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Wigner–von Neumann perturbations of a periodic potential: spectral singularities in bands

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Abstract

Wigner–von Neumann type perturbations of the periodic one-dimensional Schrödinger operator are considered. The asymptotics of the solution to the generalized eigenfunction equation is investigated. It is proven that a subordinated solution and therefore an embedded eigenvalue may occur at the points of the absolutely continuous spectrum satisfying a certain resonance (quantization) condition between the frequencies of the perturbation, the frequency of the background potential and the corresponding quasimomentum.

1. *Introduction*

The celebrated Wigner–von Neumann potential provides an example of a one-dimensional Schrödinger operator

$$-\frac{d^2}{dx^2} + V(x), \qquad \text{acting in } L_2(\mathbb{R}),$$

with an eigenvalue embedded into the absolutely continuous spectrum [**22**, **23**]. In this case the absolutely continuous spectrum fills in the unique band $[0, \infty)$ and the potential with asymptotics $c \sin(2\omega x + \varphi)/x,\ x \to \pm\infty$ may produce exactly one eigenvalue with the energy $E_0 = \omega^2$. In the case the potential has asymptotics containing several frequencies

$$\sum_{i=1}^{N} \frac{c_i \sin(2\omega_i x + \varphi_i)}{x},$$

the eigenvalues may occur at the energies $E_i = \omega_i^2, i = 1, 2, \ldots, N$ (see [**1**] and [**23**]). Additionally the first derivative of the spectral density $\rho'(\lambda)$ vanishes at the points $\lambda = E_i$ [**18**, **19**]. Spectral properties of such potentials have been studied [**2**–**4**, **26**], see also [**17**, **21**, **24**, **27**]. The structure of zeroes of $\rho'(\lambda)$ for strong perturbations has been investigated for Jacobi and Schrödinger operators [**15**, **20**]. Related questions concerning spectral concentration were studied in [**5**]. On the other hand several criteria guaranteing the absence of embedded eigenvalues can be found in [**12**, **23**]. It has been suggested recently that such bound states can be observed in certain atomic and molecular systems [**6**].

Embedded eigenvalues for the perturbed periodic one-dimensional Schrödinger operators have been studied in [**17**]. Explicit examples leading to dense point spectrum in bands have been constructed using Prüfer transform.

In this paper we are going to study this phenomenon in the case where the background operator has several or infinite number of bands of absolutely continuous spectrum. Consider the following operator

$$H = -\frac{d^2}{dx^2} + Q_{\mathrm{per}}(x) + V_0(x) + \sum_{i=1}^{N} \frac{c_i \sin(2w_i x + \varphi_i)}{(|x|+1)^{\gamma_i}}, \tag{1·1}$$

where

$$V_0 \in L^1(\mathbf{R}). \tag{1·2}$$

Let the potential Q_{per} have period T

$$Q_{\mathrm{per}}(x+T) = Q_{\mathrm{per}}(x) \tag{1·3}$$

and let it be locally integrable

$$Q_{\mathrm{per}} \in L^1_{loc}(\mathbf{R}). \tag{1·4}$$

Therefore the periodic Schrödinger operator

$$H_0 = -\frac{d^2}{dx^2} + Q_{\mathrm{per}} \tag{1·5}$$

has purely continuous spectrum, filling up a countable number of bands.

The Wigner–von Neumann type perturbation

$$V_{\mathrm{WN}}(x) = \sum_{i=1}^{N} \frac{c_i \sin(2\omega_i x + \varphi_i)}{(|x|+1)^{\gamma_i}}, \tag{1·6}$$

is described by real parameters $c_i, \omega_i, \varphi_i, \gamma_i, i = 1, 2, \ldots, N$. We assume in addition that $\gamma \equiv \min_{i=1,2,\ldots,N} \gamma_i > 1/2$.[1] The only interesting case is where $\gamma_i \leqslant 1$.

We are going to use the following notation:

$$\begin{aligned} Q(x) &= Q_{\mathrm{per}}(x) + V_0(x) + \sum_{i=1}^{N} \frac{c_i \sin(2w_i x + \varphi_i)}{(|x|+1)^{\gamma_i}}; \\ V(x) &= V_0(x) + V_{\mathrm{WN}}(x) = V_0(x) + \sum_{i=1}^{N} \frac{c_i \sin(2w_i x + \varphi_i)}{(|x|+1)^{\gamma_i}}. \end{aligned} \tag{1·7}$$

Our goal is to study the asymptotics of the solutions to the eigenfunction equation

$$-\psi''(x) + Q(x)\psi(x) = E\psi(x), \tag{1·8}$$

for E from the absolutely continuous spectrum of H_0 which turns out to coincide with the absolutely continuous spectrum of H. The asymptotic behaviour for $x \to +\infty$ and $x \to -\infty$ can be studied independently. Therefore in order to simplify the analysis we restrict our consideration to the case of one semiaxis whenever it is possible. The case where E belongs to the generalized boundary of the absolutely continuous spectrum, i.e. the points

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[1] The case $\gamma > 1/2$ is considered only in order to avoid tedious elementary calculations. The more general case $\gamma > 0$ can be studied using the same technique.

where the quasimomentum $\theta(E)$ is equal to 0, $\pm\pi$, is excluded from our consideration and will be studied in a forthcoming publication. It appears that a subordinated solution ([**8**]) on one of the half-axes exists only if

$$\frac{\omega_i T}{\pi} \notin \mathbb{Z}$$

and at least one of the following two quantization conditions

$$\frac{\omega_i T + \theta}{\pi} \in \mathbb{Z} \quad \text{or} \quad \frac{\omega_i T - \theta}{\pi} \in \mathbb{Z}$$

is satisfied. If $\omega_i T/\pi \in \mathbb{Z}$ and one of the conditions is satisfied, then $\theta = 0, \pm\pi$, i.e. the corresponding energy belongs to the generalized boundary of the absolutely continuous spectrum, but this case is excluded from our consideration.

An embedded eigenvalue may exist only if the subordinated solution is square integrable. The answer is determined by the spectrum of the principle homogenized interaction matrix $\tilde{\Gamma}$ (to be introduced in Sections 3 and 5). This real symmetric zero trace matrix is obtained by a proper Cesaro averaging of the Wigner–von Neumann potential V_{WN} using the periodic background potential Q_{per} and describes the interplay between the two potentials.

This new notion allows us to give an exhaustive answer to the question, whether subordinated solution to (1·8) exists in the case E belongs to *the generalized interior* of the absolutely continuous spectrum of H_0 determined by $\{E : \theta(E) \neq 0, \pm\pi\}$. In particular we localize all points on $\sigma_{\mathrm{ac}}(H_0)$, where embedded eigenvalues of H may occur. The existence of such eigenvalues depends on whether the corresponding subordinated solutions for $x \to \pm\infty$ are square integrable and on the local potential V_0. These eigenvalues are highly unstable and can disappear under arbitrary small local changes of V_0 destroying the matching of the two subordinated solutions on the positive and negative semiaxes. On the other hand if the two subordinated solutions (for $x \to \pm\infty$) are square integrable, we can always create an eigenvalue at the same energy point by changing V_0. Nevertheless location of possible eigenvalues is stable even under L_1-perturbations, since the character of the asymptotics is stable under such perturbations. Moreover the derivative of the spectral function $\rho'(\lambda)$ tends to zero as λ approaches this singular point (see the proper formula for $\rho'(\lambda)$ in [**5**]). This question will be discussed in a forthcoming publication.

The main tool of our studies is the asymptotic analysis of discrete dynamical systems developed originally for the spectral analysis of some classes of Jacobi operators, being essentially a generalization of the semiclassical method.

2. *Representation for the monodromy matrix*

The monodromy matrix for the Schrödinger equation (1·8) is determined as the following mapping

$$M(x, y; E) \colon \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix} \longmapsto \begin{pmatrix} \psi(y) \\ \psi'(y) \end{pmatrix}, \tag{2·1}$$

where ψ is an arbitrary solution to the differential equation.

Another monodromy matrix associated with the periodic problem

$$-\frac{d^2}{dx^2}\psi + Q_{\mathrm{per}}\psi = E\psi \tag{2·2}$$

will be denoted by $M_0(x, y; E)$. The corresponding matrix for the interval $[x, y] = [nT, (n+1)T]$ does not depend on n and will be denoted by $M_{\mathrm{per}}(E) = M_0(nT, (n+1)T; E)$.

Let us denote by M_n the monodromy matrix for the operator H and period $[nT, (n+1)T]$. Equation (1·8) can be re-written as

$$\begin{aligned} \frac{d}{dx}\begin{pmatrix} \psi \\ \psi' \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ Q(x) - E & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi' \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ Q_{\mathrm{per}}(x) - E & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi' \end{pmatrix} + V(x)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi' \end{pmatrix}. \end{aligned} \tag{2·3}$$

The following Lemma is a standard tool in the perturbation theory of ordinary differential equations [**11**].

