weyl's classification, is in the limit-point case at the singular point. We believe that this observation concerning the connection between the deficiency indices of Hain–Lüst operator and the quasi-regularity conditions is of much greater generality.

2. Semi-boundedness

It has been proven in [17] that the essential spectrum of any self-adjoint operator associated with the differential expression (1.1) is bounded if and only if the quasi-regularity conditions (1.4) are satisfied.

PROPOSITION 2.1 (Kurasov and Naboko [17, lemma 3.1]). *Under the assumptions (1.2), (1.3) on the coefficients ρ, β, m and q, the quasi-regularity conditions are fulfilled if and only if the essential spectrum of at least one (and hence any) self-adjoint extension of $\boldsymbol{L}_{\min}$ is bounded.*

Therefore, if the quasi-regularity conditions are not satisfied, every self-adjoint operators associated with (1.1) has an unbounded essential spectrum. It is natural to examine the questions under which conditions this essential spectrum is semi-bounded from below. It is more or less clear that the deficiency indices of the minimal operator are finite (this fact will be proven rigorously mathematically later). Therefore, the same conditions on the coefficients guarantee that both the minimal operator and any of its self-adjoint extensions are semi-bounded from below.

LEMMA 2.2. *The operator $\boldsymbol{L}_{\min}$ is semi-bounded from below if and only if one of the following three conditions is satisfied:*

$$\rho m - \beta^2|_{x=0} > 0, \tag{2.1}$$

$$\rho m - \beta^2|_{x=0} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\rho(x)m(x) - \beta^2(x))\bigg|_{x=0} > 0, \tag{2.2}$$

$$\rho m - \beta^2|_{x=0} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\rho(x)m(x) - \beta^2(x))\bigg|_{x=0} = 0. \tag{2.3}$$

(Note that this condition (2.3) just coincides with the quasi-regularity condition, which guarantees boundedness (from above and from below) of the essential spectrum.)

Proof. We will prove the sufficiency and necessity of these conditions separately.

(i) Sufficiency. It is sufficient to show that the quadratic form associated with the minimal operator $\boldsymbol{L}_{\min}$ is semi-bounded from below under conditions (2.1)–(2.3), i.e. that the inequality

$$\langle \boldsymbol{L}_{\min} U, U\rangle \geqslant C\|U\|^2_{\mathcal{H}} \tag{2.4}$$

holds with a certain real constant, C.

Let $U \in C_0^\infty(0,1] \oplus C_0^\infty(0,1]$ and satisfy the boundary condition (1.8) at $x = 1$. Then the quadratic form can be calculated using integration by parts:

$$\begin{aligned}\langle \boldsymbol{L}U, U\rangle &= \langle \rho u_1', u_1'\rangle + w_U(1)\overline{u_1(1)} + \langle q u_1, u_1\rangle - 2\operatorname{Re}\left\langle \frac{\beta}{x}u_2, u_1'\right\rangle + \left\langle \frac{m}{x^2}u_2, u_2\right\rangle \\ &= \left\|\frac{1}{\sqrt{\rho}}w_U\right\|^2 + h_1|u_1(1)|^2 + \left\langle \frac{\rho m - \beta^2}{\rho x^2}u_2, u_2\right\rangle + \langle q u_1, u_1\rangle. \end{aligned}\tag{2.5}$$

Note that we have used the fact that the support of U does not contain the origin and the function U satisfies the symmetric boundary condition (1.8) at $x = 1$. The second term in the second line of (2.5) vanishes in the special case $h_1 = \infty$ (the Dirichlet boundary condition at $x = 1$). Let us show that under conditions (2.1)–(2.3) the quadratic form is semi-bounded from below.

We first prove that the sum of the first two terms is bounded from below with respect to $\|U\|^2$. This proof is trivial if either $h_1 > 0$ (both terms are non-negative) or $h_1 = \infty$ (the second term is absent and the first term is non-negative).

Consider the case $h_1 < 0$. Let us prove that $|u_1(1)|^2$ is infinitesimally bounded with respect to $\|u_1'\|_{L_2(1/2,1)}$ and $\|U\|_{L_2(1/2,1)}$ (see inequality (2.6), below). We consider the obvious estimate

$$|u_1(1)|^2 \leqslant 2\int_x^1 |u_1(t)u_1'(t)|\,\mathrm{d}t + |u_1(x)|^2$$

and integrate it over the interval $[\frac{1}{2}, 1]$:

$$\begin{aligned}\tfrac{1}{2}|u_1(1)|^2 &\leqslant 2\int_{1/2}^1\int_{1/2}^1 |u_1(t)u_1'(t)|\,\mathrm{d}t\,\mathrm{d}x + \int_{1/2}^1 |u_1(x)|^2\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\bigg(\varepsilon\|u_1'\|^2_{L_2(1/2,1)} + \frac{4}{\varepsilon}\|u_1\|^2_{L_2(1/2,1)}\bigg) + \|u_1\|^2_{L_2(1/2,1)}.\end{aligned}$$

This inequality implies that

$$|u_1(1)|^2 \leqslant \varepsilon\|u_1'\|^2_{L_2(1/2,1)} + \left(\frac{4}{\varepsilon} + 2\right)\|u_1\|^2_{L_2(1/2,1)}. \tag{2.6}$$

On the other hand, the first term in (2.5) (which is clearly positive) can be estimated from below,

$$\begin{aligned}\left\|\frac{1}{\sqrt{\rho}}w_U(x)\right\|^2_{L_2(0,1)} &\geqslant \left\|\frac{1}{\sqrt{\rho}}w_U(x)\right\|^2_{L_2(1/2,1)} \\ &\geqslant \left\|\sqrt{\rho}u_1'\right\|^2_{L_2(1/2,1)} - \left\|\frac{\beta}{\sqrt{\rho}x}u_2\right\|^2 \\ &\geqslant \rho_0\|u_1'\|^2_{L_2(1/2,1)} - C_1\|U\|^2,\end{aligned} \tag{2.7}$$

using the triangle inequality and the fact that the following function is uniformly bounded:

$$\frac{\beta(x)}{\sqrt{\rho(x)}x} \leqslant C_1 \quad \text{for } x \in [\tfrac{1}{2}, 1].$$

In the case when $h_1 < 0$, choosing $\varepsilon = \rho_0/|h_1|$, we conclude that the sum of the first two terms in (2.5) is semi-bounded from below with respect to the norm in $\mathcal{H}$.

The third term,

$$\left\langle \frac{\rho m - \beta^2}{x^2\rho}u_2, u_2\right\rangle$$

in (2.5) is uniformly bounded if the quasi-regularity condition (2.3) is satisfied. If either of the conditions (2.1) or (2.2) is satisfied, then the function $(\rho m - \beta^2)/x^2\rho$

is positive in a certain neighbourhood of the origin, say $[0, r]$, so that the scalar product can be decomposed into the sum of two integrals, one positive and one uniformly bounded in the norm of $\mathcal{H}$:

$$\left\langle \frac{\rho m - \beta^2}{x^2 \rho} u_2, u_2 \right\rangle = \int_0^r \frac{\rho m - \beta^2}{x^2\rho} |u_2|^2 \, \mathrm{d}t + \int_r^1 \frac{\rho m - \beta^2}{x^2 \rho} |u_2|^2 \, \mathrm{d}t.$$

The last term, $\langle q u_1, u_1 \rangle$, is also bounded, since the function q is uniformly bounded.

We have proven that under conditions (2.1), (2.2) or (2.3) the quadratic form of the minimal operator is semi-bounded from below.

(ii) Necessity. Suppose that (2.1)–(2.3) are not satisfied, i.e. parameters of the operator satisfy one of the following two conditions:

$$\rho m - \beta^2|_{x=0} < 0, \tag{2.8}$$

$$\rho m - \beta^2|_{x=0} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)\bigg|_{x=0} < 0. \tag{2.9}$$

If one of these conditions is satisfied, then the function $\rho m - \beta^2$ is negative in a certain interval $(0, \varepsilon) \subset (0, 1)$. It may be equal to zero at the origin, but the order of the zero cannot be higher than 1. It follows that there exists a certain positive number $C_2 > 0$ such that the following inequality holds:

$$\rho m - \beta^2 < -C_2 \rho x \quad \Longrightarrow \quad \frac{\rho m - \beta^2}{x^2 \rho} < -\frac{C_2}{x}, \tag{2.10}$$

for all $x \in (0, \varepsilon)$. Using this fact, we construct a sequence of functions $U^k \in \mathrm{dom}(\boldsymbol{L}_{\min}) \subset \mathcal{H}$ with the following properties:

(a) $\|U^k\|$ is uniformly bounded;

(b) $\langle \boldsymbol{L}_{\min} U^k, U^k \rangle$ tends to $-\infty$ as $k \to \infty$.

The last term in (2.5) is uniformly bounded, since q is a bounded function, and therefore does not affect the divergence of $\langle \boldsymbol{L} U^k, U^k \rangle$ to $-\infty$. In addition, the sequence we construct will have the following property

$$w_{U^k} \equiv 0 \quad \Longleftrightarrow \quad (u_1^k)' = \frac{\beta(x)}{x \rho(x)} u_2^k. \tag{2.11}$$

The second component of U^k can be chosen equal to

$$u_2^k = \begin{cases} \sin(\ln x - \ln \varepsilon), & x \in (\varepsilon \mathrm{e}^{-2\pi k}, \varepsilon), \\ 0, & \text{otherwise}. \end{cases}$$

Then, in order to satisfy (2.11), we choose the first component equal to

$$u_1^k(x) = \int_0^x \frac{\beta(t)}{t \rho(t)} u_2^k(t) \, \mathrm{d}t.$$

It is clear that the functions u_2^k are uniformly bounded and therefore $\|u_2^k\|$ are uniformly bounded as well. Due to oscillation properties of u_2^k the functions u_1^k

are uniformly bounded as well. Indeed, for $x \in (\varepsilon e^{-2\pi k}, \varepsilon)$ the function u_1^k can be estimated by integrating by parts:

$$\begin{aligned} u_1^k(x) &= \int_{\varepsilon e^{-2\pi k}}^x \frac{\beta(t)}{t\rho(t)} \sin(\ln t - \ln \varepsilon)\, dt \\ &= -\int_{\varepsilon e^{-2\pi k}}^x \frac{\beta(t)}{\rho(t)} \frac{d}{dt}(\cos(\ln t - \ln \varepsilon))\, dt \\ &= -\frac{\beta(t)}{\rho(t)} \cos(\ln t - \ln \varepsilon)\Big|_{t=\varepsilon e^{-2\pi k}}^x + \int_{\varepsilon e^{-2\pi k}}^x \frac{d}{dt}\left(\frac{\beta(t)}{\rho(t)}\right) \cos(\ln t - \ln \varepsilon)\, dt. \end{aligned}$$

