On the Positon Solutions of the KdV Equation

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Abstract

Singular and slow decaying solutions of the KdV equation are discussed. It is shown that the positon solution coincides with the two-soliton solution considered for a special choice of parameters. Relations with the spectral properties of the scattering problem on the half line are discussed.

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1 Introduction.

The inverse scattering transformation plays an important role during the solution of nonlinear equations. Analytical solutions of such equations can be obtained in the case, when the inverse problem can be solved analytically. This paper is devoted to the analytical solutions of the KdV equation:

$$u_t = 6u\, u_x - u_{xxx}. \tag{1}$$

Gardner, Green, Kruskal and Miura [12] were the first to show the relations between the solutions of this equation and the inverse scattering problem on the line. It was shown that soliton solutions of the KdV equation are defined by the the bound states of the corresponding Schroedinger operator on the line. In this case potential is defined by the reflection coefficient, energies of the bound states and normalizing constants. Solitons are analytical solutions of the KdV equation, corresponding to the zero reflection coefficient. These solutions are defined by a finite number of parameters. One can obtain such solutions solving the inverse scattering problem in the Faddeev class of potentials $L_{1,1}$

$$\int_{-\infty}^{+\infty} (1+|x|)|u(x)|dx < \infty. \tag{2}$$

The inverse scattering problem in this case is equivalent to some finite dimensional problem. As the result, the solution can be expressed in terms of determinants of finite dimensional matrices. Corresponding potentials were named Bargmann potentials. The construction procedure can be carried out step by step using the Darboux transformation [16]. We note that soliton potentials can be obtained during the solution of the inverse scattering problem on the halfaxis for the rational reflection coefficient. Corresponding formulas are essentially the same.

We are interesting in the solutions of the KdV equation which violate Faddeev condition (2). Stationary solutions of this type were known for a long time (see [6] for review). The first solution which has an extremely slow decay like $\frac{\sin(kx+\varphi)}{x}$ at infinity $x \to \infty$ was constructed first by Matveev using some generalization of the Darboux transformation [17, 18, 19]. It was named "positon" due to the relations with the positive eigenvalues embedded into the continuous spectrum [23]. Later the generalizations of the Darboux transformation were used to derive positon solutions of Sine-Gordon [4] and mKdV [25] equations. Potentials similar to the positon solutions of the KdV equation were obtained independently at the same time by the author during the solution of the inverse scattering problem on the half line for the rational reflection coefficients [13, 14]. Relations between these two approaches were pointed out first by S.N.M.Ruijsenaars. The aim of this paper is to clarify these relations.

The second section is devoted to singular solutions of the KdV equation which can be obtained by a limit procedure for Darboux transformation. It will be shown that such solutions can be obtained by analytical continuation of the one-soliton solutions to complex values of the parameters. The third section is devoted to the calculation of the positon as a soliton pair corresponding to the negative velocity. We restrict our consideration to the case of one-positon solution which coincides with the two-soliton solution with a special choice of parameters. The forth section is devoted to the relations between the positon solution and the scattering problem on the line.

2 Singular solutions and solitons.

We will derive here singular solutions of the KdV equation using the first Darboux transformation. It will be shown that such solutions can be obtained by an analytical continuation of the soliton solution, which is defined usually by two real parameters $b_1, x_1 \in \mathbf{R}$:

$$u[1] = \frac{-2b_1^2}{\cosh^2\left(b_1(x - 4b_1^2 t) - x_1\right)}. \tag{3}$$

This is bounded solution of the KdV equation decreasing at infinity exponentially.