LEMMA 1. *Let the potentials* Q_{per} *and* V_0 *satisfy condition* (1·4) *and* (1·2). *Then the monodromy matrix for the operator* H (*given by* (1·1)) *admits the following asymptotic representation for* $n \to \infty$, $n \in \mathbb{N}$.

$$\begin{aligned} M(nT, (n+1)T; E) = M_{\mathrm{per}}(E)\Bigg(I + \sum_{i=1}^{N} \frac{c_i}{((n+1)T)^{\gamma_i}} \int_0^T \sin(2\omega_i(nT+\tau) + \varphi_i) \\ \times M_0^{-1}(0, \tau)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau)d\tau + R_n \Bigg), \end{aligned} \tag{2·4}$$

where the 2×2 *matrices* R_n *satisfy*

$$\sum_{n \in \mathbb{N}} \| R_n \| < \infty. \tag{2·5}$$

Proof. Equation (2·3) implies the following differential equations on the monodromy matrices M and M_0

$$\begin{cases} \dfrac{\partial}{\partial y} M(x, y; E) = \begin{pmatrix} 0 & 1 \\ Q_{\mathrm{per}}(y) + V(y) - E & 0 \end{pmatrix} M(x, y; E), \\ M(x, x; E) = I \end{cases};$$

$$\begin{cases} \dfrac{\partial}{\partial y} M_0(x, y; E) = \begin{pmatrix} 0 & 1 \\ Q_{\mathrm{per}}(y) - E & 0 \end{pmatrix} M_0(x, y; E), \\ M_0(x, x; E) = I \end{cases}.$$

Let us introduce the perturbation matrix $A(x, y; E)$ determined by the following equality

$$M(x, y; E) = M_0(x, y; E)\, A(x, y; E), \tag{2·6}$$

which always exists since $\det M = \det M_0 \equiv 1$. This matrix satisfies the system

$$\begin{cases} \dfrac{\partial}{\partial y} A(x, y; E) = V(y) M_0^{-1}(x, y; E)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(x, y; E) A(x, y; E), \\ A(x, x; E) = I \end{cases} \tag{2·7}$$

which can be transformed into the integral equation

$$A(x, y; E) = I + \int_x^y M_0^{-1}(x, t; E) V(t)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(x, t; E) A(x, t; E)\, dt. \tag{2·8}$$

In particular, considering $x = nT,\ y = (n+1)T;\ n \in \mathbb{Z}$ one gets

$$A(nT, (n{+}1)T; E) = I + \int_{nT}^{(n+1)T} M_0^{-1}(nT, t; E) V(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(nT, t; E) A(nT, t; E)\, dt.$$

Iteration of the equation gives the following representation for the solution

$$A(nT, (n+1)T; E) = I + \sum_{i=1}^{N} \frac{c_i}{((n+1)T)^{\gamma_i}} \int_0^T \sin(2\omega_i(nT+\tau) + \varphi_i) \\ \times M_0^{-1}(0, \tau; E) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau; E)\, d\tau + R_n(E), \qquad (2{\cdot}9)$$

where $R_n(E) = \mathcal{O}(1/n^{2\gamma}) + \mathcal{O}(\int_0^T |V_0(nT+\tau)| d\tau)$ satisfies condition (2·5), since $\gamma > 1/2$ and $V_0 \in L^1(\mathbb{R})$.

A similar result holds true for all negative integers n.

3. *Asymptotics of the generalized eigenfunctions. Absence of embedded eigenvalues for generic values of energy*

The following classical results of the theory of trigonometric series will be used.

PROPOSITION 1 (see [**28**]). *Assume $\alpha, \gamma \in \mathbb{R}$, $\gamma > 0$, then the following estimates hold*:

$$\sum_{k=n}^{\infty} \frac{e^{ik\alpha}}{k^\gamma} = O(1/n^\gamma),\ n \to \infty, \quad \textit{if and only if } \frac{\alpha}{2\pi} \notin \mathbb{Z}; \qquad (3{\cdot}1)$$

$$\sum_{k=n}^{\infty} \frac{\cos k\alpha}{k^\gamma} = O(1/n^\gamma),\ n \to \infty, \quad \textit{if and only if } \frac{\alpha}{2\pi} \notin \mathbb{Z}; \qquad (3{\cdot}2)$$

$$\sum_{k=n}^{\infty} \frac{\sin k\alpha}{k^\gamma} = O(1/n^\gamma),\ n \to \infty, \quad \textit{for any real } \alpha. \qquad (3{\cdot}3)$$

Let us study the asymptotic behavior of the solutions to the generalized eigenfunction equation (see (1·8)). To carry out this study it is enough to investigate the asymptotics of the monodromy matrix, which can be obtained using the Harris–Lutz procedure developed originally for equations with oscillating coefficients [**10**].

Suppose that the energy belongs to the generalized interior of the absolutely continuous spectrum of the periodic operator H_0. Then the original monodromy matrix is diagonalizable and therefore is similar to the rotation matrix $\mathcal{R}_\theta$

$$M_{\text{per}}(E) = W^{-1}\mathcal{R}_\theta W \equiv W^{-1} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} W, \qquad (3{\cdot}4)$$

where $W = W(E)$ is a real invertible matrix. Lemma 1 implies

$$M(nT, (n+1)T; E) \\ = W^{-1}\left[\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \left(\mathbb{I} + \sum_{i=1}^{N} \frac{c_i}{((n+1)T)^{\gamma_i}} \int_0^T \sin(2\omega_i(nT+\tau) + \varphi_i) \right.\right. \\ \left.\left. \times W M_0^{-1}(0, \tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau) W^{-1} d\tau + R_n^{(1)} \right) \right] W, \qquad (3{\cdot}5)$$

where $R_n^{(\ell)}$, $\ell = 1, 2, 3, \dots$ denote here, and in what follows, sequences of 2×2 matrices such that $\sum_n \parallel R_n^{(\ell)} \parallel < \infty$.

Using the matrix chronological product $\prod_{m=0}^{\overset{\curvearrowleft}{n}} A_m \equiv A_n A_{n-1} \cdots A_1 A_0$ the monodromy matrix for the period $[0, (n+1)T]$ can be written as follows

$$
\begin{aligned}
&M(0, (n+1)T; E) \\
&= W^{-1} \left\{ \prod_{m=0}^{\overset{\curvearrowleft}{n}} \left[\mathcal{R}_\theta \left(\mathbb{I} + \sum_{i=1}^{N} \frac{c_i}{((m+1)T)^{\gamma_i}} \int_0^T \sin(2\omega_i (mT + \tau) + \varphi_i) \right.\right.\right. \\
&\quad \left.\left.\left. \times W M_0^{-1}(0, \tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau) W^{-1} d\tau + R_m^{(1)} \right) \right] \right\} W \\
&= W^{-1} \mathcal{R}_\theta^{n+1} \left\{ \prod_{m=0}^{\overset{\curvearrowleft}{n}} \left[\mathcal{R}_\theta^{-m} \left(\mathbb{I} + \sum_{i=1}^{N} \frac{c_i}{((m+1)T)^{\gamma_i}} \int_0^T \sin(2\omega_i (mT + \tau) + \varphi_i) \right.\right.\right. \\
&\quad \left.\left.\left. \times W M_0^{-1}(0, \tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau) W^{-1} d\tau + R_m^{(1)} \right) \mathcal{R}_\theta^{m} \right] \right\} W \\
&= W^{-1} \mathcal{R}_\theta^{n+1} \left[\prod_{m=0}^{\overset{\curvearrowleft}{n}} \left(\mathbb{I} + \sum_{i=1}^{N} \frac{c_i}{((m+1)T)^{\gamma_i}} \int_0^T \sin(2\omega_i (mT + \tau) + \varphi_i) \mathcal{R}_\theta^{-m} \right.\right. \\
&\quad \left.\left. \times W M_0^{-1}(0, \tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau) W^{-1} \mathcal{R}_\theta^{m} d\tau + R_m^{(2)} \right) \right] W,
\end{aligned}
$$

where $R_m^{(2)} = \mathcal{R}_\theta^{-m} R_m^{(1)} \mathcal{R}_\theta^{m}$. Let us denote by Π_n the product in square brackets $[\dots]$ appearing in the latter formula

$$
\begin{aligned}
\Pi_n &= \prod_{m=0}^{\overset{\curvearrowleft}{n}} \left(\mathbb{I} + \sum_{i=1}^{N} \frac{c_i}{((m+1)T)^{\gamma_i}} \int_0^T \sin(2\omega_i (mT + \tau) + \varphi_i) \right. \\
&\quad \left. \times \mathcal{R}_\theta^{-m} W M_0^{-1}(0, \tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau) W^{-1} \mathcal{R}_\theta^{m} d\tau + R_m^{(2)} \right).
\end{aligned} \tag{3·6}
$$

Then we have

$$
M(0, (n+1)T; E) = W^{-1} \mathcal{R}_\theta^{n+1} \Pi_n W
$$

and therefore to understand the asymptotic behavior of $M(0, (n+1)T; E)$ it is enough to study Π_n. Consider the following representation for the integrals appearing in (3·6)

$$
\begin{aligned}
\mathbb{H}_m^{(i)} &= \frac{c_i}{T^{\gamma_i}} \int_0^T \sin\left(2\omega_i (mT + \tau) + \varphi_i\right) \\
&\quad \times \mathcal{R}_\theta^{-m} W M_0^{-1}(0, \tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau) W^{-1} \mathcal{R}_\theta^{m} d\tau \\
&\equiv \sin 2m(\omega_i T + \theta) S_{+1}^{(i)}(E) + \cos 2m(\omega_i T + \theta) C_{+1}^{(i)}(E) \\
&\quad + \sin 2m(\omega_i T) S_0^{(i)}(E) + \cos 2m(\omega_i T) C_0^{(i)}(E) \\
&\quad + \sin 2m(\omega_i T - \theta) S_{-1}^{(i)}(E) + \cos 2m(\omega_i T - \theta) C_{-1}^{(i)}(E),
\end{aligned} \tag{3·7}
$$

which can be obtained by applying directly formulas for the products of elementary trigonometric functions and the decomposition

$$
\sin(2\omega_i (mT + \tau) + \varphi_i) = \sin(2\omega_i mT) \cos(2\omega_i \tau + \varphi_i) + \sin(2\omega_i \tau + \varphi_i) \cos(2\omega_i mT).
$$

Here $S_s^{(i)}, C_c^{(i)}$ are certain 2×2 matrices. Since the left-hand side of (3·7) is a matrix with zero trace for all $m \in \mathbb{Z}$, one should expect that all new 6 matrices can be chosen having the same property. Since the matrices $W^{-1}\mathbb{H}_m^{(i)}W$ are real, all matrices $W^{-1}C_sW,\ W^{-1}S_sW$ can be chosen real as well. This fact will be confirmed later by direct calculations.