Then the functions u_1^k are uniformly bounded because the functions $\beta(t)/\rho(t)$ and $(\beta(t)/\rho(t))'$ are uniformly bounded. This implies that both $|u_1^k(1)|^2$ and $\|u_1^k\|$ are uniformly bounded. We conclude that the sequence $\|U^k\|$ is uniformly bounded. The corresponding quadratic form $\langle LU^k, U^k\rangle$ given by (2.5) tends to $-\infty$, since the first term in (2.5) vanishes, the second and fourth terms are uniformly bounded and the third term due to (2.10) can be estimated as

$$\left\langle \frac{\rho m - \beta^2}{\rho x^2} u_2, u_2 \right\rangle \leqslant -C_2 \left\langle \frac{1}{x} u_2^k, u_2^k \right\rangle = -C_2 \int_{\varepsilon e^{-2\pi k}}^{\varepsilon} \frac{\sin^2(\ln x - \ln \varepsilon)}{x}\, dx,$$

and therefore tends to $-\infty$. The sequence constructed satisfies conditions (a) and (b), but it does not belong to the domain of $\boldsymbol{L}$, since the functions are not infinitely many times differentiable at the points $x = \varepsilon e^{-2\pi k}$ and ε. To get infinitely differentiable functions, we can smooth U^k out without drastically changing the norm and the value of the quadratic form. □

Thus, we have proven that the matrix differential operator L is semi-bounded from below if and only if conditions (2.1)–(2.3) are satisfied. The assumption that the operator is semi-bounded is standard in the studies of different physical problems.

3. Deficiency indices of the minimal operator

The operator $\boldsymbol{L}^*_{\min}$ (the adjoint operator to $\boldsymbol{L}_{\min}$) is defined by the same operator matrix (1.1) on the domain of functions from $W_2^2[0,1] \oplus W_2^1[0,1] \subset \mathcal{H}$ satisfying the following two additional conditions:

$$-\frac{d}{dx}\rho(x)\frac{d}{dx}u_1 + qu_1 + \frac{d}{dx}\frac{\beta}{x}u_2 \in L_2[0,1];$$
$$-\frac{\beta}{x}\frac{d}{dx}u_1 + \frac{m}{x^2}u_2 \in L_2[0,1].$$

The following theorem has been proven in [17]. We reformulate it here with the notation used in the current paper.[1]

[1] $\boldsymbol{L}_{\min}$ was used in [17] to denote the symmetric operator determined by the differential expression L on the domain $C_0^\infty(0,1) \oplus C_0^\infty(0,1)$ including only functions with compact support separated from the point $x = 1$. In the current paper the domain of $\boldsymbol{L}_{\min}$ contains functions with support not necessarily separated from $x = 1$ but satisfying the standard boundary condition (1.8) at this endpoint.

PROPOSITION 3.1 (this follows [17, theorem 4.1]). *The operator* $\boldsymbol{L}_{\min}$ *is a symmetric operator in the Hilbert space* $\mathcal{H}$ *with finite equal deficiency indices.*

(i) *If the operator matrix* L *is singular quasi-regular (i.e.* $m(0) \neq 0$ *and quasi-regularity conditions are satisfied), then the deficiency indices of* $\boldsymbol{L}_{\min}$ *are trivial and the operator* $\boldsymbol{L}_{\min}$ *is self-adjoint.*

(ii) *If the operator matrix is regular or is not quasi-regular, then the deficiency indices of* $\boldsymbol{L}_{\min}$ *are equal to* $(1,1)$*. The self-adjoint extensions of* $\boldsymbol{L}_{\min}$ *are described by boundary conditions using the following alternatives, which cover all possibilities.*

(a) *If* $\rho(0)m(0) - \beta^2(0) \neq 0$ *or* $\beta(0) = 0$*, then the first component* u_1 *of any vector from the domain of the adjoint operator* $\boldsymbol{L}^*_{\min}$ *is continuous in the closed interval* $[0,1]$*. All self-adjoint extensions of the operator* $\boldsymbol{L}_{\min}$ *are described by the standard boundary condition*[2] *at* $x = 0$*,*

$$\omega_U(0) = h_0 u_1(0), \quad h_0 \in \boldsymbol{R} \cup \{\infty\}. \tag{3.1}$$

(b) *If*

$$\rho(0)m(0) - \beta^2(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)(0) \neq 0 \quad \text{and} \quad \beta(0) \neq 0,$$

then the first component, u_1*, of any vector from the domain of the adjoint operator* $\boldsymbol{L}^*_{\min}$ *admits the asymptotic representation*

$$u_1(x) = k\omega_U(0)\ln x + c_U + o(1) \quad \text{as } x \to 0, \tag{3.2}$$

where

$$k = -\frac{\beta^2(0)}{\rho(0)}\left(\frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)\Big|_{x=0}\right)^{-1}$$

and c_U *is an arbitrary constant depending on* U*. Then all self-adjoint extensions of the operator* $\boldsymbol{L}_{\min}$ *are described by the non-standard boundary condition (see footnote 2)*

$$\omega_U(0) = h_0 c_U, \quad h_0 \in \boldsymbol{R} \cup \{\infty\}. \tag{3.3}$$

Information concerning the deficiency indices of $\boldsymbol{L}_{\min}$ *and self-adjoint local boundary conditions is collected in table 1.*

This proposition implies in particular that deficiency indices of $\boldsymbol{L}_{\min}$ are always finite and equal. Therefore, there always exists a family of self-adjoint operators associated with the differential expression L. Every operator from such a family is an extension of $\boldsymbol{L}_{\min}$ and the essential spectrum does not depend on the particular extension chosen.

[2] In the case when $h_0 = \infty$, the corresponding boundary condition should be written as $u_1(0) = 0$ or $c_U = 0$.

Table 1. *The deficiency indices of $\boldsymbol{L}_{\min}$ and self-adjoint local boundary conditions*

		$\rho(0)m(0)-\beta^2(0)=0$	
	$\rho(0)m(0)-\beta^2(0)\neq 0$	$\left.\dfrac{\mathrm{d}}{\mathrm{d}x}(\rho m-\beta^2)\right\vert_{x=0}\neq 0$	$\left.\dfrac{\mathrm{d}}{\mathrm{d}x}(\rho m-\beta^2)\right\vert_{x=0}=0$
$\beta(0)=0$	indices $(1,1)$ standard b.c. (3.1)	indices $(1,1)$ standard b.c. (3.1)	indices $(1,1)$ standard b.c. (3.1)
$\beta(0)\neq 0$	indices $(1,1)$ standard b.c. (3.1)	indices $(1,1)$ non-standard b.c. (3.3)	indices $(0,0)$ self-adjoint

4. The Friedrichs extension

We have seen that the differential expression L does not necessarily determine a unique self-adjoint operator in $\mathcal{H}$. In the case when $\boldsymbol{L}_{\min}$ is semi-bounded it is natural to associate with L the Friedrichs extension of $\boldsymbol{L}_{\min}$. This extension is studied in this section. Note that this problem has been studied in a more general context in [16], but we provide a detailed analysis for the operator under investigation.

THEOREM 4.1. *The Friedrichs extension of the symmetric operator, $\boldsymbol{L}_{\min}$, is described by boundary conditions at $x=0$ depending on their type (and the properties of the coefficients) as follows.*

(i) *If the operator $\boldsymbol{L}_{\min}$ is self-adjoint, then no boundary condition at the origin is needed and the Friedrichs extension just coincides with $\boldsymbol{L}_{\min}$. This case occurs if the coefficients of the operator matrix satisfy the following conditions:*

$$\rho m-\beta^2|_{x=0}=0,\quad \left.\frac{\mathrm{d}}{\mathrm{d}x}(\rho m-\beta^2)\right|_{x=0}=0\quad \text{and}\quad \beta(0)\neq 0. \tag{4.1}$$

(ii) *If the extensions of $\boldsymbol{L}_{\min}$ are described by the standard boundary condition at the origin (see (3.1)), then the Friedrichs extension corresponds to the condition*

$$u_1(0)=0. \tag{4.2}$$

This case occurs if the coefficients of the operator matrix satisfy one of the following two conditions:

$$\rho m-\beta^2|_{x=0}>0 \tag{4.3}$$

or

$$\rho m-\beta^2|_{x=0}=0,\quad \left.\frac{\mathrm{d}}{\mathrm{d}x}(\rho m-\beta^2)\right|_{x=0}>0\quad \text{and}\quad \beta(0)=0. \tag{4.4}$$

(iii) *If the extensions of $\boldsymbol{L}_{\min}$ are described by the non-standard boundary condition at the origin (see (3.3)), then the Friedrichs extension corresponds to the condition*

$$w_U(0)=0. \tag{4.5}$$

This case occurs if the coefficients of the operator matrix satisfy the following conditions:

$$\rho m - \beta^2|_{x=0} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)\bigg|_{x=0} > 0 \quad \textit{and} \quad \beta(0) \neq 0. \tag{4.6}$$

Proof. The statement formulated in part (i) is trivial and is included for the sake of completeness only. We consider the two remaining cases separately, but the same idea will be used. It will be proven that every function from the domain of the Friedrichs extension necessarily satisfies one of the boundary conditions ((4.2) or (4.5) depending on their type) describing self-adjoint extensions of $\boldsymbol{L}_{\min}$. This will be sufficient to determine the boundary conditions describing the Friedrichs extension, since the operator $\boldsymbol{L}_{\min}$ is closed and has deficiency indices $(1,1)$. Indeed, every function satisfying the boundary conditions corresponding to two different self-adjoint extensions necessarily belongs to the domain of the original symmetric operator $\boldsymbol{L}_{\min}$. Therefore, to establish the boundary conditions describing the Friedrichs extension, it is sufficient to prove that the functions from the domains of these extensions satisfy (4.2) and (4.5), respectively.

