Math. Proc. Camb. Phil. Soc. (2010), **148**, 331 © *Cambridge Philosophical Society* 2009
doi:10.1017/S030500410999034X
First published online 26 November 2009

Inverse problems for Aharonov–Bohm rings

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(*Received* 7 *April* 2009; *revised* 14 *September* 2009)

Abstract

The inverse problem for Schrödinger operators on metric graphs is investigated in the presence of magnetic field. Graphs without loops and with Euler characteristic zero are considered. It is shown that the knowledge of the Titchmarsh–Weyl matrix function (Dirichlet-to-Neumann map) for just two values of the magnetic field allows one to reconstruct the graph and potential on it provided a certain additional no-resonance condition is satisfied.

1. *Introduction*

Differential operators on metric graphs is a rapidly developing area of modern mathematical physics attracting the attention of a large scientific community, especially during recent years. Such operators were used by theoretical physicists in order to describe quantum transport in small rings and in other simple networks, for example in the presence of magnetic field [**4**, **5**, **6**, **19**, **20**, **69**, **71**, **73**]. Quantum oscillations discovered in such systems have now been observed experimentally (see for example [**34**, **68**, **72**]). Many papers are devoted now to rigorous studies of second order differential operators on graphs and their spectral and scattering properties are rather well-understood from the mathematical point of view [**22–25**, **29**, **30**, **35**, **40–42**, **44**, **47**, **48**, **50**, **58**, **60**, **61**]. One book, several review articles and collection volumes appeared recently reflecting the interest of the mathematical and physical community in such problems [**12**, **45**, **46**, **64**]. These studies show clearly that methods originally developed for both ordinary and partial differential operators can be applied successfully. One may observe a striking similarity to differential operators on Riemannian manifolds, but operators on graphs possess non typical properties as well. We mention just two such phenomena:

(i) The operators on graphs may have eigenfunctions vanishing on certain edges, *i.e.* the null-sets may have non-zero Lebesgue measure.
(ii) The scattering matrices may exhibit total reflection, *i.e.* the reflection coefficients may have absolute value equal to one.

To solve inverse problems one may use different techniques originally developed for partial and ordinary differential equations, but certain modifications are needed. In our opinion

† Supported in part by Swedish Research Council Grant #80525401.

many difficulties in solving the inverse problem are related to the special properties of equations on graphs mentioned above. This paper, devoted to the solution of the inverse problem for graphs with zero Euler characteristic, is an illustration to the above statement.

The inverse problems for differential operators on graphs have been studied from the beginning of the 80s. Such problems in fact contain three entirely connected problems:

(i) reconstruction of the metric graph;
(ii) reconstruction of the differential operators on the edges (real potentials in the case of the Schrödinger operator);
(iii) reconstruction of the boundary conditions at the vertexes.

It has been realized that the inverse problem in general does not have unique solution [**11**, **33**, **50**]. Several articles are devoted to the solution of the inverse problem in the case of trees [**2**, **7**, **8**, **10**, **16**, **29**, **30**, **36**, **37**, **41**, **74–76**] and this problem is rather well developed, even if certain questions still remain open. For example it is not clear how to reconstruct the boundary conditions in the case other than the star graph. The methods used to solve this problem are almost one-dimensional, which is related to the fact that on trees one may consider Cauchy problem (which is often impossible if the graph possess cycles). An illustration to this statement is the article [**2**], where it has been shown that the inverse problem for trees can be solved if the reduced by one dimension Titchmarsh–Weyl function (Dirichlet-to-Neumann map) (see Section 4·1 for the precise definition) is known, *i.e.* the Titchmarsh–Weyl matrix function associated with all except one boundary vertex. That paper is based on the generalization of Boundary Control method [**9**], which in the case of dimension one is closely connected with the inverse scattering approach based on the A-amplitude [**3**, **31**, **65**, **70**].