Using these notation we can introduce new 2×2 energy depending matrices $\mathbb{T}_m$,

$$\mathbb{T}_m = \sum_{i=1}^{N} \frac{1}{(m+1)^{\gamma_i}} \left[\sum_{s=-1}^{1} \left(\sin 2m(\omega_i T + s\theta) S_s^{(i)}(E) + \cos 2m(\omega_i T + s\theta) C_s^{(i)}(E) \right) \right];$$

$$\Pi_n = \prod_{m=0}^{\overset{\frown}{n}} \left(\mathbb{I} + \mathbb{T}_m + R_m^{(2)} \right). \tag{3·8}$$

Now we are ready to use an approach similar to the one of Harris–Lutz [**10**, **14**, **16**]. The role of this procedure is to eliminate the oscillating terms in $\mathbb{T}_m$.

Let $\mathbb{G}_n$ be certain (to be specified later) 2×2 real matrices such that the matrices $\mathbb{I} + \mathbb{G}_n,\ n = 0, 1, 2, \ldots$ are invertible. Then (3·8) can be transformed identically into

$$\Pi_n = (\mathbb{I} + \mathbb{G}_n) \left[\prod_{m=0}^{\overset{\frown}{n}} \left\{ (\mathbb{I} + \mathbb{G}_m)^{-1} \left(\mathbb{I} + \mathbb{T}_m + R_m^{(2)} \right) (\mathbb{I} + \mathbb{G}_{m-1}) \right\} \right] (\mathbb{I} + \mathbb{G}_{-1})^{-1}. \tag{3·9}$$

Suppose in addition that

$$\| \mathbb{G}_m \| = O(1/m^{\gamma}), m \to \infty, \tag{3·10}$$

where $\gamma \equiv \min_i \gamma_i$. Since $\gamma > 1/2$, elementary calculations show that

$$\left\{ (\mathbb{I} + \mathbb{G}_m)^{-1} \left(\mathbb{I} + \mathbb{T}_m + R_m^{(2)} \right) (\mathbb{I} + \mathbb{G}_{m-1}) \right\} = \left(\mathbb{I} + \mathbb{T}_m - \mathbb{G}_m + \mathbb{G}_{m-1} + R_m^{(3)} \right),$$

since $\| \mathbb{T}_m \| = \mathcal{O}(1/m^{\gamma}),\ m \to \infty$.

Let us extract the nonoscillating in m term from the following matrix sequence

$$\mathbb{H}_m^{(i)} = [\text{oscillating part}] + \Gamma^{(i)}(E) \equiv \mathbb{H}_m^{osc,(i)} + \Gamma^{(i)}(E), \tag{3·11}$$

where the oscillating part contains all terms with $s : (\omega_i T + s\theta(E))/\pi \notin \mathbb{Z}$. Obviously the decomposition is unique. The independent of m matrix $\Gamma^{(i)}$ can be calculated using the formula

$$\Gamma^{(i)}(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{H}_m^{(i)}, \tag{3·12}$$

and will be called the *homogenized interaction matrix* corresponding to the frequency ω_i. This matrix is going to play an important role in what follows and can be calculated using the following formula obtained from (3·12) and (3·7)

$$\Gamma^{(i)} = c_i T^{1-\gamma_i} \lim_{x \to \infty} \frac{1}{x} \int_0^x \sin(2\omega_i \tau + \varphi_i) W M_0^{-1}(0, \tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau) W^{-1} d\tau. \tag{3·13}$$

Let us choose now the matrices $\mathbb{G}_m$ from the principle recurrent equation

$$\mathbb{T}_m^{osc} - \mathbb{G}_m + \mathbb{G}_{m-1} \equiv 0, \tag{3·14}$$

which has to be satisfied starting from a sufficiently large value of m. Here we use the notation

$$\mathbb{T}_m^{osc} = \sum_{i=1}^{N} \frac{c_i}{((m+1)T)^{\gamma_i}} \mathbb{H}_m^{osc,(i)}. \tag{3·15}$$

This special choice of $\mathbb{G}_m$ allows us to eliminate all oscillating terms in $\mathbb{T}_m$. An explicit solution to this equation tending to zero at infinity is

$$\mathbb{G}_m = - \sum_{k=m+1}^{\infty} \mathbb{T}_k^{osc}. \tag{3·16}$$

Convergence of the series (3·16) and estimate (3·10) follow from Proposition 1. Therefore $\| \mathbb{G}_m \| < 1$ for sufficiently large m and $\mathbb{I} + \mathbb{G}_m$ is invertible. Without loss of generality we can suppose that this matrix is always invertible, since the first few terms in the product (3·9) do not affect the character of the asymptotics.

Let M be chosen such that $\sum_{k=m+1}^{\infty} \| T_k \| < 1/2$ for all $m \geqslant M$. To satisfy the invertibility condition we put $\mathbb{G}_m = 0$ for all $m = 0, 1, \ldots, M$ and use formula (3·16) to determine the rest of matrices $\mathbb{G}_m$.

We get the representation

$$\Pi_n = (\mathbb{I} + \mathbb{G}_n) \left[\prod_{m=0}^{\overset{\curvearrowleft}{n}} \left(\mathbb{I} + \sum_{i=1}^{N} \frac{1}{(m+1)^{\gamma_i}} \Gamma^{(i)}(E) + R_m^{(3)} \right) \right]. \tag{3·17}$$

The asymptotic analysis carried out allows to study the behavior of the eigenfunctions for generic values of E from the generalized interior of the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(H_0)$ of the background operator H_0.

Theorem 2. *Suppose that*:
- (i) $E \in \sigma_{\mathrm{ac}}(H_0)$;
- (ii) *for all* $i = 1, 2, \ldots, N$ *both* $(\theta(E) \pm T\omega_i)/\pi \notin \mathbf{Z}$;
- (iii) *The monodromy matrix* M_{per} *is diagonalizable*;
- (iv) $\omega_i T/\pi \notin \mathbb{Z},\ i = 1, 2, \ldots, N.$

Then the equation

$$-\frac{d^2}{dx^2}\psi + Q(x)\psi = E\psi \tag{3·18}$$

has no decaying solution and therefore no embedded eigenvalue with the energy E occurs. Moreover all solutions have oscillating behaviour, the same as for the background operator (*see* (3·20)).

Remarks.
- (i) Assumption (i) means that the spectrum of M_{per} lies on the unit circle, or in other words $|\mathrm{Sp}\, M_{\mathrm{per}}(E)| \leqslant 2$.
- (ii) Assumption (ii) is generically (with respect to E) satisfied on the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(H_0)$ for fixed values of ω_i.
- (iii) Assumption (iii) means that $M_{\mathrm{per}}(E)$ is not a Jordan box, or in other words it is not similar to one of the matrices $\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For this condition to be satisfied it is sufficient that $|\mathrm{Sp} M_{\mathrm{per}}(E)| \equiv 2\cos\theta < 2$, i.e. the energy E does not belong to the generalized boundary of $\sigma_{\mathrm{ac}}(H_0)$ defined as $\{E \in \sigma_{\mathrm{ac}}(H_0) : \theta(E) = 0, \pi\}$.

(iv) Assumption (iv) means that there is no resonance between the frequencies of the perturbation $2\omega_i$ and the one of the crystal $\omega_{\text{crys}} = 2\pi/T$. Using this notation (iv) reads as follows

$$\frac{2\omega_i}{\omega_{\text{crys}}} \notin \mathbb{Z}.$$

(v) In accordance with Gilbert–Pearson theory [**8**, **9**] the absolutely continuous spectra of the operators H and H_0 coincide including their local multiplicities (so far under the condition ((iv))).[2]

Proof. Under the assumptions of the theorem all matrices $\Gamma^{(i)}(E)$ are equal to 0, since no nonoscillating terms in the sequence $H_m^{(i)}$ appear due to conditions (ii) and (iv). Thus the sequence Π_n converges as $n \to \infty$ to a certain matrix Π_∞, which is invertible because $M(0, t; E)$ is ivertible for any t. This implies the following asymptotics for the monodromy matrix

$$\begin{aligned} M(0, (n+1)T; E) &= W^{-1}(E)\mathcal{R}_\theta^{n+1}\Pi_\infty W(E)(\mathbb{I} + o(1)) \\ &= M_{\text{per}}^{n+1}(E)W^{-1}(E)\Pi_\infty W(E)(\mathbb{I} + o(1)). \qquad (3{\cdot}19) \end{aligned}$$

By standard reasoning for ODE [**7**, **11**] one can get the following asymptotics for all t not necessarily equal to nT, $n \in \mathbb{Z}$

$$M(0, t; E) = M_0(0, t; E)W^{-1}(E)\Pi_\infty W(E)(\mathbb{I} + o(1)). \qquad (3{\cdot}20)$$

Therefore all solutions to the perturbed equation have just the same asymptotics as for the periodic one. These solutions are asymptotically pure oscillating (no decay occurs) for all allowed values of E.