(ii) To construct the Friedrichs extension we must consider the closure of the domain $\mathrm{dom}(\boldsymbol{L}_{\min})$ with respect to the following quadratic form, which is positive for all sufficiently large values of A:

$$\begin{aligned}[U,U] &\equiv \langle(\boldsymbol{L}_{\min}+A)U,U\rangle = \langle \boldsymbol{L}_{\min}U,U\rangle + A\|U\|^2 \\ &= \left\|\frac{1}{\sqrt{\rho}}w_U\right\|^2 + \left\langle \frac{\rho m-\beta^2}{x^2\rho}u_2,u_2\right\rangle + \langle qu_1,u_1\rangle + h_1|u_1(1)|^2 + A\|U\|^2. \end{aligned} \tag{4.7}$$

The terms $\langle qu_1,u_1\rangle$ and $h_1|u_1(1)|^2$ can be estimated through the other terms. Indeed, the estimate for $\langle qu_1,u_1\rangle$ is trivial:

$$|\langle qu_1,u_1\rangle| \leqslant \max|q(x)|\|u_1\|^2 \leqslant \max|q(x)|\|U\|^2.$$

To obtain the estimate for $h_1|u_1(1)|^2$, consider the triangle inequality,

$$\begin{aligned}\left\|\frac{1}{\sqrt{\rho}}w_U\right\|^2 &\geqslant \left\|\frac{1}{\sqrt{\rho}}w_U\right\|^2_{L_2(1/2,1)} \\ &\geqslant \tfrac{1}{2}\sqrt{\rho_0}\|u_1'\|^2_{L_2(1/2,1)} - \text{const.}\left\|\frac{1}{x}u_2\right\|^2_{L_2(1/2,1)}.\end{aligned}$$

Using (2.6), this implies that

$$\begin{aligned}|u_1(1)|^2 &\leqslant \varepsilon\|u_1'\|^2_{L_2(1/2,1)} + \left(\frac{4}{\varepsilon}+2\right)\|u_1\|^2_{L_2(1/2,1)} \\ &\leqslant \varepsilon\frac{2}{\sqrt{\rho_0}}\left\|\frac{1}{\sqrt{\rho}}w_U\right\|^2 + \text{const.}(\varepsilon)\|U\|^2.\end{aligned}$$

It follows that the quadratic form $[U,U]$ given by (4.7) for sufficiently large values of A is equivalent to the quadratic form

$$Q(U,U) = \left\| \frac{1}{\sqrt{\rho}} w_U \right\|^2 + \left\langle \frac{\rho m - \beta^2}{x^2 \rho} u_2, u_2 \right\rangle + A\|U\|^2,$$

where A is a certain positive real number. (The parameter A here may differ slightly from that used in (4.7).)

Let us now study the two possible cases, (4.3) and (4.4), separately.

CASE 1. Let condition (4.3) be satisfied: $\rho m - \beta^2|_{x=0} > 0$. On a certain interval $(0,\varepsilon)$, $\varepsilon > 0$ the function $(\rho m - \beta^2)/\rho$ is strictly positive. Consider any sequence $U^k \in \operatorname{dom}(\boldsymbol{L}_{\min})$, $k = 1, 2, \ldots$, having support on $(0,\varepsilon)$ and converging in the norm given by $Q(U,U)$. Since the form Q can be estimated from below by

$$Q(U,U) \geqslant C_3 \left\| \frac{1}{x} u_2 \right\|^2_{L_2(0,\varepsilon)}, \quad C_3 > 0,$$

and therefore for any sequence U^k converging with respect to $Q(U,U)$, the sequence u_2^k/x is a Cauchy sequence in $L_2(0,\varepsilon)$. On the other hand, the same quadratic form can also be estimated as

$$Q(U,U) \geqslant \left\| \frac{1}{\sqrt{\rho}} w_U \right\|^2_{L_2(0,\varepsilon)} = \left\| -\sqrt{\rho} u_1' + \frac{\beta}{\sqrt{\rho}} \frac{1}{x} u_2 \right\|^2_{L_2(0,\varepsilon)}. \tag{4.8}$$

Therefore, $-\sqrt{\rho}(u_1^k)'$ is a Cauchy sequence in $L_2(0,\varepsilon)$. Taking into account the fact that $Q(U,U) \geqslant C_4 \|u_1\|^2_{L_2(0,\varepsilon)}$, $C_4 > 0$, we conclude that u_1^k is a Cauchy sequence with respect to the norm of $W_2^1(0,\varepsilon)$. Every function u_1^k is equal to zero at the origin and therefore the limit function u_1 satisfies the Dirichlet boundary condition (4.2).

The assumption that the support of U^k belongs to $(0,\varepsilon)$ is not very restrictive. Let U^k be any sequence from $\operatorname{dom}(\boldsymbol{L}_{\min})$ converging in the norm $Q(U,U)$. Consider in addition any cut-off function $\psi \in C_0^\infty(0,1]$, identically equal to 1 on the interval $[\varepsilon, 1]$. Then the sequence ψU^k converges to a function from $\operatorname{dom}(\boldsymbol{L}_{\min})$ and therefore the sequence $(1-\psi)U^k$ is a Cauchy sequence with respect to $Q(U,U)$ having support on the interval $(0,\varepsilon)$. We have proven that every such sequence converges to a function satisfying Dirichlet condition at the origin. It follows that the limit of U^k satisfies the same condition. Thus, the Friedrichs extension of the operator $\boldsymbol{L}_{\min}$ is described by (4.2) in this case.

CASE 2. Let condition (4.4) be satisfied:

$$\rho m - \beta^2|_{x=0} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x}(\rho m - \beta^2)\Big|_{x=0} > 0 \quad \text{and} \quad \beta(0) = 0.$$

Again there exists $\varepsilon > 0$ such that the function $\rho m - \beta^2 > 0$ for $x \in (0,\varepsilon)$. Consider an arbitrary sequence $U^k \in \operatorname{dom}(\boldsymbol{L}_{\min})$, $k = 1, 2, \ldots$, converging in the norm given by $Q(U,U)$. We assume again that supports of all functions U^k belong to $(0,\varepsilon)$. Condition (4.4) imply that, for a certain $\varepsilon > 0$, the estimate

$$Q(U,U) \geqslant C_5 \left\| \frac{1}{\sqrt{x}} u_2 \right\|^2_{L_2(0,\varepsilon)}, \quad C_5 > 0, \tag{4.9}$$

holds and therefore $u_2^k/\sqrt{x}$ is a Cauchy sequence in $L_2(0,\varepsilon)$. The estimate (4.8) can be modified as

$$Q(U,U) \geqslant \left\| -\sqrt{\rho}u_1' + \frac{\beta}{\sqrt{\rho}}\frac{1}{\sqrt{x}} \times \frac{1}{\sqrt{x}}u_2 \right\|^2_{L_2(0,\varepsilon)}.$$

Since $\beta \in C^1(0,1)$ and $\beta(0) = 0$, the function

$$\frac{\beta}{\sqrt{\rho}}\frac{1}{\sqrt{x}}$$

is bounded and therefore $-\sqrt{\rho}(u_1^k)'$ is a Cauchy sequence in $L_2(0,\varepsilon)$. Taking into account the fact that u_1^k is a Cauchy sequence, we conclude that u_1^k converges with respect to the norm of $W_2^1(0,\varepsilon)$ and therefore satisfies the Dirichlet boundary condition (4.2) at the origin. The same reasoning as in case 1 may be applied to modify the proof for sequences U^k not necessarily having support in $(0,\varepsilon)$. Thus, the Friedrichs extension of the operator $\boldsymbol{L}_{\min}$ is also described by (4.2).

(iii) Under condition (4.6) the form $[U,U]$ is again equivalent to the form $Q(U,U)$. As in case 2, there exists a certain $\varepsilon > 0$, such that $\rho m - \beta^2 > 0$ for $x \in (0,\varepsilon)$. Consider any sequence $U^k \in \operatorname{dom}(\boldsymbol{L}_{\min})$, $k = 1,2,\ldots$, converging in the norm given by $Q(U,U)$. The estimate (4.9) holds and it follows that $u_2^k/\sqrt{x}$ is a Cauchy sequence in $L_2(0,\varepsilon)$. On the other hand, $w_{U^k}/\sqrt{\rho}$ is also a Cauchy sequence in $L_2(0,\varepsilon)$. Since

$$\sqrt{x}\frac{1}{\sqrt{\rho}}w_{U^k} = -\sqrt{\rho}\sqrt{x}u_1^{k\prime} + \frac{\beta}{\sqrt{\rho}}\frac{1}{\sqrt{x}}u_2,$$

it follows that $\sqrt{x}u_1^{k\prime}$ is a Cauchy sequence in $L_2(0,\varepsilon)$. It follows that the functions belonging to the domain of the Friedrichs extension in particular satisfy

$$\sqrt{x}u_1' \in L_2(0,\varepsilon). \tag{4.10}$$

Recall that, in the case under investigation, every function from the domain of the adjoint operator $\boldsymbol{L}^*_{\min}$, as well as its Friedreichs extension, possesses the asymptotic representation (3.2). Every function possessing this representation satisfies (4.10) if and only if $w_U(0) = 0$, i.e. only if the function satisfies the non-standard boundary condition (4.5). To see this, let us use the estimates

$$u_1(\varepsilon) - u_1(x) = \int_x^\varepsilon u'(x)\,\mathrm{d}x = \int_x^\varepsilon \frac{1}{\sqrt{x}}\sqrt{x}u_1'(x)\,\mathrm{d}x$$
$$\Longrightarrow |u(\varepsilon) - u(x)| \leqslant \sqrt{\ln\varepsilon - \ln(x)}\|\sqrt{x}u_1'\|_{L_2(0,+\varepsilon)}.$$

It follows that the Friedrichs extension is the extension described by the boundary condition (4.5). □

5. The essential spectrum: the quasi-regularity conditions are not fulfilled

This is the main section of the article and it is devoted to the calculation of the essential spectrum of any self-adjoint operator associated with the differential expression L. This problem has been solved in the case in which the quasi-regularity conditions are satisfied [17]. These conditions guarantee that the essential spectrum of the operator is bounded. Therefore, in this section we concentrate our attention on the case where the quasi-regularity conditions (1.4) are not satisfied, but conditions (2.1)–(2.3) are fulfilled.