The inverse problem for graphs with cycles is not so well understood and only few partial results are known. Probably the most interesting result obtained so far is that the spectrum of the Laplace operator (differential operator $-d^2/dx^2$ on the edges with standard matching conditions (2·3) at the vertexes) uniquely determines the underlying graph, provided the lengths of the edges are rationally independent. This fact was first proved by B. Gutkin, T. Kottos and U. Smilansky in [**33**, **44**] and improved later by P. Kurasov and M. Nowaczyk in [**47**, **49**, **59**]. To prove this fact, the trace formula, first considered by J.-P. Roth [**67**], has been used. Rigorous treatment of the problem allowed one to establish another important fact, namely that the spectrum of a Schrödinger operator on a compact finite graph determines the Euler characteristic of the underlying geometric graph. The Euler characteristic χ is equal to $M-N$, where M is the number of vertexes and N is the number of edges. It gives in particular the number of generators in the fundamental group and is deeply connected with the index theory [**26**]. In addition the total length of the graph can trivially be calculated from the spectral asymptotics which is of Weyl type. One may say that the size and complexity of the graph is uniquely determined by the spectrum of the Schrödinger operator on it. It has been proved by V. Kostrykin and R. Schrader [**43**], that for almost all boundary conditions the scattering matrix uniquely determines the graph and its metric structure. Similarly L. Freidlander proved in [**25**], that the spectrum of operators on metric graphs is generically simple. These papers do not give any explicit criteria which guarantees the unique solvability of the inverse problem.

The situation with the second inverse problem mentioned above (concerning recovering of the potential) is even more complicated. Of course the Titchmarsh–Weyl matrix allows

one to calculate the potential on all boundary edges and even on all branches (this follows directly from the results of [**2**]). On the other hand calculation of the potential on the graph's kernel (the part of the graph which remains when all branches are cut away, see Section 2·1 for details) is a difficult problem. The only fact known to the author is that in general the potential cannot be reconstructed if the graph has a loop [**62**], or if the kernel has internal symmetries [**13**, **50**].

The problem to recover the matching conditions in the presence of cycles has been studied in [**43**], where it was shown that the boundary conditions generically can be recovered up to trivial gauge transformation. As will be shown for magnetic Schrödinger operators, introduction of the magnetic potential is equivalent to a certain change in the matching conditions. Therefore in this paper we restrict our consideration to the case of standard matching conditions.

We note that for a class of second order differential operators the inverse problem has been considered in [**21**]. Any differential operator on a metric graph can also be seen as a multichannel operator. The corresponding inverse problem was studied recently by Yu. I. Lyubarskii and V. A. Marchenko [**53**].

In this paper we suggest a completely new approach to the inverse problem for graphs. Since the Titchmarsh–Weyl matrix, or Dirichlet-to-Neumann map, appeared to be an efficient tool to solve the inverse problem on graphs, we are going to discuss the possibility of recovering the graph and potential on it from this matrix which is going to play the role of spectral data. As the counterexamples show, these data are not enough to solve the inverse problem [**13**, **33**, **62**]. Therefore we are going to extend the set of spectral data by introducing a magnetic field. It is not so hard to show that the spectrum of the magnetic Schrödinger operator on a metric graph does not depend on the particular form of the magnetic field, but does depend on the fluxes of the magnetic field through the cycles in the graph (see Proposition 2·1). Therefore it is natural to ask the question whether the inverse problem can be solved if the Titchmarsh–Weyl function is known for different values of the magnetic flux. One may think about an experimenter measuring the Dirichlet-to-Neumann map for graphs first without any magnetic field, and then in the presence of the magnetic field. We give an exhausting answer to this problem in the case of the simplest graphs having just one cycle (*i.e.* Euler characteristic equal to zero). It appears that if the graph contains a loop, then the potential on the loop in general cannot be recovered even from the extended set of spectral data described above, also the freedom allowed is reduced. On the other hand, for graphs without loops the potential is determined by the new spectral data, but under certain no-resonance conditions. It is striking that this no-resonance condition is closely related to the properties of differential operators on graphs described above (see Section 7·2). We consider our studies as a first step in the research programme aiming to solve the inverse problem for graphs with arbitrary Euler characteristic. One may see direct relations between our studies and classical inverse spectral results in dimension one. It is well known that the potential in the Sturm–Liouville operator on a finite interval is uniquely determined by two spectra: Dirichlet–Dirichlet and Dirichlet–Neumann spectra for example [**14**, **15**, **51**, **52**, **54**, **55**, **63**]. As we have already mentioned, introduction of the magnetic field corresponds to the change in the boundary condition. Therefore our results can also be formulated as the potential that the graph is uniquely determined by the Titchmarsh–Weyl function given for different matching conditions, but allowed change in the matching conditions is not arbitrary and this makes the problem difficult. (Otherwise one

may consider the matching conditions decomposing the graph into the set of independent intervals and use the above mentioned classical result for finite intervals.)