In the first step we recall some facts about Darboux transformation [16] for the case of the zero initial potential. Starting from the Schroedinger equation

$$-\psi_{xx} = \lambda\psi \tag{4}$$

on the real axis we get the following potential after the first Darboux transformation:

$$u_1(a_1, b_1, x) = -2\frac{\partial^2}{\partial x^2} \ln W_{11}(a_1, b_1, x) \tag{5}$$

where W_{11} is Wronskian of two solutions of the equation (4)

$$W_{11}(a_1, b_1, x) = \frac{1}{a_1^2 - b_1^2} W[f_0(\imath a_1, x), \varphi_0(\imath b_1, x)];$$

$$f_0(\imath a_1, x) = e^{-a_1 x}; \varphi_0(\imath b_1, x) = \frac{\sinh(b_1 x)}{2b_1}.$$

We suppose that constants a_1, b_1 are real and positive. Then the formula (5) gives us a reflectionless potential:

$$u_1(a_1, b_1, x) = \frac{2b_1^2}{\sinh^2(b_1 x - \imath\delta_1(\imath b_1))}, \tag{6}$$

where $\delta_1(k)$ denotes the phase of the Bluaschke product defined by the constant a_1 :

$$e^{\imath 2\delta_1(k)} = \frac{k - \imath a_1}{k + \imath a_1}. \tag{7}$$

This potential has a quadratic singularity on the negative halfaxis at the point $b_1 x = \imath\delta_1(\imath b_1)$. We note that $\imath\delta_1(k)$ is real for pure imaginary k. Solution of the Schroedinger equation with this potential can be expressed in terms of elementary functions

$$\psi_1(a_1, b_1, k, x) = e^{\imath kx} \frac{e^{b_1 x}\left(\frac{1}{b_1-a_1} - \frac{1}{b_1+\imath k}\right) + e^{-b_1 x}\left(\frac{1}{b_1+a_1} - \frac{1}{b_1-\imath k}\right)}{\frac{e^{b_1 x}}{b_1-a_1} + \frac{e^{-b_1 x}}{b_1+a_1}}$$

Asymptotics of the calculated solution

$$\psi(k, x) \sim_{x\to+\infty} e^{\imath kx}\left(1 - \frac{b_1 - a_1}{b_1 + \imath k}\right)$$
$$\psi(k, x) \sim_{x\to-\infty} e^{\imath kx}\left(1 - \frac{b_1 + a_1}{b_1 - \imath k}\right) \tag{8}$$

shows that the reflection coefficient is identically equal to zero $R(k) \equiv 0$, the transition coefficient is given by the following expression

$$T(k) = \frac{\imath k - b_1}{\imath k + b_1} \tag{9}$$

Calculated singular potential is reflectionless, but it does not belong to the standard class $L_{1,1}$ (see (2)) due to the quadratic singularity. Corresponding solution of the KdV equation (1) can be obtained by substitution of $\varphi_0(\imath b_1, x)$ by a time dependent oscillatory solution of (4)

$$\varphi_0(\imath b_1, x, t) = \frac{\sinh(b_1(x - 4b_1^2 t))}{2b_1}. \tag{10}$$

This solution is moving to the right with the constant velocity:

$$u_1(a_1, b_1, x, t) = \frac{2b_1^2}{\sinh^2\left(b_1(x - 4b_1^2 t) - \imath\delta_1(\imath b_1)\right)}. \tag{11}$$

Constructed singular solution coincides with the soliton solution (3), considered for the complex values of the parameter $x_1 = \imath\pi/2 + \imath\delta_1(\imath b_1)$.

Another important example of the singular solution can be obtained as a limit of the calculated solution for $b \to 0$. The pointwise limits of the potential and corresponding scattering solution can be calculated [6]

$$u_1(a_1, 0, x, t) = \frac{2a_1^2}{(1 + a_1 x)^2};$$

$$\psi(a_1, 0, k, x) = e^{\imath kx} \frac{1 + \frac{a_1^2}{k^2} + a_1 x(1 + \frac{a_1}{\imath k})}{1 + a_1 x} \tag{12}$$

It is easy to verify that the limit scattering solution is a solution of the Schroedinger equation with the limit potential everywhere excluding point $x = -1/a_1$, where potential has a singularity. But both potential and scattering solution can be continued analytically to the complex neighborhood of this point. The scattering matrix for the calculated potential coincides with the limit scattering matrix, i.e. the reflection coefficient is equal to zero and the transition coefficient is constant $T(k) \equiv 1$. Such potentials were named superreflectionless in [17, 18]. Calculated potential represent a stationary solution of the KdV equation decreasing at infinity like $1/x^2$. Hence this solution does not belong to the class $L_{1,1}$ not only due to the singularity but due to the slow decay at infinity also.