THEOREM 5.1. *Let $\boldsymbol{L}_{\min}$ be semi-bounded from below (i.e. let the assumptions of lemma 2.2 be satisfied). Suppose that the quasi-regularity conditions (1.4) are not satisfied. Then the essential spectrum of any self-adjoint extension $\boldsymbol{L}$ of the operator $\boldsymbol{L}_{\min}$ is given by*

$$\sigma_{\mathrm{ess}}(\boldsymbol{L}) = \overline{\mathcal{R}\left\{\frac{m-(\beta^2/\rho)}{x^2}\right\}}. \tag{5.1}$$

Proof. Let us make the change of variables

$$x \to y \begin{cases} x = \mathrm{e}^{-y}, \\ \mathrm{d}x = -\mathrm{e}^{-y}\,\mathrm{d}y = -x\,\mathrm{d}y, \end{cases} \tag{5.2}$$

which transforms the interval $[0,1]$ into the semi-axis $[0,\infty)$. The points 0 and 1 are mapped into the points ∞ and 0 respectively. This change of variables determines the following unitary correspondence between the Hilbert spaces $L_2(0,1)$ and $L_2(0,\infty)$:

$$\left.\begin{aligned} \Phi : \psi(x) &\mapsto \tilde{\psi}(y) = \psi(\mathrm{e}^{-y})\mathrm{e}^{-y/2}; \\ \Phi^{-1} : \tilde{\psi}(y) &\mapsto \psi(x) = \frac{1}{\sqrt{x}}\tilde{\psi}(-\ln x). \end{aligned}\right\} \tag{5.3}$$

The differential operator L is transformed into the following differential operator, acting on two-component functions on $[0,\infty)$:

$$K = \begin{pmatrix} -\dfrac{\mathrm{d}}{\mathrm{d}y}\dfrac{\rho}{x^2}\dfrac{\mathrm{d}}{\mathrm{d}y} + \left(q(x) + \dfrac{\rho'_x}{2x} - \dfrac{3\rho}{4x^2}\right) & -\dfrac{\mathrm{d}}{\mathrm{d}y}\dfrac{\beta}{x^2} + \dfrac{\beta}{2x^2} \\ \dfrac{\beta}{x^2}\dfrac{\mathrm{d}}{\mathrm{d}y} + \dfrac{\beta}{2x^2} & \dfrac{m}{x^2} \end{pmatrix} \equiv \begin{pmatrix} A & C^* \\ C & D \end{pmatrix}. \tag{5.4}$$

In what follows we use both variables x and y simultaneously, and hope that this will not lead to misunderstanding.

Let us consider the minimal (symmetric) operator $\boldsymbol{K}_{\min}$ as the closure of the differential operator K considered on the domain of functions from $C_0^\infty[0,\infty) \oplus C_0^\infty[0,\infty)$ (functions which are arbitrarily many times differentiable with compact support on $[0,\infty)$ not necessarily separated from the origin) satisfying the standard boundary condition at the origin, which can be recalculated from (1.8):

$$\tilde{\omega}_U(0) = \tilde{h}_1\tilde{u}_1(0), \quad \tilde{h} \in \mathbb{R} \cup \{\infty\}. \tag{5.5}$$

The analysis of the operator $\boldsymbol{K}_{\min}$ is equivalent to the analysis of the operator $\boldsymbol{L}_{\min}$ carried out in the preceding sections, since these two operators are connected by the unitary transformation (5.3). Hence, the deficiency indices of the operator $\boldsymbol{K}_{\min}$ are $(0,0)$ or $(1,1)$, depending on the properties of the coefficients as $y \to \infty$. It is not difficult to reformulate these conditions, but we will not do that, since our aim is to calculate the essential spectrum, which does not depend on the particular extension of the minimal operator; all extensions have just the same essential spectrum, since the deficiency indices are finite.

Consider the resolvent equation

$$(\boldsymbol{K}_{\min} - \mu)^{-1} F = U$$

for sufficiently small negative values of $\mu \ll -1$. For smooth F and U,

$$F \in \mathcal{R}(\boldsymbol{K}_{\min}|_{\mathrm{dom}(\boldsymbol{K}_{\min}) \cap C_0^\infty[0,\infty) \oplus C_0^\infty[0,\infty)}) \quad \text{and} \quad U \in C_0^\infty[0,\infty) \oplus C_0^\infty[0,\infty),$$

the equation can be written as

$$f_1 = (A-\mu)u_1 + C^* u_2, \qquad f_2 = Cu_1 + (D-\mu)u_2.$$

Using the fact that the operator $D - \mu = (m/x^2) - \mu$ is invertible for sufficiently small negative $\mu \ll -1$ ($m|_{x=0} > 0$ or $m'|_{x=0} > 0$ if $m|_{x=0} = 0$), component u_2 can be excluded from the system by first resolving the second equation,

$$u_2 = (D-\mu)^{-1} f_2 - (D-\mu)^{-1} C u_1,$$

and then substituting this expression into the first equation

$$f_1 = ((A-\mu) - C^*(D-\mu)^{-1}C)u_1 + C^*(D-\mu)^{-1} f_2.$$

Hence, in order to calculate u_1, we need to invert the so-called Hain–Lüst operator

$$T(\mu) = (A - \mu I) - C^*(D-\mu I)^{-1} C.$$

Let us consider the minimal operator $T_{\min}(\mu)$ corresponding to this differential expression, defined on the functions from the domain $C_0^\infty(0,\infty)$. This operator can be written in the form

$$\begin{aligned} T_{\min}(\mu)(\mu) &= -\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\rho}{x^2} - \frac{\beta^2}{x^2(m-\mu x^2)}\right)\frac{\mathrm{d}}{\mathrm{d}y} \\ &\quad + \left\{ q(x) + \frac{\rho_x'}{2x} - \frac{3\rho}{4x^2} - \frac{\beta^2}{4x^2(m-\mu x^2)} - x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\beta^2}{2x^2(m-\mu x^2)}\right) - \mu \right\} \\ &= -\frac{\mathrm{d}}{\mathrm{d}y} V_\mu \frac{\mathrm{d}}{\mathrm{d}y} + W_\mu, \end{aligned} \tag{5.6}$$

where we use the following notation:

$$V_\mu = \frac{\hat{v}_\mu}{x^2}, \qquad W_\mu = \frac{1}{4}\frac{\hat{v}_\mu}{x^2} + \frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\hat{v}_\mu}{x^2}\right) + q(x) - \mu, \tag{5.7}$$

$$\hat{v}_\mu = \frac{\rho m - \beta^2 - \rho\mu x^2}{m - \mu x^2}. \tag{5.8}$$

In what follows it will be convenient to separate the three possible sets of parameters which guarantee that the operator $\boldsymbol{L}_{\min}$ is semi-bounded from below and the quasi-regularity conditions are not satisfied.

(A)

$$(\rho m - \beta^2)|_{x=0} > 0. \tag{5.9}$$

(This condition implies in particular that $m(0) > 0$.)

(B)

$$(\rho m - \beta^2)|_{x=0} = 0, \quad (\rho m - \beta^2)'_x|_{x=0} > 0 \quad \text{and} \quad m(0) = 0. \tag{5.10}$$

(This condition implies in particular that $m'_x(0) > 0$.)

(C)

$$(\rho m - \beta^2)|_{x=0} = 0, \quad (\rho m - \beta^2)'|_{x=0} > 0 \quad \text{and} \quad m(0) > 0. \tag{5.11}$$

In what follows we refer to these as cases A, B or C.

LEMMA 5.2. *Under conditions of theorem 5.1 the action of the operator $T_{\min}(\mu)$ can be written using one of the following two representations:*

$$T_{\min}(\mu) = \mathrm{e}^y\left(- \frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} + w_\mu \right)\mathrm{e}^y, \tag{5.12}$$

with

$$w_\mu = \tfrac{1}{4}\hat{v}_\mu - \tfrac{1}{2}x(\hat{v}_\mu)'_x + (q-\mu)x^2, \qquad v_\mu = \hat{v}_\mu, \tag{5.13}$$

and

$$T_{\min}(\mu) = \mathrm{e}^{y/2}\left(- \frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} + w_\mu \right)\mathrm{e}^{y/2}, \tag{5.14}$$

with

$$w_\mu = (q-\mu)x, \qquad v_\mu = \frac{\hat{v}_\mu}{x}. \tag{5.15}$$

REMARK 5.3. The first representation, (5.12), will be used in cases A and B. In case C the function $\hat{v}_\mu$ is vanishing at zero and therefore it is natural to use the function $v_\mu = \hat{v}_\mu/x$ (instead of $v_\mu = \hat{v}_\mu$). This leads to the second representation, (5.14). Therefore, in what follows we use the definition (5.13) for the function v_μ in cases A and B and definition (5.15) in case C.

Proof of lemma 5.2. We prove representations (5.12) and (5.14), separately starting from the former. Consider (5.6) for the Hain–Lüst operator,

$$T_{\min}(\mu) = -\frac{\mathrm{d}}{\mathrm{d}y}\frac{v_\mu}{x^2}\frac{\mathrm{d}}{\mathrm{d}y} + W_\mu = -\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^y v_\mu \mathrm{e}^y \frac{\mathrm{d}}{\mathrm{d}y} + W_\mu.$$

Using the following commutation relation for the operator of multiplication by a certain differentiable function $\varphi(y)$ and the operator of the first differentiation

$$\frac{\mathrm{d}}{\mathrm{d}y}\varphi = \varphi\frac{\mathrm{d}}{\mathrm{d}y} + \varphi'_y, \tag{5.16}$$

the expression for $T_{\min}(\mu)$ can be transformed as follows:

$$
\begin{aligned}
T_{\min}(\mu) & \\
&= -\mathrm{e}^y \frac{\mathrm{d}}{\mathrm{d}y} v_\mu \mathrm{e}^y \frac{\mathrm{d}}{\mathrm{d}y} - \mathrm{e}^y v_\mu \mathrm{e}^y \frac{\mathrm{d}}{\mathrm{d}y} + W_\mu \\
&= -\mathrm{e}^y \frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} \mathrm{e}^y - \mathrm{e}^y v_\mu \frac{\mathrm{d}}{\mathrm{d}y} \mathrm{e}^y + \mathrm{e}^y \frac{\mathrm{d}}{\mathrm{d}y} v_\mu \mathrm{e}^y + \mathrm{e}^y v_\mu \mathrm{e}^y + W_\mu \\
&= \mathrm{e}^y \bigg(-\frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} + (v_\mu)'_y + v_\mu + x \bigg(\frac{1}{4} \frac{v_\mu}{x^2} + \frac{1}{2} x \frac{\mathrm{d}}{\mathrm{d}x} \bigg(\frac{v_\mu}{x^2} \bigg) + q(x) - \mu \bigg) x \bigg) \mathrm{e}^y .
\end{aligned}
$$

Taking into account the fact that

$$
\frac{\mathrm{d}}{\mathrm{d}y} = -x \frac{\mathrm{d}}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{v_\mu}{x^2} \right) = \frac{(v_\mu)'_x}{x^2} - \frac{2 v_\mu}{x^3}
$$

we get the desired representation,

$$
T_{\min}(\mu) = \mathrm{e}^y \Big(-\frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} + \tfrac{1}{4} v_\mu - \tfrac{1}{2} x (v_\mu)'_x + (q - \mu) x^2 \Big) \mathrm{e}^y .
$$

To get the representation (5.14) we use similar calculations to obtain first

$$
T_{\min}(\mu) = \mathrm{e}^{y/2} \Big(-\frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} + \tfrac{1}{2} (v_\mu)'_y + \tfrac{1}{4} v_\mu + x W_\mu \Big) \mathrm{e}^{y/2},
$$

and then

$$
T_{\min}(\mu) = \mathrm{e}^{y/2} \Big(-\frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} + (q - \mu) x \Big) \mathrm{e}^{y/2} .
$$

□

The following lemma proves that the function v_μ is always positive definite for negative μ with sufficiently large absolute value.