As a corollary to the theorem we can re-establish the classical result on Wiegner–von Neumann potential having the asymptotics $C \sin 2\omega x/x$, $x \to +\infty$. In that case the original periodic potential is zero and its period T can be chosen arbitrarily. Then the dispersion relation is $\theta(E) = \sqrt{E}T(\text{mod}2\pi)$. Theorem 2 implies that no decaying solution exists if T can be chosen so that the following three numbers are not integers

$$T\frac{\omega}{\pi},\ T\frac{\omega + \sqrt{E}}{\pi},\ T\frac{\omega - \sqrt{E}}{\pi},$$

which is always possible if $\omega \neq 0, \omega \neq \pm\sqrt{E}$. The case $\omega = 0$ is excluded from our consideration. It follows that decaying solution may appear for $E = \omega^2$ only.

Summing up we conclude that under conditions (ii) and (iv) (i.e. for generic values of $E \in \sigma_{\text{ac}}(H_0)$) the asymptotic behavior of the solutions to (3·18) have the same type as for the background periodic operator. Therefore the character of the spectrum remains unchanged for these values of the energy [**8**]. In the following sections we are going to study two special cases:

(a) compatibility between the frequency of the crystal $\omega_{\text{crys}} = 2\pi/T$ and frequencies ω_i of the perturbation;
(b) the resonance between the quasimomentum $\theta(E)$ and the ratio of the frequencies $\omega_i/\omega_{\text{crys}}$.

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[2] In the case of the trivial background potential $Q_{\text{per}} \equiv 0$, this question has been studied in [**25**], see also [**24**].

4. *Compatible periods of crystal and perturbation*

In this section we are going to study the first special case. Let us assume that the perturbation term contains several frequencies compatible with the frequency of the original crystal. Without loss of generality we can suppose that these are frequencies $\omega_i, i = 1, 2, \ldots, N_1,\ N_1 \leqslant N$. Then in the general situation (with respect to the energy) no embedded eigenvalue appears, which is surprising: the compatibility between the periods of the crystal and perturbation does not change the character of the asymptotics.

THEOREM 3. *Suppose that the following assumptions are satisfied*:

(i) $E \in \sigma_{\mathrm{ac}}(H_0)$, *and* $|\mathrm{Sp}\,(M_{\mathrm{per}}(E))| = 2|\cos\theta(E)| < 2$, *i.e. the energy* E *belongs to the generalized interior of* $\sigma_{\mathrm{ac}}(H_0)$;

(ii) $(T\omega_i \pm \theta(E))/\pi \notin \mathbb{Z},\ i = N_1+1, N_1+2, \ldots, N$;

(iii) $\omega_i T/\pi \in \mathbb{Z},\ i = 1, 2, \ldots, N_1$, *and* $\omega_i T/\pi \notin \mathbb{Z},\ i = N_1+1, N_1+2, \ldots, N$.

Then equation (3·18) *has neither decaying nor growing solution and therefore* E *is not an embedded eigenvalue of* H. *Moreover the original* (*pure oscillating*) *asymptotics holds*: *for every solution* ψ *of the Schrödinger equation* (1·8) *there exists a solution* ψ_0 *of the original equation such that* $\psi(x) - \psi_0(x) = o(1),\ x \to \infty$. *In particular the following solution exists*

$$\begin{pmatrix} \psi^{\pm} \\ \psi^{\pm\prime} \end{pmatrix}(nT) = \exp\big\{ \pm i\,(n\theta(E)) \big\}(\vec{e}^{\,\pm} + o(1)), \tag{4·1}$$

where $M_{\mathrm{per}}(E)\vec{e}^{\,\pm} = e^{\pm i\theta(E)}\vec{e}^{\,\pm}$.

Proof. Consider the following representation for the propagator matrix (see Lemma 1)

$$\begin{aligned} M(0, (n+1)T; E) = \prod_{m=0}^{\overset{\curvearrowleft}{n}} \Bigg\{ M_{\mathrm{per}}(E)\Bigg(\mathbb{I} + \sum_{i=1}^{N} \frac{c_i}{((m+1)T)^{\gamma_i}} \int_0^T \sin(2\omega_i(nT+\tau) + \varphi_i) \\ \times M_0^{-1}(0,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau)\, d\tau + R_n \Bigg)\Bigg\} \\ = W^{-1}(E)\mathcal{R}_\theta^{n+1}\Pi_n W(E), \end{aligned} \tag{4·2}$$

where Π_n is given by (3·6) and can be written as

$$\Pi_n = \prod_{m=0}^{\overset{\curvearrowleft}{n}} \big(\mathbb{I} + \mathbb{T}_m' + \mathbb{T}_m'' + R_m^{(2)} \big), \tag{4·3}$$

and

$$\begin{aligned} \mathbb{T}_m' &= \sum_{i=1}^{N_1} \frac{c_i}{((m+1)T)^{\gamma_i}} \int_0^T \sin\,(2\omega_i(mT+\tau) + \varphi_i) \\ &\qquad \times \mathcal{R}_\theta^{-m} W M_0^{-1}(0,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) W^{-1} \mathcal{R}_\theta^m\, d\tau, \\ \mathbb{T}_m'' &= \sum_{i=N_1+1}^{N} \frac{c_i}{((m+1)T)^{\gamma_i}} \int_0^T \sin\,(2\omega_i(mT+\tau) + \varphi_i) \\ &\qquad \times \mathcal{R}_\theta^{-m} W M_0^{-1}(0,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) W^{-1} \mathcal{R}_\theta^m\, d\tau. \end{aligned} \tag{4·4}$$

This formula can be treated applying Harris-Lutz method (see Section 3) adapted for frequencies ω_i, $i = N_1 + 1, \ldots, N$ only

$$\Pi_n = (\mathbb{I} + \mathbb{G}''_n) \prod_{m=0}^{\overset{\curvearrowleft}{n}} \left\{ (\mathbb{I} + \mathbb{G}''_m)^{-1} \big(\mathbb{I} + \mathbb{T}'_m + \mathbb{T}''_m + R^{(2)}_m \big) (\mathbb{I} + \mathbb{G}''_{m-1}) \right\} (\mathbb{I} + \mathbb{G}''_{-1})^{-1}.$$

We choose matrices $\mathbb{G}''_m$ satisfying the relation

$$\mathbb{T}''_m - \mathbb{G}''_m + \mathbb{G}''_{m-1} = 0. \tag{4·5}$$

Since the frequencies ω_i satisfy conditions (2)(ii) and (iii), the following estimate can be proved as before

$$\| \mathbb{G}''_m \| = O(1/m^{\gamma}). \tag{4·6}$$

This form of Harris–Lutz type procedure gives the following representation for the propagation matrix $M(0, (n+1)T; E)$, which can be simplified by first exchanging the order of the matrices $W^{-1}(E)\mathcal{R}_\theta^{n+1}$ and $\mathbb{I} + \mathbb{G}''_n$ (due to the uniform boundedness of $\| W^{-1}(E)\mathcal{R}^n_\theta \|$, $n = 1, 2, \ldots$), and then returning back to the representation used in formula (2·4). Additionally we include the multiplies $(\mathbb{I} + \mathbb{G}''_{-1})^{-1}$ into the reminder

$$\begin{aligned}
&M(0, (n+1)T; E) \\
&\quad = W^{-1}(E)\mathcal{R}_\theta^{n+1}(\mathbb{I} + \mathbb{G}''_n) \Bigg(\prod_{m=0}^{\overset{\curvearrowleft}{n}} \left\{ \mathbb{I} + \mathbb{T}'_m + R^{(4)}_m \right\} \Bigg) W(E) \\
&\quad = (\mathbb{I} + o(1)) W^{-1}(E)\mathcal{R}_\theta^{n+1} \Bigg(\prod_{m=0}^{\overset{\curvearrowleft}{n}} \left\{ \mathbb{I} + \mathbb{T}'_m + R^{(4)}_m \right\} \Bigg) W(E) \\
&\quad = (\mathbb{I} + o(1)) \prod_{m=0}^{\overset{\curvearrowleft}{n}} \Bigg\{ M_{\text{per}}(E) + \sum_{i=1}^{N_1} \frac{c_i}{((m+1)T)^{\gamma_i}} \\
&\qquad \times \int_0^T \sin(2\omega_i(mT + \tau) + \varphi_i) M_0(\tau, T) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau)\, d\tau + R^{(5)}_m \Bigg\}
\end{aligned}$$

$$\begin{aligned}
&= (\mathbb{I} + o(1)) \prod_{m=0}^{\overset{\curvearrowleft}{n}} \Bigg\{ M_{\text{per}}(E) + \sum_{i=1}^{N_1} \frac{c_i}{((m+1)T)^{\gamma_i}} \\
&\qquad \times \int_0^T \sin(2\omega_i \tau + \varphi_i) M_0(\tau, T) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau)\, d\tau + R^{(5)}_m \Bigg\} \\
&= (\mathbb{I} + o(1)) \prod_{m=0}^{\overset{\curvearrowleft}{n}} \Bigg\{ M_{\text{per}}(E) + \sum_{i=1}^{N_1} \frac{1}{(m+1)^{\gamma_i}} \mathbb{F}_i + R^{(5)}_m \Bigg\},
\end{aligned} \tag{4·7}$$

where we used that $2\omega_i T/\pi \in \mathbb{Z}$, $i = 1, 2, \ldots, N_1$, (assumption (iii)) and the following notation for m-independent real 2×2 matrices

$$\mathbb{F}_i = \frac{c_i}{T^{\gamma_i}} \int_0^T \sin(2\omega_i \tau + \varphi_i) M_0(\tau, T) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0, \tau)\, d\tau,\ i = 1, 2, \ldots, N_1. \tag{4·8}$$

This representation for the propagation matrix allows one to use the discrete version of the Levinson theorem ([**13**])[3]:

PROPOSITION 4 (*Discrete version of Levinson theorem for* 2×2 *matrices*). *Consider the recurrent equation*

$$\vec{X}_{n+1} = (\mathbb{V}_n + \mathbb{R}_n)\vec{X}_n, \ \ \vec{X}_n \in \mathbb{C}^2, \tag{4·9}$$

satisfying the following assumptions:

(i) *the matrices* $\mathbb{V}_n + \mathbb{R}_n$ *are invertible*;

(ii) $\sum_{n=1}^{\infty} \parallel \mathbb{R}_n \parallel < \infty$;

(iii) $\{\mathbb{V}_n\}_{n=0}^{\infty}$ *belongs to Stolz class* D^1, *i.e.*

$$\sum_{n=1}^{\infty} \parallel \mathbb{V}_n - \mathbb{V}_{n-1} \parallel < \infty,$$

which implies that the limit $\lim_{n\to\infty} \mathbb{V}_n = \mathbb{V}_\infty$ *exists*;

(iv) *the matrix* $\mathbb{V}_\infty$ *has two eigenvectors and two different nonzero eigenvalues*

$$\mathbb{V}_\infty \vec{e}^{\,j} = \lambda^j \vec{e}^{\,j}, \ \ j = 1, 2,$$

$$\lambda^1 \neq \lambda^2, \ \ \lambda^j \neq 0;$$

(v) (*Levinson condition*)[4]
for certain $n_0 \geqslant 1$ *either*

$$\left| \frac{\prod_{k=n_0}^{n} \lambda_k^1}{\prod_{k=n_0}^{n} \lambda_k^2} \right| \longrightarrow_{n\to\infty} \infty \text{ and } \left| \frac{\prod_{k=n_1}^{n_2} \lambda_k^1}{\prod_{k=n_1}^{n_2} \lambda_k^2} \right| > \delta > 0 \text{for any } n_2 \geqslant n_1 \geqslant n_0; \tag{4·10}$$

or

$$\left| \frac{\prod_{k=n_1}^{n_2} \lambda_k^1}{\prod_{k=n_1}^{n_2} \lambda_k^2} \right| \leqslant C \text{ for any } n_2 \geqslant n_1 \geqslant n_0 \tag{4·11}$$

holds, where $\lambda_k^j \neq 0$ *are the eigenvalues of* $\mathbb{V}_k$ *ordered so that* $\lambda_k^j \to_{k\to\infty} \lambda^j$, $j = 1, 2$.

Then there exists a basis $\vec{X}_n^j$ *of solutions to* (4·9) *having the asymptotics*

$$\vec{X}_n^j = \left(\prod_{k=n_0}^{n} \lambda_k^j \right) \big(\vec{e}^{\,j} + o(1) \big), \text{ as } s \to \infty, \ i = 1, 2. \tag{4·12}$$

The product in the last formula of (4·7) can be considered as a matrix solution to the recurrent equation (4·9) with

$$\mathbb{V}_n = M_{\mathrm{per}}(E) + \sum_{i=1}^{N_1} \frac{1}{(n+1)_i^{\gamma}} \mathbb{F}_i, \ \ \mathbb{R}_n = R_n^{(5)}, \ \ \mathbb{V}_\infty = M_{\mathrm{per}}(E).$$

Since $|\lambda^1| = |\lambda^2| = 1$ (this holds true since $E \in \sigma_{\mathrm{ac}}(H_0)$) one has to check that the Levinson condition (v) is satisfied. Taking into account condition (i), and the fact that all matrices $\mathbb{F}_i$ and M_{per} are real, one concludes that $|\lambda_k^1| = |\lambda_k^2|$ for sufficiently large k and condition (4·11) is trivially satisfied. Here we used that the eigenvalues of $M_{\mathrm{per}}(E)$ are different and

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[3] For classical continuous Levinson theorem see, for example, [**7**].

[4] The Levinson condition (5) is always satisfied when $|\lambda^1| \neq |\lambda^2|$.

are lying on the unit circle (elliptic case) and therefore the ellipticity is stable under small real perturbations.

In order to apply Proposition 4 let us calculate the eigenvalues of the matrices $\mathbb{V}_n$

$$\mathrm{Sp}\mathbb{V}_n = \mathrm{Sp}\, M_{\mathrm{per}}(E) + \sum_{i=1}^{N_1} \frac{1}{(n+1)^{\gamma_i}} \mathrm{Sp}\, \mathbb{F}_i = \mathrm{Sp}\, M_{\mathrm{per}}(E) \equiv 2\cos\theta(E),$$

since (see (4·8))

$$\begin{aligned}\mathrm{Sp}\, \mathbb{F}_i &= \frac{c_i}{T^{\gamma_i}} \int_0^T \sin(2\omega_i\tau + \varphi_i)\mathrm{Sp}\left\{ M_0(\tau, T) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) \right\} d\tau \\ &= \frac{c_i}{T^{\gamma_i}} \int_0^T \sin(2\omega_i\tau + \varphi_i)\mathrm{Sp}\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,T) \right\} d\tau \\ &= \frac{c_i}{T^{\gamma_i}} \left[\int_0^T \sin(2\omega_i\tau + \varphi_i) d\tau \right] \mathrm{Sp}\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,T) \right\} \\ &= 0,\end{aligned}$$

because $\int_0^T \sin(2\omega_i\tau + \varphi_i)d\tau = 0$ for $i = 1, 2, \ldots, N_1$ according to assumption (iii).

Considering $\mathbb{V}_n$ as a perturbation of $M_{\mathrm{per}}(E)$, $\det M_{\mathrm{per}}(E) = 1$, one gets

$$\det \mathbb{V}_n = 1 + \sum_{i=1}^{N_1} \frac{1}{(n+1)^{\gamma_i}} \mathrm{Sp}\, \left(M_{\mathrm{per}}^{-1}(E)\mathbb{F}_i \right) + O\left(\frac{1}{n^{2\gamma}} \right).$$

By formula (4·8)

$$\mathrm{Sp}\, \left(M_{\mathrm{per}}^{-1}(E)\mathbb{F}_i \right) = \frac{c_i}{T^{\gamma_i}} \int_0^T \sin(2\omega_i t + \varphi_i)\mathrm{Sp}\, \left(M_0^{-1}(0,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) \right) d\tau = 0,$$

since $\mathrm{Sp}\,(M_0^{-1}(0,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau)) = \mathrm{Sp}\, \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) = 0$. It follows that

$$\det \mathbb{V}_n = 1 + O\left(\frac{1}{n^{2\gamma}} \right)$$

and therefore $\det \mathbb{V}_n - 1 \in \ell_1$ since $\gamma > 1/2$. Therefore the eigenvalues of $\mathbb{V}_n$ are ℓ_1 perturbations of those of $M_{\mathrm{per}}(E)$ due to condition (i), although $\mathbb{V}_n$ is not an ℓ_1 perturbation of $M_{\mathrm{per}}(E)$. Then formula (4·12) implies the asymptotic representation (4·1).

Since condition (ii) is generically satisfied for $E \in \sigma_{\mathrm{ac}}(H_0)$ Gilbert–Pearson theory implies that $\sigma_{\mathrm{ac}}(H) = \sigma_{\mathrm{ac}}(H_0)$ including local multiplicities.

5. *Homogenized interaction matrix and subordinated solutions*

It will be convenient in what follows to rearrange the sum of homogenized interaction matrices as follows. Let us separate the set $\{\gamma_i\}_{i=1}^N$ into classes $\tilde{\gamma}_k$, $k = 1, 2, \ldots, M (M \leqslant N)$ corresponding to different exponents:

$$\tilde{\gamma}_1 < \tilde{\gamma}_2 < \cdots < \tilde{\gamma}_M. \tag{5·1}$$

Using these notations the sum of homogenized perturbation matrices is given by

$$\sum_{i=1}^{N} \frac{1}{(m+1)^{\gamma_i}} \Gamma^{(i)} = \sum_{k=1}^{M} \frac{1}{(m+1)^{\tilde{\gamma}_k}} \tilde{\Gamma}_k, \tag{5·2}$$

where

$$\tilde{\Gamma}_k = \sum_{i:\gamma_i=\tilde{\gamma}_k} \Gamma^{(i)}. \tag{5·3}$$

Among the matrices $\tilde{\Gamma}_k$ several can be equal to zero. Consider the smallest k such that $\tilde{\Gamma}_k$ is different from zero. This matrix $\tilde{\Gamma}$ and the corresponding exponent $\tilde{\gamma}$ will be called the *principle homogenized interaction matrix* and the *principle exponent* respectively. In the case all $\tilde{\Gamma}_k = 0$ the principle homogenized interaction matrix is equal to zero.

LEMMA 2. *Suppose that $\omega_i T/\pi \notin \mathbb{Z}$. Then the homogenized interaction matrix $\Gamma^{(i)}$ is symmetric, real and has zero trace.*

Proof. The homogenized interaction matrix $\Gamma^{(i)}$ admits the representation

$$\Gamma^{(i)} = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{c_i}{T^{\gamma_i}} \int_0^T \sin(2\omega_i(mT+\tau)+\varphi_i)\mathcal{R}_\theta^{-m} W M_0^{-1}(0,\tau) \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) W^{-1} \mathcal{R}_\theta^m \, d\tau. \tag{5·4}$$

It is clear that the matrix $\Gamma^{(i)}$ is real and has zero trace, since $\mathrm{Sp} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$.