Another class of the soliton-like solutions is connected with the possibility to consider complex values of the parameter b_1. For example if parameter b_1 is purely imaginary $b_1 = \imath\chi, \chi \in \mathbf{R}$, then the Darbous transformation gives us a real valued function

$$u_1(a_1, \imath\chi, x, t) = \frac{-8\chi^2}{a_1^2 + \chi^2} \left(\frac{e^{\imath\chi(x+4\chi^2 t)}}{a_1 - \imath\chi} - \frac{e^{-\imath\chi(x+4\chi^2 t)}}{a_1 + \imath\chi} \right)^{-2} =$$

$$= 2\chi^2 \sin^{-2}(\chi(x + 4\chi^2 t) + \delta_1(\chi)). \tag{13}$$

This is a periodic solution with infinitely many singularities on the real axis, mooving to the left, i.e. in the direction opposite to one for the standard soliton.

3 Positons and solitons.

We consider in this section the second Darboux transformation, which was used to derive two soliton solutions of the KdV equation. It will be shown that the limit of such solution considered for a special choice of parameters coincides with a positon solution [17, 18, 19].

The second Darboux transformation gives the following solution of KdV equation

$$u_2(a_1, a_2, b_1, b_2, x, t) = -2\frac{\partial^2}{\partial x^2} \ln \det \mathbf{W}(a_1, a_2, b_1, b_2, x, t); \tag{14}$$

where $\mathbf{W}$ is a 2×2 matrix with the components

$$W_{nm} = \frac{1}{a_n^2 - b_m^2} W[f_0(\imath a_n, x), \varphi_0(\imath b_m, x, t)];$$

$$f_0(\imath a_n, x) = e^{-a_n x}; \varphi_0(\imath b_m, x, t) = \frac{\sinh(b_m(x - 4b_m^2 t))}{2b_m}.$$

This solution is real for a special choice of the parameters $a_{1,2}, b_{1,2}$. These constants have to be real or conjugated one to another complex numbers. The two soliton solution can be obtained in the same way. The constructed solution decreases at infinity exponentially for $b_{1,2}$ with the positive real part. We will consider the limit of this solution when $\Re b_j = \epsilon \to 0, \Im b_{1,2} = \pm b_0$. The determinant of the matrix $\mathbf{W}$ is equal to zero for b_j pure imaginary:

$$\det \mathbf{W} \sim_{\epsilon \to 0} \epsilon \, \frac{e^{-a_1 x} e^{-a_2 x} \imath (a_2 - a_1)}{b_1 b_2 (a_1^2 + b_0^2)(a_2^2 + b_0^2)}$$

$$\left\{ 2b_0(x + 3b_0^2 t) + 2B - \sin(2b_0(x + b_0^2 t) + 2\delta_2(b_0)) \right\} \tag{15}$$

where the following notations were used

$$B = B(b_0, a_1, a_2) = \frac{b_0(a_1 + a_2)(a_1 a_2 + b_0^2)}{(a_1^2 + b_0^2)(a_2^2 + b_0^2)},$$

$$e^{2\imath\delta_2(k)} = \frac{(k - \imath a_1)}{(k + \imath a_1)} \frac{(k - \imath a_2)}{(k + \imath a_2)} \tag{16}$$