LEMMA 5.4. *Let conditions of theorem 5.1 be satisfied. Then the function v_μ is positive definite for sufficiently small $\mu \ll -1$, i.e. there exist $c > 0$ and $\mu_0 \in \mathbb{R}$, such that*

$$
\mu \leqslant \mu_0 \quad \Longrightarrow \quad v_\mu(x) \geqslant c. \tag{5.17}
$$

Proof of lemma 5.4. The function v_μ is given by different formulae (5.13) and (5.15) in cases A–C. Therefore, let us separate the proof into three parts corresponding to these three situations.

CASE A. Let condition (5.9) be satisfied. Then the function v_μ is given by

$$
v_\mu = \frac{\rho m - \beta^2 - \rho \mu x^2}{m - \mu x^2}. \tag{5.18}
$$

Choose negative μ satisfying the following two inequalities:

$$
\mu < \frac{m(x)}{x^2} \quad \text{and} \quad \mu < \frac{1}{\rho_0} \frac{\rho(x) m(x) - \beta^2(x)}{x^2}. \tag{5.19}
$$

This is possible, since the functions $m(x)$ and $\rho(x)m(x) - \beta^2(x)$ are continuous in $[0,1]$ and attain positive values at the origin. This implies in particular that the functions are positive in a certain interval $[0,\varepsilon]$, $\varepsilon > 0$. Therefore, with regard to negative values of the spectral parameter μ, it can be chosen satisfying the inequalities (5.19) in the interval $x \in [\varepsilon, 1]$, where the quotients are continuous functions and therefore are bounded from below. (The same reasoning will be used in cases B and C below.)

Under these conditions both the numerator and denominator of the function v_μ are continuous positive definite functions. Thus, the function v_μ is also positive definite.

CASE B. Let condition (5.10) be satisfied. The function v_μ is again given by (5.18). Choose μ satisfying the following two inequalities

$$\mu < \frac{m'(x)}{2x} \quad \text{and} \quad \mu < \frac{1}{\rho_0}\frac{(\rho(x)m(x) - \beta^2(x))'}{2x}, \tag{5.20}$$

which is possible, since the functions $m'(x)$ and $(\rho(x)m(x) - \beta^2(x))'$ are continuous and attain positive values at the origin.

Under these conditions the function v_μ is given by a quotient of two functions which are positive definite for any positive x. Moreover, the limit of v_μ as $x \to 0$ is positive:

$$\lim_{x\to 0} v_\mu(x) = \frac{(\rho m - \beta^2)'_x|_{x=0}}{m'_x|_{x=0}} > 0.$$

Therefore, the function v_μ can be considered as a continuous function on the compact interval $[0,1]$ and therefore attains its minimum, which is clearly positive, since the function has positive limits at the endpoints of the interval $[0,1]$ and is positive everywhere inside the open interval $(0,1)$. Therefore, this function is positive definite.

CASE C. Let condition (5.11) be satisfied. The function v_μ is now given by formula (5.15):

$$v_\mu = \frac{\rho m - \beta^2 - \rho\mu x^2}{x(m - \mu x^2)}.$$

Choose μ satisfying the two inequalities

$$\mu < \frac{m(x)}{x^2} \quad \text{and} \quad \mu < \frac{1}{\rho_0}\frac{(\rho(x)m(x) - \beta^2(x))'}{2x} \tag{5.21}$$

and apply the same arguments as in cases A and B.

The function v_μ is then again given by a quotient of two functions which are positive for any $x \in (0,1]$. Moreover, the limit of v_μ as $x \to 0$ is positive:

$$\lim_{x\to 0} v_\mu(x) = \frac{(\rho m - \beta^2)'_x|_{x=0}}{m(0)} > 0.$$

The same argument as in case B leads to the conclusion that the function v_μ is positive definite. □

LEMMA 5.5. *Let the conditions of theorem 5.1 be satisfied. Then in cases A and B the function w_μ is positive definite for sufficiently small $\mu \ll -1$, i.e. the inequality*

$$w_\mu(x) \geqslant c > 0 \tag{5.22}$$

is satisfied with a certain positive constant c. In case C the function w_μ/x is positive definite, i.e. the inequality

$$w_\mu(x) \geqslant cx, \tag{5.23}$$

is satisfied, where c is a certain positive constant.

Proof of lemma 5.5. Let us consider cases A–C separately.

CASES A AND B. The function w_μ is given by

$$w_\mu = \tfrac{1}{4}\hat{v}_\mu - \tfrac{1}{2}x(\hat{v}_\mu)'_x + (q-\mu)x^2.$$

The function $h(x) \equiv \frac{1}{4}\hat{v}_\mu - \frac{1}{2}x(\hat{v}_\mu)'_x$ is a continuous function on $[0,1]$ attaining positive value at the origin $h(0) = \frac{1}{4}v_\mu(0) > 0$. Therefore, there exists μ_1 such that $h(x) - \mu_1 x^2$ is positive definite, i.e. there exists $c > 0$ such that $h(x) - \mu_1 x^2 > c$. It follows that for $\mu < \mu_1 - \|q\|_\infty$ the function w_μ is positive definite, i.e. satisfies (5.22).

CASE C. The function w_μ is now given by

$$w_\mu = (q-\mu)x.$$

Choosing $\mu < -\|q\|_\infty$ we guarantee that $w_\mu > cx$, where c is a certain positive constant. □

LEMMA 5.6. *Let the conditions of theorem 5.1 be satisfied. The operator $T_{\min}(\mu)(\mu)$ is then positive definite for all sufficiently small $\mu \ll -1$ uniformly with respect to μ, i.e. the following estimate is valid*

$$T_{\min}(\mu) \geqslant c > 0. \tag{5.24}$$

Proof of lemma 5.6. CASES A AND B. Consider the quadratic form of the operator $T_{\min}(\mu)$. Let $u \in C_0^\infty$. Then

$$\begin{aligned}\langle u, T_{\min}(\mu)u\rangle &= \left\langle \frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^y u, v_\mu \frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^y u\right\rangle + \langle \mathrm{e}^y u, w_\mu \mathrm{e}^y u\rangle \\ &\geqslant \min w_\mu(x)\|\mathrm{e}^y u\|^2 \geqslant c\|u\|^2,\end{aligned}$$

where the constant c is taken from the estimate (5.22) from lemma 5.5. The estimate is valid uniformly with respect to μ, provided it is sufficiently small.

CASE C. The quadratic form may be estimated as

$$\begin{aligned}\langle u, T_{\min}(\mu)u\rangle &= \left\langle \frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^{y/2} u, v_\mu \frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^{y/2} u\right\rangle + \langle \mathrm{e}^{y/2} u, w_\mu \mathrm{e}^{y/2} u\rangle \\ &\geqslant \inf \frac{w_\mu(x)}{x}\|u\|^2 \geqslant c\|u\|^2,\end{aligned}$$

where c is the positive constant from (5.23). □

This lemma implies in particular that the operator $T_{\min}(\mu)$ is boundedly invertible. Let us denote by $T(\mu)$ the Friedrichs extension of the minimal operator $T_{\min}(\mu)$: an extension having the same lower bound as the minimal operator. This extension will be called Hain–Lüst operator in what follows.

Consider the resolvent operator

$$\begin{aligned} M(\mu) &\equiv (K_{\min} - \mu)^{-1} \\ &= \begin{pmatrix} T^{-1}(\mu) & -T^{-1}(\mu)[C^*(D-\mu I)^{-1}] \\ -[(D-\mu I)^{-1}C]T^{-1}(\mu) & (D-\mu I)^{-1} + [(D-\mu I)^{-1}C]T^{-1}(\mu)[C^*(D-\mu I)^{-1}] \end{pmatrix} \\ &=: \begin{pmatrix} G_{11}(\mu) & G_{12}(\mu) \\ G_{21}(\mu) & G_{22}(\mu) \end{pmatrix}. \end{aligned} \tag{5.25}$$

In what follows we show that the operators $G_{11}(\mu)$, $G_{12}(\mu)$ and $G_{21}(\mu)$ are compact for sufficiently small negative values of the spectral parameter μ, which will imply that the essential spectrum of $M(\mu)$ is determined exclusively by $G_{22}(\mu)$. (This is in contrast to the case where the quasi-regularity conditions (1.4) are satisfied. In that case the operator $G_{11}(\mu)$ also contributes to the essential spectrum [17].)

LEMMA 5.7. *Let conditions of theorem 5.1 hold. The operator $G_{11}(\mu) \equiv T_{\min}^{-1}(\mu)$ is then compact for sufficiently small values of $\mu \ll -1$.*

Proof of lemma 5.7. To prove this lemma we will show that the operator $T_{\min}^{-1}(\mu)$ maps every bounded set to a compact set. In other words, we will show that every set bounded in the graph norm of $T_{\min}(\mu)$ is compact in $L_2[0,\infty)$ for sufficiently small negative values of μ. Let u, $\|u\| \leqslant 1$ be mapped by $T_{\min}^{-1}(\mu)$ to v,

$$v = T_{\min}^{-1}(\mu)u \quad \Longrightarrow \quad T_{\min}(\mu)v = u \quad \Longrightarrow \quad \langle v, T_{\min}(\mu)v\rangle = \langle v, u\rangle.$$

Then lemma 5.6 implies that every set bounded in the graph norm is bounded in the norm associated with the quadratic form of the operator

$$c\|v\|^2 \leqslant |\langle v, T_{\min}(\mu)v\rangle| \leqslant \|v\|\|u\| \quad \Longrightarrow \quad \langle v, T_{\min}(\mu)v\rangle \leqslant C,$$

since $\|u\| \leqslant 1$. We conclude that, in order to prove this lemma it is sufficient to show that every subset of $\operatorname{dom}(T_{\min}(\mu))$ bounded with respect to the quadratic form $\langle v, T_{\min}(\mu)v\rangle$ is relatively compact in the Hilbert space $L_2[0,\infty)$. In other words, it is sufficient to show that the set of functions

$$\left\{ u \in C_0^\infty[0,\infty) \;\middle|\; u(0) = 0,\ \left\langle \mathrm{e}^{\alpha y}\left(-\frac{\mathrm{d}}{\mathrm{d}y}v_\mu\frac{\mathrm{d}}{\mathrm{d}y} + w_\mu\right)\mathrm{e}^{\alpha y}u, u\right\rangle \leqslant 1 \right\}$$

is relatively compact, where $\alpha = 1$ in cases A and B, and $\alpha = \frac{1}{2}$ in case C. Taking into account estimates on the functions v_μ and w_μ (lemmas 5.4 and 5.5) we conclude that, for sufficiently small μ, to prove the compactness of $T_{\min}(\mu)$ it is sufficient to show the relative compactness of the following sets.