Note that a real matrix A is symmetric if and only if $\mathrm{Sp} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A \right\} = 0$. Therefore let us consider

$$\begin{aligned} \mathrm{Sp} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Gamma^{(i)} \right\} &= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \int_0^T \sin(2\omega_i(mT+\tau)+\varphi_i) \\ &\quad \times \mathrm{Sp} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{R}_\theta^{-m} W M_0^{-1}(0,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) W^{-1} \mathcal{R}_\theta^m \right\} d\tau \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \int_0^T \sin(2\omega_i(mT+\tau)+\varphi_i) \\ &\quad \times \mathrm{Sp} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W M_0^{-1}(0,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) W^{-1} \right\} d\tau, \end{aligned}$$

since the matrices $\mathcal{R}_\theta^{-m}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ commute. Let us use notation

$$g(\tau) = \mathrm{Sp} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W M_0^{-1}(0,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) W^{-1} \right\} \in L_1(0,T).$$

Then elementary calculations imply the following Cesaro limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n} \int_0^T \sin(2\omega_i(mT+\tau)+\varphi_i) g(\tau)\, d\tau = 0,$$

provided $\omega_i T/\pi \notin \mathbb{Z}$.

The principle homogenized interaction matrix $\tilde{\Gamma}$ determining the asymptotic behaviour of the solutions has the same properties: it is symmetric, real and has zero trace. It follows that its determinant is nonpositive. Let us denote the eigenvalues and eigenvectors of $\tilde{\Gamma}$ by $\pm\mu$, $\mu \geqslant 0$ and $\vec{e}_{\pm}$, respectively.

THEOREM 5. *Suppose that the following assumptions are satisfied*:

(i) $E \in \sigma_{\mathrm{ac}}(H_0)$, *and* $\left|\mathrm{Sp}\,(M_{\mathrm{per}}(E))\right| = 2\cos\theta < 2$, *i.e. the energy* E *belongs to the generalized interior of* $\sigma_{ac}(H_0)$;

(ii) $(\omega_i T + \theta(E))/\pi \in \mathbb{Z}$ *or* $(\omega_i T - \theta(E))/\pi \in \mathbb{Z}$ *for all* $i = 1, 2, \ldots, N_2$ *and* $(\omega_i T \pm \theta(E))/\pi \notin \mathbb{Z}, i = N_2 + 1, \ldots, N, N_2 \leqslant N$;

(iii) $\omega_i T/\pi \notin \mathbb{Z}, i = 1, 2, \ldots, N$;

Let $\tilde{\Gamma}$ *be the principle homogenized interaction matrix with the eigenvalues* $\pm\mu$, $\mu \geqslant 0$.

If $\mu \neq 0$ ($\tilde{\Gamma} \neq 0$) *then there exists a subordinated on* $\mathbb{R}_+$ *solution having the following asymptotics depending on the principle exponent* $\tilde{\gamma}$:

(a) *if* $(1/2 <)\tilde{\gamma} < 1$, *then*

$$\left|\begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}\right|_{\mathbb{C}^2} \sim \exp\left\{-\frac{\mu}{1-\tilde{\gamma}}\left(\frac{x}{T}\right)^{1-\tilde{\gamma}}\right\} \in L_2(\mathbb{R}_+); \tag{5·5}$$

(b) *if* $\tilde{\gamma} = 1$, *then*

$$\left|\begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}\right|_{\mathbb{C}^2} \sim \frac{1}{x^{\mu}}, \tag{5·6}$$

and therefore belongs to $\ell_2(\mathbb{R}_+)$ *only if* $\mu > 1/2$.

If $\mu = 0$, *i.e.* $\tilde{\Gamma} = 0$, *then no subordinated solution on* $\mathbb{R}_+$ *exists, the asymptotics of the solutions coincides with the free one, and therefore every nonzero solution* ψ *to* (1·8) *satisfies the estimate*

$$0 < c \leqslant \left|\begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}\right|_{\mathbb{C}^2} \leqslant C.$$

Proof.

Step 1. Following step by step the first part of the proof of Theorem 2 the terms containing frequencies $\omega_i, i = N_2 + 1, \ldots, N$ can be removed from the consideration. Therefore investigation of $M(0, (n+1)T; E) = W^{-1}(E)\mathcal{R}_\theta^{n+1}\Pi_n W(E)$ reduces to the study of

$$\begin{aligned}\Pi_n = (\mathbb{I} + o(1))\Bigg\{\prod_{m=0}^{\overset{\curvearrowleft}{n}}\Bigg(\mathbb{I} + \sum_{i=1}^{N_2}\frac{c_i}{((m+1)T)^{\gamma_i}}\mathcal{R}_\theta^{-m}W(E)\\ \times\int_0^T \sin(2\omega_i(mT+\tau)+\varphi_i)M_0^{-1}(0,\tau)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}M_0(0,\tau)W^{-1}(E)\mathcal{R}_\theta^m + R_m^{(6)}\Bigg)\Bigg\},\end{aligned} \tag{5·7}$$

where the reminder $R_m^{(6)}$ absorbs the influence of all perturbation terms having frequencies ω_i, $i = N_2 + 1, \ldots, N$.

Step 2. Proceeding as in Section 3 (see (3·11)) let us extract the nonoscillating terms

$$\mathbb{H}_m^{(i)} = [\text{oscillating part}] + \Gamma^{(i)}(E) \equiv \mathbb{H}_m^{osc,(i)} + \Gamma^{(i)}(E),\ i = 1, 2, \ldots, N_2,$$

where the matrices $\Gamma^{(i)}$ are given by (3·12). Applying the Harris–Lutz type procedure once more (see the proof of Theorem 2, in particular (4·5) and (4·6)) one can get rid of all oscillating terms and re-write Π_n in the form

$$\Pi_n = (\mathbb{I} + o(1)) \left\{ \prod_{m=0}^{\overset{\frown}{n}} \left(\mathbb{I} + \sum_{i=1}^{N_2} \frac{1}{(m+1)^{\gamma_i}} \Gamma^{(i)} + R_m^{(7)} \right) \right\}. \tag{5·8}$$

Step 3. The homogenized interaction matrices $\Gamma^{(i)}$ are real, symmetric and have zero trace. The principle matrix $\tilde{\Gamma} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ has the same properties and therefore its discriminant $\operatorname{disc} \tilde{\Gamma} = -\det \tilde{\Gamma} = a^2 + b^2 \geqslant 0$ is nonnegative, which implies that the matrix has one nonpositive and one nonnegative eigenvalue.

Step 4. Let us come back to the analysis of the asymptotics of the product

$$\prod_{m=0}^{\overset{\frown}{n}} \left(\mathbb{I} + \sum_{i=1}^{N_2} \frac{1}{(m+1)^{\gamma_i}} \Gamma^{(i)} + R_m^{(7)} \right)$$

and extract the leading homogenized interaction matrix

$$\sum_{i=1}^{N_2} \frac{1}{(m+1)^{\gamma_i}} \Gamma^{(i)} = \frac{1}{(m+1)^{\tilde{\gamma}}} \left(\tilde{\Gamma} + \sum_{i:\tilde{\gamma}_i > \tilde{\gamma}} \frac{1}{(m+1)^{\tilde{\gamma}_i - \tilde{\gamma}}} \tilde{\Gamma}_i \right). \tag{5·9}$$

To investigate the asymptotics of the corresponding product let us use the following

PROPOSITION 6 ([**13**, theorem 1·7]).
Consider the recurrent relation

$$\vec{X}_{n+1} = (\mathbb{V}_n + \mathbb{R}_n) \vec{X}_n, \ n = 0, 1, 2, \ldots, \vec{X}_n \in \mathbb{C}^2, \tag{5·10}$$

where $\mathbb{V}_n = a_n \mathbb{I} + p_n \mathbb{S}$:

(i) $\sum_n \parallel \mathbb{S}_{n+1} - \mathbb{S}_n \parallel < \infty$;
(ii) $a_n, p_n \in \mathbb{R}$; $a_n \to a_\infty \dotplus 0$, $p_n \to 0$, as $n \to \infty$;
(iii) $\sum_n \parallel \mathbb{R}_n \parallel < \infty$;
(iv) $\det(\mathbb{V}_n + \mathbb{R}_n) \dotplus 0$.

Assumption (*i*) *implies the existence of the limit* $\lim_{n\to\infty} \mathbb{S}_n = \mathbb{S}_\infty$. *Denote by* $\mu_n^{(i)}, \vec{e}_n^{(i)}$ *and* $\mu_\infty^{(i)}, \vec{e}_\infty^{(i)}$ *the eigenvalues and eigenvectors of* $\mathbb{S}_n$ *and* $\mathbb{S}_\infty$ *respectively chosen in such a way that the sequences* $\mu_n^{(i)}$ *converge to* $\mu_\infty^{(i)}, i = 1, 2$.
(a) *If* $\operatorname{disc} \mathbb{S}_\infty < 0$ *then there exists a basis* $\vec{X}_n^{(1)}, \vec{X}_n^{(2)}$ *of solutions to* (5·10) *such that*

$$\vec{X}_n^{(i)} = \left(\prod_{m=n_0}^{n-1} \left(a_m + p_m \mu_m^{(i)} \right) \right) \vec{e}^{(i)}, \ i = 1, 2, \tag{5·11}$$

for a certain n_0.
(b) *If* $\operatorname{disc} \mathbb{S}_\infty > 0$ *and the sequence* p_n *is either positive or negative, then there exists subordinated solution*

$$\vec{X}_n^{(1)} = \left(\prod_{m=n_0}^{n-1} \left(a_m + p_m \mu_m^{(1)} \right) \right) \vec{e}^{(1)}, \tag{5·12}$$

where $\operatorname{sign}(p_n/a_\infty)(\mu_\infty^{(2)} - \mu_\infty^{(1)}) > 0$.