But the pointwise limit of the solution is not trivial

$$u_2(a_1, a_2, \imath b_0, -\imath b_0, x, t) =$$

$$16b_0^2 \frac{1 - (b_0(x + 12b_0^2 t) + B) \sin 2\left(b_0(x + 4b_0^2 t) + \delta_2(b_0)\right) - \cos 2\left(b_0(x + 4b_0^2 t) + \delta_2(b_0)\right)}{\left(2b_0(x + 12b_0^2 t) + 2B - \sin 2\left(b_0(x + 4b_0^2 t) + \delta_2(b_0)\right)\right)^2}, \tag{17}$$

This form of the solution of the KdV equation coincides with the one positon solution constructed first in [18]. Our form of the positon depends on three real parameters $\{a_1, a_2, b_0\}$ or $\{\Re a_1, \Im a_1, b_0\}$ again. For $\Re a_j > 0$ the singularity of the positon is situated on the negative halfaxis at the initial moment. By changing the constants a_1, a_2 one can change position of the positon on the real line at the initial moment.

Our calculations show that one positon solution coincides with the pair of soliton solutions. This soliton pair is formed by two periodic singular solutions of the type (13). These two solutions are periodic moving to the left with the same velocity, but the limit solution decreases at infinity like:

$$u_2(a_1, a_2, \imath b_0, -\imath b_0, x, t) \sim_{x \to \infty} \frac{-4b_0 \sin 2\left(b_0(x + 4b_0^2 t) + \delta_2(b_0)\right)}{x} \tag{18}$$

Both singular periodic solutions are functional invariant solutions of the KdV. Positon changes the form, but in general it moves to the left with the same velocity as two singular solutions. One can say, that two solitons exchange the mass or the energy. More detailed description of the positon properties can be found in [17]. We recall only that positons represent another class of superreflectionless potentials. The transition coefficient is equal to 1 identically again.

4 Superreflectionless potentials and scattering problem on the halfline.

The inverse scattering problem in one dimension was first investigated for potentials satisfying the Faddeev condition (2) (see [9, 11, 20]). It was shown that the scattering matrix defines potential uniquely if no bound states are present. Later several examples of potentials with the same scattering matrix were found [1, 2, 3, 5, 8, 21]. All these potentials decrease at infinity like $\frac{const}{x^2}$, and thus violate condition (2). Constructed superreflectionless potentials present a family of potentials with the same scattering matrix without any bound state. It is another one example of potentials for which the inverse scattering problem does not have a unique solution in it's standard formulation. The inverse scattering problem for such potentials was considered for the case of the scattering problem on the half line with Dirichlet boundary condition at the origin[13, 14]. It was shown that such potentials correspond to the rational reflection coefficients $R(k)$ of a special form: the Levinson's theorem is violated for them:

$$R_2(k) = \frac{k - \imath a_1}{k + \imath a_1}\frac{k - \imath a_2}{k + \imath a_2}. \tag{19}$$

The Jost function $F(k)$ for such potentials has a singularity on the real axis:

$$F(k) = \frac{k + \imath a_1}{k - b_0}\frac{k + \imath a_2}{k + b_0}, \Im b_0 = 0. \tag{20}$$

The inverse scattering problem for such reflection coefficients can be solved using some approximation procedure. One can suppose that the singularity of the Jost function is situated in the lower half plane of the spectral parameter k at the points $k = \pm b_0 - \epsilon$ and consider the limit, when this singularity approaches the real line $\epsilon \to 0$. This procedure is similar to the limit procedure for Darboux transformation used in this paper. Scattering solutions and potentials were calculated. It was shown that all potentials of the form $u_2(a_1, a_2, \imath b_0, -\imath b_0, x, 0)$ have one and the same scattering matrix $R_2(k)$, and as the result the solution of the inverse scattering problem for such potentials is not unique even on the half line. The singularity of the Jost function on the real axis can not be reconstructed from the reflection coefficient. The spectral nature of this nonuniqueness is not the same as in the case of the eigenvalue. The spectral density, corresponding to these potentials is equal to zero at the points $k = \pm b_0$.

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