For cases A and B

$$S \equiv \left\{ u \in C_0^\infty[0,\infty) \;\middle|\; u(0) = 0,\ \left\|\frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^y u)\right\|^2 + \|\mathrm{e}^y u\|^2 \leqslant 1 \right\} \tag{5.26}$$

and for case C

$$S \equiv \left\{ u \in C_0^\infty[0,\infty) | u(0) = 0, \ \left\| \frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^{y/2}u) \right\|^2 + \|u\|^2 \leqslant 1 \right\}. \tag{5.27}$$

Let us construct a compact ε-net for these sets. Consider a set of cut-off functions χ_N with the following properties

$$\chi_N \in C_0^\infty[0,\infty), \quad \chi_N(y) = \begin{cases} 1, & y \leqslant N, \\ 0, & y \geqslant N+1. \end{cases}$$

Then the tails $(1-\chi_N)u$ of functions u from the sets S can be estimated as follows.

For cases A and B,

$$\|\mathrm{e}^y u\| \leqslant 1 \quad \Longrightarrow \quad \mathrm{e}^N \|(1-\chi_N)u\| \leqslant 1 \quad \Longrightarrow \quad \|(1-\chi_N)u\| \leqslant \mathrm{e}^{-N}.$$

For case C,

$$\begin{aligned} &\mathrm{e}^{y/2}u(y) \\ &\quad = \int_0^y (\mathrm{e}^{\tilde{y}/2}u(\tilde{y}))' \,\mathrm{d}\tilde{y} \\ &\quad \Longrightarrow |\mathrm{e}^{y/2}u(y)| \leqslant \int_0^y |(\mathrm{e}^{\tilde{y}/2}u(\tilde{y}))'| \,\mathrm{d}\tilde{y} \leqslant \sqrt{\int_0^y 1 \,\mathrm{d}\tilde{y} \int_0^y |(\mathrm{e}^{\tilde{y}/2}u(\tilde{y}))'|^2 \,\mathrm{d}\tilde{y}} \leqslant \sqrt{y} \\ &\quad \Longrightarrow |u(y)| \leqslant \sqrt{y}\mathrm{e}^{-y/2} \Longrightarrow \int_N^\infty |u(y)|^2 \,\mathrm{d}y \leqslant (N+1)\mathrm{e}^{-N} \\ &\quad \Longrightarrow \|(1-\chi_N)u\| \leqslant \sqrt{N+1}\mathrm{e}^{-N/2}. \end{aligned}$$

These estimates show that, taking sufficiently large N, the sets $\chi_N S \equiv \{\chi_N u \mid u \in S\}$ approximate the sets S with arbitrary precision ε. The sets $\chi_N S$ are bounded in the metrics of $W_2^1[0, N+1]$. Since the embedding of $W_2^1[0, N+1]$ into $L_2[0, N+1]$ is compact for any finite N (Rellich's theorem), these sets form a compact ε-net for the set S. Hence, the operator $T_{\min}(\mu)$ is compact. □

LEMMA 5.8. *Under the conditions of theorem 5.1, the operators*

$$\frac{\mathrm{d}}{\mathrm{d}y} T_{\min}^{-1}(\mu) \quad \textit{and} \quad T_{\min}^{-1}(\mu) \frac{\mathrm{d}}{\mathrm{d}y}$$

are compact for sufficiently small negative values of μ.

Proof of lemma 5.8. The two operators under consideration,

$$\frac{\mathrm{d}}{\mathrm{d}y} T_{\min}^{-1}(\mu) \quad \text{and} \quad T_{\min}^{-1}(\mu) \frac{\mathrm{d}}{\mathrm{d}y},$$

are formally mutually adjoint. Therefore, it is sufficient to proof the compactness of only one of them, say $\mathrm{d}T_{\min}^{-1}(\mu)/\mathrm{d}y$. This operator is compact if and only if it maps every bounded set, say $B_1 \equiv \{u; \|u\| \leqslant 1\}$ onto a compact set. For arbitrary u, consider the function $v = T_{\min}^{-1}(\mu)u$. Then the operator $\mathrm{d}T_{\min}^{-1}(\mu)/\mathrm{d}y$ is compact if the sets S determined by (5.26) and (5.27) are relatively compact in the norm of

$W_2^1[0,\infty)$. In the proof of lemma 5.7 we showed that the sets $\chi_N S$ form an ε-net for the set S in the norm of $L_2[0,\infty)$. Let us show that these sets form an ε-net even in the norm of $W_2^1[0,\infty)$. It remains to show that, for sufficiently large N, the first derivatives can be estimated uniformly. Let $\alpha = 1$ in cases A and B and let $\alpha = \frac{1}{2}$ in case C. Then we have

$$\begin{aligned}\left\|\frac{\mathrm{d}}{\mathrm{d}y}[(1-\chi_N)v]\right\| &= \left\|\frac{\mathrm{d}}{\mathrm{d}y}[(1-\chi_N)\mathrm{e}^{-\alpha y}\mathrm{e}^{\alpha y}v]\right\| \\ &\leqslant \|(1-\chi_N)'v\| + \|(1-\chi_N)(-\alpha)v\| + \left\|(1-\chi_N)\mathrm{e}^{-\alpha y}\frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^{\alpha y}v)\right\| \\ &\leqslant (c+\alpha)\|v\|_{L_2[N,\infty)} + \mathrm{e}^{-\alpha N}\left\|\frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^{\alpha y}v)\right\|,\end{aligned}$$

where $c = \max|\chi_N'|$. We have already proven that the first term tends to zero as $N\to\infty$. The norms in the second term are uniformly bounded, since, by lemmas 5.4 and 5.5,

$$\langle T_{\min}(\mu)v, v\rangle \geqslant c\left\|\frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^{y}v)\right\|.$$

We conclude that the sets $\chi_N S$ form ε-net for the sets S.

It remains to prove that these sets $\chi_N S$ are relatively compact. In cases A and B the following estimate holds, due to lemmas 5.4 and 5.5:

$$\langle T_{\min}(\mu)v, v\rangle \geqslant c\left(\left\|\frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^{y}u)\right\|^2 + \|\mathrm{e}^{y}u\|^2\right).$$

This estimate implies that $T_{\min}^{-1/2}(\mu)$ is a bounded operator from $L_2[0,N]$ onto $W_2^1[0,N]$. It follows that the operator $T_{\min}^{-1}(\mu)$ is a bounded operator from $L_2[0,N]$ onto $W_2^2[0,N]$. Therefore, the operator $\mathrm{d}T_{\min}^{-1}(\mu)/\mathrm{d}y$ is a bounded operator from $L_2[0,N]$ onto $W_2^1[0,N]$ and it is a compact operator as an operator in $L_2[0,N]$, since the embedding of $W_2^1[0,N]$ in $L_2[0,N]$ is compact. The proof of case C is similar. □

The last two lemmas imply that the operators $G_{12}(\mu)$ and $G_{21}(\mu)$ are compact. Indeed, the operator $G_{12}(\mu)$ can be written as

$$G_{12}(\mu) = T_{\min}^{-1}(\mu)\frac{\mathrm{d}}{\mathrm{d}y}\frac{\beta}{m-\mu x^2} - T_{\min}^{-1}(\mu)\frac{\beta}{2(m-\mu x^2)}.$$

In cases A and C the C^1-function $m(x)$ has a positive value at the origin. Hence, the function $m-\mu x^2$ is positive definite for sufficiently small values of μ. Therefore, the operator $B = \beta/(m-\mu x^2)$ is bounded. In case B the derivative $m'(0)$ is positive and therefore the function $m-\mu x^2$ has a first-order zero at the origin. The function $\beta(x)$ has also at least first-order zero at the origin and therefore the function B is uniformly bounded for sufficiently small $\mu \ll -1$. Thus, the operator

$$G_{12}(\mu) = T_{\min}^{-1}(\mu)\frac{\mathrm{d}}{\mathrm{d}y}B - \tfrac{1}{2}T_{\min}^{-1}(\mu)B$$

is compact as well as the operator

$$G_{21}(\mu) = -B\frac{\mathrm{d}}{\mathrm{d}y}T^{-1}_{\min}(\mu) - \tfrac{1}{2}BT^{-1}_{\min}(\mu).$$

Thus, the essential spectrum of the matrix operator $M(\mu)$ coincides with the essential spectrum of the operator $G_{22}(\mu)$ up to the spectral point $\lambda = 0$, which may be ignored during the calculation of the essential spectrum of the operator K, because it in fact corresponds to the spectral point ∞ for the operator $\boldsymbol{L}$.

In our calculations we will use Calkin calculus. We say that any two operators A and B are equal in Calkin algebra if their difference is a compact operator. The following notation for the equivalence relation in Calkin algebra will be used throughout the paper:

$$A \doteq B.$$

Consider the cases A and B first. In these cases the operator $G_{22}(\mu)$ has the form

$$\begin{aligned} G_{22}(\mu) &= \frac{1}{D-\mu I} + \frac{1}{D-\mu I}CT^{-1}(\mu)C^*\frac{1}{D-\mu I} \\ &= \frac{x^2}{m-\mu x^2} + \frac{\beta}{m-\mu x^2}\Big(\frac{\mathrm{d}}{\mathrm{d}y}+\frac{1}{2}\Big)T^{-1}(\mu)\Big(-\frac{\mathrm{d}}{\mathrm{d}y}+\frac{1}{2}\Big)\frac{\beta}{m-\mu x^2} \\ &\doteq \frac{x^2}{m-\mu x^2} - \frac{\beta}{m-\mu x^2}\frac{\mathrm{d}}{\mathrm{d}y}T^{-1}(\mu)\frac{\mathrm{d}}{\mathrm{d}y}\frac{\beta}{m-\mu x^2} \\ &\doteq \frac{x^2}{m-\mu x^2} + \frac{\beta \mathrm{e}^{-y}}{m-\mu x^2}\frac{\mathrm{d}}{\mathrm{d}y}\Big(\frac{\mathrm{d}}{\mathrm{d}y}v_\mu\frac{\mathrm{d}}{\mathrm{d}y} - w_\mu\Big)^{-1}\frac{\mathrm{d}}{\mathrm{d}y}\frac{\beta \mathrm{e}^{-y}}{m-\mu x^2}, \end{aligned}$$

where we have used representation (5.12), commutation relation (5.16) and the compactness of the operators

$$\frac{\beta}{m-\mu x^2}\mathrm{e}^{-y}T^{-1}(\mu)\frac{\mathrm{d}}{\mathrm{d}y}\frac{\beta}{m-\mu x^2}$$

and

$$\frac{\beta}{m-\mu x^2}\frac{\mathrm{d}}{\mathrm{d}y}T^{-1}(\mu)\mathrm{e}^{-y}\frac{\beta}{m-\mu x^2}.$$