We can apply this proposition to the recurrent relation

$$\vec{X}_{n+1} = \left(\mathbb{I} + \sum_{i=1}^{N_2} \frac{1}{(n+1)^{\gamma_i}} \Gamma^{(i)} + R_n^{(7)}\right)\vec{X}_n, \tag{5·13}$$

using the following identification of the parameters

$$a_n \equiv 1,\ p_n = \frac{1}{(n+1)^{\tilde{\gamma}}},\ \mathbb{S}_n = \tilde{\Gamma} + \sum_{i:\tilde{\gamma}_i > \tilde{\gamma}} \frac{1}{(n+1)^{\tilde{\gamma}_i - \tilde{\gamma}}} \tilde{\Gamma}_i;\ \mathbb{S}_\infty = \tilde{\Gamma};\ \mathbb{R}_n = R_n^{(7)}. \tag{5·14}$$

Since the disc $\tilde{\Gamma} = -\det \tilde{\Gamma} \geqslant 0$, (the matrix is symmetric, real and has zero trace), let us separate two different cases:

(a) (generic) $\det \tilde{\Gamma} < 0 \Rightarrow \mu > 0$.

Recurrent equation (5·13) has principle solution (by Proposition 6) having the asymptotics

$$\vec{X}_{n+1} = \left[\prod_{m=n_0}^{n} \left(1 - \frac{\mu}{(m+1)^{\tilde{\gamma}}}\right)\right] \left[\vec{e}_- + o(1)\right].$$

The evolution matrix admits the representation

$$\begin{aligned} M(0, (n+1)T; E) &= W^{-1}\mathcal{R}_\theta^{n+1}\Pi_n W \\ &= W^{-1}\mathcal{R}_\theta^{n+1}(\mathbb{I} + \mathbb{B}_n)\left\{\prod_{m=0}^{n}\left(\mathbb{I} + \sum_{i=1}^{N_2} \frac{1}{(m+1)^{\gamma_i}}\Gamma_i + R_m^{(7)}\right)\right\} W, \end{aligned}$$

where $\mathbb{B}_n = o(1)$. It is clear that the matrices $\mathbb{I} + \mathbb{B}_n$ are invertible, since the evolution matrix is invertible. Consider the sequence of vectors

$$\vec{X}_{n+1} = (\mathbb{I} + \mathbb{B}_n)^{-1}\mathcal{R}_\theta^{-(n+1)} W M(0, (n+1)T; E) W^{-1}\vec{Y},\ \vec{Y} \in \mathbb{C}^2.$$

It satisfies recurrent relation (5·13) and therefore, by Proposition 6, for a proper choice of the vector $\vec{Y}$ the sequence $\vec{X}_n$ has the asymptotics

$$\vec{X}_{n+1} = \left\{\prod_{m=n_0}^{n}\left(1 - \frac{\mu}{(m+1)^{\tilde{\gamma}}}\right)\right\}\left[\vec{e}_- + o(1)\right]. \tag{5·15}$$

For such choice of $\vec{Y}$ we have

$$\begin{aligned} M(0, (n+1)T; E) W^{-1}\vec{Y} &= \left\{\prod_{m=n_0}^{n}\left(1 - \frac{\mu}{(m+1)^{\tilde{\gamma}}}\right)\right\} W^{-1}\mathcal{R}_\theta^{(n+1)}(\vec{e}_- + o(1)) \\ &= \left\{\prod_{m=n_0}^{n}\left(1 - \frac{\mu}{(m+1)^{\tilde{\gamma}}}\right)\right\} M_0(0, (n+1)T; E)(W^{-1}\vec{e}_- + o(1)). \end{aligned}$$

Using perturbation theory this asymptotic formula can easily be interpolated between the discrete values of $x = nT$, $n \in \mathbb{N}$. Indeed we have

$$M(0, x; E) W^{-1}\vec{Y} = \left\{\prod_{m=n_0}^{[\frac{x}{T}]}\left(1 - \frac{\mu}{(m+1)^{\tilde{\gamma}}}\right)\right\} M_0(0, x; E)(W^{-1}\vec{e}_- + o(1)). \tag{5·16}$$

It follows that there exists a solution to the Schrödinger equation (3·18) with the asymptotics

$$\begin{aligned}\begin{pmatrix}\psi(x)\\ \psi'(x)\end{pmatrix} &= \left\{\prod_{m=n_0}^{[\frac{x}{T}]}\left(1-\frac{\mu}{(m+1)^{\tilde\gamma}}\right)\right\} M_0(0,x;E)(W^{-1}\vec{e}_- + o(1))\\ &= \left\{\prod_{m=n_0}^{[\frac{x}{T}]}\left(1-\frac{\mu}{(m+1)^{\tilde\gamma}}\right)\right\} (M_0(0,x;E)W^{-1}\vec{e}_- + o(1)), \qquad (5\cdot17)\end{aligned}$$

where

$$\prod_{m=n_0}^{[\frac{x}{T}]}\left(1-\frac{\mu}{(m+1)^{\tilde\gamma}}\right) = \begin{cases}\text{const}\, e^{-\frac{\mu(t/T)^{1-\tilde\gamma}}{1-\tilde\gamma}}(1+o(1)), \ \tilde\gamma<1;\\ \text{const}\,\dfrac{1}{t^{\mu}}(1+o(1)), \qquad \tilde\gamma=1.\end{cases} \qquad (5\cdot18)$$

(b) $\det\tilde\Gamma = 0 \Rightarrow \tilde\Gamma = 0$.
This implies that all matrices $\tilde\Gamma_i$ are equal to zero. It is obvious from (5·13) that the asymptoics of all solutions to (3·18) coincides with the asymptotics for the original equation, since

$$\Pi_n = (\mathbb{I}+o(1))\left\{\prod_{m=1}^{n}\left(\mathbb{I}+R_m^{(7)}\right)\right\}$$

converges to a certain invertible matrix. Hence for such values of E every solution to (1·8) has pure oscillating asymptotics, in particular no decaying solution occurs.

The homogenized interaction matrices can be calculated explicitly using the monodromy matrix for the background periodic potential and the parameters $\gamma_j, c_i, \omega_i, \varphi_i$ describing the perturbation. Let us introduce the matrices not depending on m and having zero trace

$$\begin{aligned}A(\omega_i,E) &= \frac{c_i}{T^{\gamma_i}}W\int_0^T \cos(2\omega_i\tau+\varphi_i)M_0^{-1}(0,\tau)\begin{pmatrix}0&0\\1&0\end{pmatrix}M_0(0,\tau)d\tau W^{-1},\\ B(\omega_i,E) &= \frac{c_i}{T^{\gamma_i}}W\int_0^T \sin(2\omega_i\tau+\varphi_i)M_0^{-1}(0,\tau)\begin{pmatrix}0&0\\1&0\end{pmatrix}M_0(0,\tau)d\tau W^{-1}.\end{aligned} \qquad (5\cdot19)$$

Then for $i = 1, 2, \ldots, N_2$ the perturbation integral in (5·7) can be written as follows with the help of the matrices just introduced (using elementary trigonometric formula)

$$\begin{aligned}&\frac{c_i}{T^{\gamma_i}}\mathcal{R}_\theta^{-m}W(E)\int_0^T \sin(2\omega_i(mT+\tau)+\varphi_i)M_0^{-1}(0,\tau)\begin{pmatrix}0&0\\1&0\end{pmatrix}M_0(0,\tau)W^{-1}(E)\mathcal{R}_\theta^m\\ &\qquad = \mathcal{R}_\theta^{-m}\left\{\sin(2\omega_i mT)A(\omega_i,E)+\cos(2\omega_i mT)B(\omega_i,E)\right\}\mathcal{R}_\theta^m. \qquad (5\cdot20)\end{aligned}$$

Let us denote the real entries of the matrices $A(\omega_i,E)$ and $B(\omega_i,E)$ as follows

$$A(\omega_i,E) = \begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}; \ B(\omega_i,E) = \begin{pmatrix}a_i' & b_i'\\ c_i' & -a_i'\end{pmatrix}, \qquad (5\cdot21)$$

using the fact that these matrices are real and have zero trace. These matrices depend non-trivially on the background periodic potential and in a more trivial way on the parameters

ω_i, φ_i of the Wigner–von Neumann perturbation. To calculate the Cesaro limit (3·12) with

$$\mathbb{H}_m^{(i)} = R_\theta^{-m}\,(\sin 2\omega_i mT\ A(\omega_i, E) + \cos 2\omega_i mT\ B(\omega_i, E))\,R_\theta^m,$$

$$R_\theta^m = \begin{pmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{pmatrix},$$

one needs to calculate the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (m\theta) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (m\theta) \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} (2\omega_i mT).$$

The limit is not equal to zero if and only if at least one of the resonance relations $(\omega_i T \pm \theta)/\pi \in \mathbb{Z}$ is satisfied.