In what follows we will use the calculus of pseudodifferential operators and therefore the Fourier transform on $\mathbb{R}$. Let us consider a new operator $M(\mu)$ defined in the space $L_2(\mathbb{R}) \oplus L_2(\mathbb{R})$ by the same matrix differential expression (5.25), where all functions involved are extended as even functions to the whole real axis. The essential spectrum of the new operator coincides up to multiplicity with the essential spectrum of the original operator [24]. Using the operator $p = (1/\mathrm{i})\mathrm{d}/\mathrm{d}y$, the operator $\hat{G}_{22}(\mu)$, the extension of the operator $G_{22}(\mu)$ to the whole real line, can now be written as

$$\hat{G}_{22} \doteq \frac{x^2}{m-\mu x^2} + \frac{\beta \mathrm{e}^{-y}}{m-\mu x^2}p\frac{1}{pv_\mu p + w_\mu}p\frac{\beta \mathrm{e}^{-y}}{m-\mu x^2}.$$

Let us use the following fact, proven in [17].

PROPOSITION 5.9 (Kurasov and Naboko [17, lemma 8]). *Let the real-valued function $f(y)$ be uniformly positive and bounded,*

$$0 < c \leqslant f(y) \leqslant C,$$

for some $c, C \in \mathbb{R}_+$. Let the function $g(y)$ be bounded and the operator

$$L \equiv p f(y) p + g(y)$$

be self-adjoint and invertible in $L_2(\mathbb{R})$. Suppose that the operator $pL^{-1}p$ be bounded. Then, for any bounded function $h(y)$ such that $\lim_{y\to\infty} h(y) = 0$, the following equality holds in Calkin algebra:

$$pL^{-1}ph \doteq \frac{h}{f}.$$

To apply this proposition, consider the function $h = \beta \mathrm{e}^{-y}/(m - \mu x^2)$. This function tends to zero as $y \to \infty (x \to 0)$, since in cases A and C the denominator is positive definite and in case C the function $\beta/(m - \mu x^2)$ is uniformly bounded.

Applying this proposition to the operator $\hat{G}_{22}$, we get

$$\begin{aligned} \hat{G}_{22} &\doteq \frac{x^2}{m - \mu x^2} + \frac{\beta x}{m - \mu x^2} \frac{1}{v_\mu} \frac{x\beta}{m - \mu x^2} \\ &= \frac{x^2}{m - \mu x^2} + \frac{\beta x}{m - \mu x^2} \frac{m - \mu x^2}{\rho m - \beta^2 - \rho\mu x^2} \frac{x\beta}{m - \mu x^2} \\ &= \frac{\rho x^2}{\rho m - \beta^2 - \rho\mu x^2} \\ &= \left(\frac{m}{x^2} - \frac{\beta^2}{\rho x^2} - \mu \right)^{-1}. \end{aligned} \tag{5.28}$$

Using Weyl's theorem for compact perturbations [15] and the Dunford spectral mapping theorem [5], we conclude that the essential spectrum of any self-adjoint extension of the operator $K_{\min}$ and therefore any self-adjoint extension of $L_{\min}$ can be calculated as

$$\sigma_{\mathrm{ess}}(L) = \mathrm{range} \left\{ \frac{m - (\beta^2/\rho)}{x^2} \right\}.$$

In case C, similar calculations can be carried out. The operator $G_{22}(\mu)$ given by

$$\begin{aligned} G_{22}(\mu) = \frac{x^2}{m - \mu x^2} &+ \frac{1}{m - \mu x^2} \left(\beta \frac{\mathrm{d}}{\mathrm{d}y} + \tfrac{1}{2}\beta \right) \mathrm{e}^{-y/2} \\ &\times \left(-\frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} + w_\mu \right)^{-1} \mathrm{e}^{-y/2} \left(-\frac{\mathrm{d}}{\mathrm{d}y}\beta + \tfrac{1}{2}\beta \right) \frac{1}{m - \mu x^2} \end{aligned}$$

may be continued as a pseudodifferential operator to the whole axis:

$$\begin{aligned} \hat{G}_{22}(\mu) &\doteq \frac{x^2}{m - \mu x^2} + \frac{1}{m - \mu x^2} \beta p \mathrm{e}^{-y/2} (p v_\mu p + w_\mu)^{-1} \mathrm{e}^{-y/2} p \beta \frac{1}{m - \mu x^2} \\ &\doteq \frac{x^2}{m - \mu x^2} + \frac{1}{m - \mu x^2} \beta \mathrm{e}^{-y/2} p (p v_\mu p + w_\mu)^{-1} p \mathrm{e}^{-y/2} \beta \frac{1}{m - \mu x^2} \end{aligned}$$

$$\begin{aligned}
&\doteq \frac{x^2}{m-\mu x^2} + \frac{1}{m-\mu x^2}\beta\sqrt{x}\frac{1}{v_\mu}\sqrt{x}\beta\frac{1}{m-\mu x^2}\\
&= \frac{x^2}{m-\mu x^2} + \frac{1}{m-\mu x^2}\beta^2\frac{x^2(m-\mu x^2)}{\rho m-\beta^2-\rho\mu x^2}\frac{1}{m-\mu x^2}\\
&= \left(\frac{m}{x^2} - \frac{\beta^2}{\rho x^2} - \mu\right)^{-1},
\end{aligned}$$

where we have again used the commutation relations (5.16) and representations (5.15) and (5.7) for the function v_μ. This accomplishes the proof. (Note that in the case considered the entries G_{11}, G_{12} and G_{21} are compact operators. This case differs drastically from that in which the quasi-regularity conditions are satisfied [17]. In the latter case a certain more detailed technique had to be used (the lemma on the essential spectrum of the triple sum of operators in Banach space [17, 25, 26]).) □

It is clear that the set

$$\mathcal{R}\left(\frac{\rho m-\beta^2}{\rho x^2}\right)$$

is unbounded from above in all three cases A–C. On the other hand, if the quasi-regularity conditions are satisfied, then the essential spectrum is bounded from above as well as from below but, in addition to the branch of the essential spectrum described, a new branch of the essential spectrum is present. This new branch of the essential spectrum has been called the singularity spectrum, since it cannot be obtained as a limit of the essential spectra of differential operators given by restrictions of the differential operators L to intervals $(\varepsilon, 1]$. This singularity spectrum is absent if the quasi-regularity conditions are not fulfilled. In the following section we discuss the relations between the singularity spectrum and properties of the Hain–Lüst operator.

6. Weyl circles for the Hain–Lüst operator and quasi-regularity conditions

In this section we investigate the relations between the quasi-regularity conditions and the properties of the Hain–Lüst operator, which is a second-order differential operator on the interval $x \in [0,1]$. This operator is given by the following differential expression:

$$T(\mu) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\rho m-\beta^2}{m-\mu x^2} - \frac{\rho\mu}{m-\mu x^2}x^2\right)\frac{\mathrm{d}}{\mathrm{d}x} + q(x) - \mu. \tag{6.1}$$

In order to avoid inessential difficulties, we study this operator for sufficiently small values of the parameter μ, i.e. we assume that $\mu \leqslant \mu_0 \ll -1$. The minimal operator $T_{\min}(\mu)$ determined by (6.1) is defined on $C_0^\infty(0,1)$, the set of smooth functions with compact support separated from the endpoints of the interval $x \in [0,1]$. The maximal operator, adjoint to $T_{\min}(\mu)$ in $L_2[0,1]$, is defined by the same differential expression (6.1) on the domain $\{f \in L_2[0,1] : T(\mu)f \in L_2[0,1]\}$. The operator $T(\mu)$ is formally symmetric and it can be made self-adjoint by introducing proper boundary conditions at the endpoints. The endpoint $x = 1$ is always regular and we assume that a certain symmetric condition is imposed at this point. The point

$x = 0$ is singular and, in order to investigate it, Weyl's limit point-limit circle theory [6, 28] will be used. These studies will tell us whether it is necessary to introduce an additional boundary condition at the origin in order to make $T(\mu)$ self-adjoint (provided μ is negative and sufficiently small). It will be more convenient to use the y-representation (5.2). In this representation the singular point $x = 0$ corresponds to $y = +\infty$.

THEOREM 6.1. *Let the standard assumptions (1.2) and (1.3) on the coefficients of the operator L given by (1.1) be satisfied. Suppose that the operator $\boldsymbol{L}$ is semi-bounded. Then the Hain–Lüst operator $T(\mu)$ is in the limit-point case at $x = 0$ ($y = \infty$) for sufficiently small values of μ, i.e. for $\mu \ll -1$, if and only if the quasi-regularity conditions (1.4) are satisfied, but the operator is not regular. In other words, the Hain–Lüst operator is in the limit circle case at $x = 0$ ($y = \infty$) for sufficiently small values of $\mu \ll -1$ if and only if either the quasi-regularity conditions (1.4) are not satisfied or the operator L is regular.*

Proof. We will consider five different cases covering all possible values of the coefficients. The cases A–C coincide with those introduced in the proof of theorem 5.1. Case D covers all coefficients which satisfy quasi-regularity conditions, but are not regular. The last case (case E) is added for the sake of completeness and corresponds to the regular operator L.

CASES A AND B.