Explicit calculations show that the homogenized interaction matrix can be expressed as follows in the three different cases covering all possibilities allowed by the assumptions

(a)

$$\begin{cases} \omega_i T + \theta \notin \pi\mathbb{Z} \\ \omega_i T - \theta \in \pi\mathbb{Z} \\ \omega_i T \notin \pi\mathbb{Z} \end{cases} \Rightarrow \Gamma^{(i)} = \pm\frac{1}{4} \begin{pmatrix} -(b_i + c_i) + 2a_i' & a_i + b_i' + c_i' \\ a_i + b_i' + c_i' & b_i + c_i - 2a_i' \end{pmatrix}$$

$$\Longrightarrow \operatorname{disc}\Gamma^{(i)} = -\det\Gamma^{(i)} = \frac{1}{16}((b_i + c_i - 2a_i')^2 + (a_i + b_i' + c_i')^2) \geqslant 0.$$

(b)

$$\begin{cases} \omega_i T + \theta \in \pi\mathbb{Z} \\ \omega_i T - \theta \notin \pi\mathbb{Z} \\ \omega_i T \notin \pi\mathbb{Z} \end{cases} \Rightarrow \Gamma^{(i)} = \pm\frac{1}{4} \begin{pmatrix} (b_i + c_i) + 2a_i' & -a_i + b_i' + c_i' \\ -a_i + b_i' + c_i' & -(b_i + c_i) - 2a_i' \end{pmatrix}$$

$$\Longrightarrow \operatorname{disc}\Gamma^{(i)} = -\det\Gamma^{(i)} = \frac{1}{16}((b_i + c_i + 2a_i')^2 + (-a_i + b_i' + c_i')^2) \geqslant 0.$$

(c)

$$\begin{cases} \omega_i T + \theta \in \pi\mathbb{Z} \\ \omega_i T - \theta \in \pi\mathbb{Z} \\ \omega_i T \notin \pi\mathbb{Z} \end{cases} \Rightarrow \Gamma^{(i)} = \frac{1}{2} \begin{pmatrix} (b_i + c_i) & -a_i \\ -a_i & -(b_i + c_i) \end{pmatrix}$$

$$\Longrightarrow \operatorname{disc}\Gamma^{(i)} = -\det\Gamma^{(i)} = \frac{1}{4}((a_i')^2 + (b_i + c_i)^2) \geqslant 0.$$

Note that $\det\Gamma^{(i)} = 0$ if and only if $\Gamma^{(i)} = 0$. The cases (a) and (b) are similar. In case (c) it is necessary that $\theta(E) = \pm\pi/2$ and therefore the energy E belongs to the generalized middle point of the energy band.

In order to clarify the results obtained, let us consider the simplest case $N = 1$, i.e. the perturbation term has only one frequency ω_i. The following table summarizes our analysis applied to this case.

	1	2	3	4	5	6	7	8
$\frac{\omega_i T + \theta}{\pi}$	$\in \mathbb{Z}$	$\in \mathbb{Z}$	$\notin \mathbb{Z}$	$\notin \mathbb{Z}$	$\in \mathbb{Z}$	$\notin \mathbb{Z}$	$\in \mathbb{Z}$	$\notin \mathbb{Z}$
$\frac{\omega_i T - \theta}{\pi}$	$\in \mathbb{Z}$	$\notin \mathbb{Z}$	$\in \mathbb{Z}$	$\notin \mathbb{Z}$	$\in \mathbb{Z}$	$\in \mathbb{Z}$	$\notin \mathbb{Z}$	$\notin \mathbb{Z}$
$\frac{\omega_i T}{\pi}$	$\in \mathbb{Z}$	$\in \mathbb{Z}$	$\in \mathbb{Z}$	$\in \mathbb{Z}$	$\notin \mathbb{Z}$	$\notin \mathbb{Z}$	$\notin \mathbb{Z}$	$\notin \mathbb{Z}$
	A	**B**	**B**	**C**	**D**	**E**	**E**	**C**

(5·22)

(i) **A** – the energy E belongs to the generalized boundary of σ_{ac}, i.e. $\theta(E) = 0, \pm\pi$;
(ii) **B** – impossible situation;
(iii) **C** – all solutions are oscillating at infinity (see Theorem 3);
(iv) **D** – the energy E belongs to the generalized middle point of the band of σ_{ac}, i.e. $\theta(E) = \pm\pi/2$, there is a possibility for subordinated solution (see Theorem 5);
(v) **E** – there is a possibility for subordinated solution (see Theorem 5).

The same table can be used even in the case of several frequencies ω_i, $i = 1, 2, \ldots, N$. Each frequency ω_i determines two values of the energy on each band $E(\theta)$, $-\pi \leqslant \theta \leqslant \pi$ of the absolutely continuous spectrum:

$$\frac{\omega_i T + \theta}{\pi} \in \mathbb{Z} \quad \text{and} \quad \frac{\omega_i T - \theta}{\pi} \in \mathbb{Z}.$$

Only for these values of the energy a subordinated solution to (1·8) may occur for the generalized interior of the absolutely continuous spectrum of H_0. In order to decide whether the subordinated solution exists or not one needs to calculate the principle homogenized interaction matrix $\tilde{\Gamma}(E)$.

In the general situation when different frequencies determine different singular points the principle homogenized interaction matrix $\tilde{\Gamma}$ coincides with the corresponding homogenized interaction matrix $\Gamma^{(i)}$ and each frequency separately determines the existence of a subordinated solution.

The singular points corresponding to different frequencies coincide if and only if

$$\frac{(\omega_i - \omega_j)T}{\pi} \in \mathbb{Z} \quad \text{or} \quad \frac{(\omega_i + \omega_j)T}{\pi} \in \mathbb{Z}.$$

In this case the frequencies with the largest exponent γ_i determines the asymptotics, provided $\Gamma^{(i)} \neq 0$. In the case two frequencies, say ω_1 and ω_2, correspond to the same exponent $\tilde{\gamma} = \gamma_1 = \gamma_2$, the principle homogenized interaction matrix $\tilde{\Gamma}$ is equal to the sum $\Gamma^{(1)} + \Gamma^{(2)}$ and can be equal to zero even if $\Gamma^{(1)} \neq 0 \neq \Gamma^{(2)}$. Therefore compatible frequencies corresponding to the same exponent cannot be treated independently.

Note that the matrices $\Gamma^{(i)}(E)$ are zero for all nonresonance energies, provided the periods of the crystal and perturbation are not compatible. Summing up, our analysis gives a full account of the problem for the energies belonging to the generalized interior of the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(H_0)$.

6. *Penetrability of energy levels into spectral bands*

In the previous consideration we studied the subordinated solution to (1·8) when $x \to +\infty$. Similar analysis can be carried out for the negative semi-axis $x \to -\infty$. Therefore in what follows we are going to denote the corresponding principle exponents and homogenized interaction matrices by $\tilde{\gamma}_\pm$ and $\tilde{\Gamma}_\pm$ for $x \to \pm\infty$ respectively.

Consider any E_0 from the generalized interior of $\sigma_{ac}(H_0)$. In the case both $\tilde{\gamma}_+$ and $\tilde{\gamma}_-$ are less than 1 the corresponding $\tilde{\Gamma}_\pm(E_0)$ are different from zero (by definition) and therefore by adjusting V_0 one obtains an operator H with embedded eigenvalue at the point E_0. The construction of a proper V_0 for a one particular value E_0 is elementary. In order to obtain operators with several embedded eigenvalues (including infinitely many) one may use the approach of [**24**].

Note that $\tilde{\Gamma}_\pm$ depend on the coupling parameters $c_j : \gamma_j = \tilde{\gamma}$, but are independent of V_0. Therefore the embedded eigenvalue can be produced for arbitrary small coupling parameters $c_j \to \alpha c_j$, α being arbitrarily small. Hence a weak Wigner–von Neumann perturbation can produce an eigenvalue in this case independently of its position on the spectral band.

However if at least one of $\tilde{\gamma}_\pm$ is equal to 1 the following necessary condition has to be satisfied.

THEOREM 7. *Let* $\tilde{\gamma}_+(E_0) = 1$ *or* $\tilde{\gamma}_-(E_0) = 1$, E_0 *belongs to the generalized interior of* $\sigma_{\mathrm{ac}}(H_0)$. *Then the point* E_0 *can be an eigenvalue of* H *only if the coupling parameters satisfy the following estimate*

$$\sum_{i:\gamma_i=1} |c_i| \geqslant (2 \parallel W(E_0) \parallel \parallel W^{-1}(E_0) \parallel \ ({\sup}^2_{-T\leqslant\tau\leqslant T} \parallel M_0(0,\tau) \parallel)^2)^{-1}. \tag{6·1}$$

Proof. The necessary condition for the existence of a subordinated solution with L^2-asymptotics when $x \to +\infty$ is that $\parallel \tilde{\Gamma}_+ \parallel = \mu > 1/2$. The norm of the principle homogenized interaction matrix can be estimated as follows (see (3·13) and (5·3))

$$\begin{aligned}\parallel \tilde{\Gamma}_+ \parallel = & \left\| \lim_{x\to\infty} \frac{1}{x} \int_0^x \sum_{i:\gamma_i=1} c_i \sin(2\omega_i\tau + \varphi_i) W(E_0) M_0^{-1}(0,\tau) \right. \\ & \left. \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) W^{(-1)}(E_0) d\tau \right\| \\ \leqslant & \sum_{i:\gamma_i=1} |c_i| \sup_{\tau>0} \left\| W(E_0) M_0^{-1}(0,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} M_0(0,\tau) W^{(-1)}(E_0) d\tau \right\| \\ \leqslant & \sum_{i:\gamma_i=1} |c_i| \parallel W(E_0) \parallel \parallel W^{-1}(E_0) \parallel \\ & \times \sup_{0<\tau<T} \parallel M_0^{-1}(0,\tau) \parallel \sup_{0<\tau<T} \parallel M_0(0,\tau) \parallel,\end{aligned}$$

since

$$WM_0(0,\tau)W^{-1} = \mathcal{R}_\theta^{[\tau/T]} WM_0(0,\tau - [\tau/T]T)W^{-1},$$

where $[\cdot]$ denotes the integer part. The same estimate holds true for the negative semiaxis. Thus one gets estimate (6·1) if one takes into account that $M_0^{-1}(0,\tau) = M_0(-\tau, 0)$.

The estimate (6·1) shows that in order to create an eigenvalue inside a band of $\sigma_{\mathrm{ac}}(H_0)$ the sum $\sum_{i:\gamma_i=1} |c_i|$ cannot be small. Since $\| W(E) \| \| W^{-1}(E) \|$ has to be singular on the boundary of absolutely continuous spectrum $\partial(\sigma_{\mathrm{ac}}(H_0))$ the estimate shows that it is "easier" to create an eigenvalue close to the boundary of the band.

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