Suppose that conditions (5.9) or (5.10) are satisfied. Then, in accordance with lemma 5.2, the operator $T(\mu)$ can be written in the form (5.12),

$$T(\mu) = \mathrm{e}^y \biggl(-\frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} + w_\mu \biggr) \mathrm{e}^y,$$

with the functions v_μ and w_μ given by (5.13). These functions satisfy the inequalities

$$v_\mu(x) \geqslant c > 0, \qquad w_\mu(x) \geqslant c > 0,$$

for sufficiently small values of μ, due to lemmas 5.4 and 5.5. Let us study the asymptotics as $y \to \infty$ of the solutions to the equation

$$T(\mu) f = \mathrm{i} f, \tag{6.2}$$

which is a second-order differential equation. Straightforward calculations transform this equation into the standard form

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} f + \biggl(2 + \biggl(\frac{\mathrm{d}}{\mathrm{d}y} v_\mu\biggr)\frac{1}{v_\mu}\biggr)\frac{\mathrm{d}}{\mathrm{d}y} f + \biggl(\frac{1}{2}\biggl(\frac{\mathrm{d}}{\mathrm{d}y} v_\mu\biggr)\frac{1}{v_\mu} + \frac{3}{4} + \frac{\mu - q + \mathrm{i}}{\mathrm{e}^{2y} v_\mu}\biggr) f = 0. \tag{6.3}$$

The functions $(\mathrm{d}v_\mu/\mathrm{d}y)/v_\mu$ and $(\mu - q + \mathrm{i})/\mathrm{e}^{2y} v_\mu$ are exponentially small as $y \to \infty$ and therefore it is natural to expect that the asymptotics of the solutions are just the same as for the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} f + 2\frac{\mathrm{d}}{\mathrm{d}y} f + \tfrac{3}{4} f = 0 \quad \Longrightarrow \quad f(y) = C_1 \mathrm{e}^{-y/2} + C_2 \mathrm{e}^{-3y/2}.$$

Let us discuss how to prove this fact for general second-order differential equation [20]

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}f + (a_1 + g_1(y))\frac{\mathrm{d}}{\mathrm{d}y}f + (a_2 + g_2(y))f = 0, \tag{6.4}$$

where a_1, a_2 are certain real constants and $g_{1,2}$ are real-valued functions tending to zero exponentially fast as $y \to \infty$. The following analysis is standard and is included to make our presentation self-contained. One may get rid of the first derivative by introducing the new function h as follows

$$f(y) = \exp\left\{ -\frac{1}{2}\left(a_1 y + \int^y g_1(\tilde{y})\,\mathrm{d}\tilde{y}\right)\right\} h(y). \tag{6.5}$$

Equation (6.4) transforms as

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}h + (-\tfrac{1}{4}(a_1 + g_1)^2 - \tfrac{1}{2}g'_{1y} + a_2 + g_2)h = 0. \tag{6.6}$$

It will be more convenient to introduce the following notation:

$$\left.\begin{aligned} k &= \sqrt{\tfrac{1}{4}a_1^2 - a_2}, \\ g &= \tfrac{1}{2}a_1 g_1 + \tfrac{1}{4}g_1^2 + \tfrac{1}{2}g'_{1y} - g_2. \end{aligned}\right\} \tag{6.7}$$

Using this notation, equation (6.6) can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}h - k^2 h - g(y)h = 0. \tag{6.8}$$

We are interested in proving that the solutions to this equation have the same asymptotics as the solution to the free equation

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}h - k^2 h = 0 \quad \Longrightarrow \quad h = C_1 \mathrm{e}^{kx} + C_2 \mathrm{e}^{-kx}.$$

Suppose that the potential g satisfies the estimate

$$|g(y)| \leqslant a(y)\mathrm{e}^{-2ky}, \tag{6.9}$$

where a is a certain L_1-function. Then the growing solution to equation (6.8) satisfies the Volterra-type equation

$$h(y) = \mathrm{e}^{-ky} + \int_y^\infty \frac{1}{2c}(\mathrm{e}^{k(y-\tilde{y})} - \mathrm{e}^{-k(y-\tilde{y})})g(\tilde{y})h(\tilde{y})\,\mathrm{d}\tilde{y}. \tag{6.10}$$

This integral equation has a solution having the asymptotics $h_1(y) \sim \mathrm{e}^{-ky}$ if the potential q satisfies the estimate (6.9) and $k > 0$. In this case the solution may be obtained by the method of successive approximations. Then it is straightforward to show that there is another linear independent solution with the asymptotics $h_2(y) \sim \mathrm{e}^{ky}$. This follows easily from the fact that h_1 and h_2 are solutions to one and the same second-order differential equation and therefore their Wronskian is constant.

Let us return to the study of equation (6.3). Comparison with equation (6.4) gives us

$$a_1 = 2, \quad g_1 = \left(\frac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\frac{1}{v_\mu},$$

$$a_2 = \tfrac{3}{4}, \quad g_2 = \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\frac{1}{v_\mu} + \frac{\mu - q + \mathrm{i}}{v_\mu}\mathrm{e}^{-2y},$$

$$\Longrightarrow \begin{cases} k = \sqrt{\tfrac{1}{4}4 - \tfrac{3}{4}} = \tfrac{1}{2}, \\ g = \dfrac{1}{2}\left(\dfrac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\dfrac{1}{v_\mu} + \dfrac{1}{4}\left(\left(\dfrac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\dfrac{1}{v_\mu}\right)^2 \\ \qquad + \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}y}\left(\left(\dfrac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\dfrac{1}{v_\mu}\right) - \dfrac{\mu - q + \mathrm{i}}{v_\mu}\mathrm{e}^{-2y}. \end{cases}$$

The potential g can be simplified as

$$g = \left(-\frac{1}{4}\left(\frac{v'_{\mu x}}{v_\mu}\right)^2 + \frac{1}{2}\frac{v''_{\mu xx}}{v_\mu} - \frac{\mu - q + \mathrm{i}}{v_\mu}\right)\mathrm{e}^{-2y}.$$

Since the function v_μ never vanishes for sufficiently small $\mu \ll -1$ and all functions are twice differentiable, the expression in brackets is uniformly bounded. It follows that g satisfies the estimate

$$|g(y)| \leqslant \text{const.}\, \mathrm{e}^{-2y}.$$

Hence, solutions to (6.8) have asymptotics

$$h_1 \sim \mathrm{e}^{y/2} \quad \text{and} \quad h_2 \sim \mathrm{e}^{-y/2}.$$

The corresponding solutions to (6.2) have the following behaviour:

$$f_1 \sim \mathrm{e}^{-y/2} \quad \text{and} \quad f_2 \sim \mathrm{e}^{-3y/2}. \tag{6.11}$$

Both solutions are square integrable in the neighbourhood of $y = +\infty$, i.e. the case of Weyl's limit circle occurs.

Case C. This case can be investigated using a similar method. The Hain–Lüst operator can be written in the form (5.14),

$$T(\mu) = \mathrm{e}^{y/2}\left(-\frac{\mathrm{d}}{\mathrm{d}y} v_\mu \frac{\mathrm{d}}{\mathrm{d}y} + w_\mu\right)\mathrm{e}^{y/2}$$

with the coefficients satisfying (5.15)

$$v_\mu \geqslant c > 0, \quad w_\mu \geqslant cx, \quad c > 0,$$

for sufficiently small values of μ. Then equation (6.2) can be transformed into

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} f + \left(1 + \left(\frac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\frac{1}{v_\mu}\right)\frac{\mathrm{d}}{\mathrm{d}y} f + \left(\frac{1}{4} + \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\frac{1}{v_\mu} + \frac{\mu - q + \mathrm{i}}{\mathrm{e}^y v_\mu}\right) f = 0.$$

Comparison with equation (6.4) gives

$$
\begin{aligned}
a_1 &= 1, \quad g_1 = \left(\frac{\mathrm{d}}{\mathrm{d}y} v_\mu\right) \frac{1}{v_\mu},\\
a_2 &= \tfrac{1}{4}, \quad g_2 = \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\frac{1}{v_\mu} + \frac{\mu - q + \mathrm{i}}{v_\mu}\mathrm{e}^{-y},\\
&\Longrightarrow \begin{cases} k = \sqrt{\tfrac{1}{4}1 - \tfrac{1}{4}} = 0,\\ g = \dfrac{1}{4}\left(\left(\dfrac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\dfrac{1}{v_\mu}\right)^2 + \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}y}\left(\left(\dfrac{\mathrm{d}}{\mathrm{d}y} v_\mu\right)\dfrac{1}{v_\mu}\right) - \dfrac{\mu - q + \mathrm{i}}{v_\mu}\mathrm{e}^{-y}. \end{cases}
\end{aligned}
$$

The potential g can be simplified as

$$g = \left(-\tfrac{1}{4}\mathrm{e}^{-y}\left(\frac{v'_{\mu x}}{v_\mu}\right)^2 + \frac{1}{2}\frac{v'_{\mu x}}{v_\mu} + \tfrac{1}{2}\mathrm{e}^{-y}\frac{v''_{\mu xx}}{v_\mu} - \frac{\mu - q + \mathrm{i}}{v_\mu}\right)\mathrm{e}^{-y}.$$

The expression in brackets is again uniformly bounded. It follows that g satisfies the estimate

$$|g(y)| \leqslant \text{const.}\, \mathrm{e}^{-y}.$$

In this case $k = 0$ and our analysis cannot be applied directly, but the modification needed is standard. The solution to the differential equation (6.8) satisfies the asymptotics

$$h \sim C_1 y + C_2.$$

The corresponding solutions to (6.2) have the following behaviour:

$$f \sim (C_1 y + C_2)\mathrm{e}^{-y/2}$$

and both solutions are square integrable in the neighbourhood of $y = +\infty$, i.e. the operator $T(\mu)$ is in the limit circle case.

Case D. Assume that the coefficients satisfy quasi-regularity conditions (1.4) but the operator is not regular. This implies in particular that $m|_{x=0} \neq 0$. Otherwise the quasi-regularity conditions would imply that $\beta|_{x=0} = 0$ and therefore m has second-order zero at the origin.

The coefficients V_μ and W_μ of the Hain–Lüst operator (5.6),

$$T(\mu) = -\frac{\mathrm{d}}{\mathrm{d}y} V_\mu \frac{\mathrm{d}}{\mathrm{d}y} + W_\mu,$$

are uniformly bounded and positive definite functions for sufficiently small values of $\mu \ll -1$. It follows that the quadratic form of $T(\mu)$ is equivalent to the quadratic form of the second-derivative operator $-(\mathrm{d}^2/\mathrm{d}y^2) + 1$. The latter operator is in the limit-point case at $y = +\infty$. Hence, the Hain–Lüst operator is in the limit-point case as well.

Case E. Suppose that the operator L is regular, i.e. the functions β and m have first-order and second-order zeros at the origin, respectively. In this case $\hat{v}_\mu$ is uniformly bounded and positive definite for sufficiently small values of μ, i.e. $\mu \ll -1$. Therefore, this case is similar to cases A and B considered above. The asymptotics of the solution is given by the same formula, (6.11), and all solutions are square integrable. □

Theorem 6.1 implies that the singularity spectrum for the operator L appears if and only if the Hain–Lüst operator is in the limit-point case at the singular point. We believe that this observation is crucial for the existence of the singularity spectrum, even in a more general setting. It is planned to continue study of this phenomenon for the more sophisticated singular matrix differential operator.

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