Our approach is based on the combination of the Boundary Control method applied to trees, as it was developed in [**2**], and the classical inverse spectral problem for periodic Schrödinger operator developed by V. A. Marchenko and I. V. Ostrovsky [**56**, **57**] and extended later in [**27**, **28**]. Therefore, after presenting the main definitions in Section 2 we describe the fascinating Marchenko–Ostrovsky theory in Section 3. We do not plan to prove all necessary facts, but we try to explain the motivation behind and reformulate the main theorem in the form suitable for our purposes. In Section 4 rigorous definition of the Titchmarsh–Weyl function is given and its relations to the dynamical Dirichlet-to-Neumann map are clarified. Using the response operator we prove in Section 5 that the branches of the graph and the potential there can be recovered from the Titchmarsh–Weyl function. In Section 6 we consider the two simplest graphs having cycles proving that potential in general cannot be reconstructed even using the described set of spectral data. Section 7 is devoted to our main example: the graph formed by one cycle and two edges attached to it. It appears that this graph and the potential on it can be reconstructed uniquely under a certain mild no-resonance condition. Relations between the no-resonance condition and scattering properties of the graph are discussed as well. In the last section we discuss generalization of these uniqueness theorem to the case of arbitrary graphs with Euler characteristic zero. It is important to underline that we not only prove uniqueness results, but give explicit reconstruction algorithms.

2. *Magnetic Schrödinger operator*

2·1. *General definitions*

In this section we give a definition for the magnetic Schrödinger operator with standard matching conditions. Let Γ be a metric graph obtained by taking N intervals (edges) $\Delta_j = [x_{2j-1}, x_{2j}] \subset \mathbb{R}$ and separating the set of end points $\mathbf{V} = \{x_j\}_{j=1}^{2N}$ into equivalence classes V_m, $m = 1, 2, \ldots, M$: $\bigcup_{m=1}^{M} V_m = \mathbf{V}$, $V_m \bigcap V_l = \varnothing$, $m \neq l$. The equivalence classes will be called vertexes and the endpoints belonging to the same equivalence class are identified. Since every interval is a subset of the real line $\mathbb{R}$, one natural distance can be introduced on Γ leading to the natural notion of a continuous function. Note that even functions discontinuous at the vertexes will be considered. For such functions $f(x_j)$, $j = 1, 2, \ldots, 2N$ will denote the limits taken from inside the corresponding interval. A graph is called connected if there is a continuous path connecting any two points on it. Almost all graphs considered in this paper are formed by a finite number of compact intervals. The space of square integrable functions $L_2(\Gamma)$ possesses the decomposition

$$L_2(\Gamma) = \oplus \sum_{n=1}^{N} L_2(\Delta_n),$$

and therefore does not 'feel' the connectivity of the graph.

The number of elements in an equivalence class (vertex) V_m is called **valence** and in the case of no loops it is equal to the number of intervals that are meeting each other in the vertex. By a loop we mean an edge with both endpoints belonging to the same vertex (see Figure 2).

Geometric properties of a graph are characterized first of all by the *number of connected components*, the *Euler characteristic* $\chi = M - N$ and the *total length* $\mathcal{L} = \sum_{n=1}^{N} d_n$, where $d_n = x_{2n} - x_{2n-1}$ are the lengths of the edges.

Every finite metric graph can be seen as a kernel (containing no vertexes of valence one) and several branches (metric trees) attached to it. The graph's kernel can be obtained by pruning the original graph and sometimes is called reduced graph.

Definition 1. The **boundary** $\partial\Gamma$ of a metric graph Γ is the set of all vertexes having valence one, *i.e.* connected to a single edge.

All edges connected by one end point to a boundary vertex are called **boundary edges**. The operation of deleting all boundary edges in a graph is called **pruning**. It will be convenient to agree that pruning the graph formed by just one edge we get the trivial graph formed by just one vertex and having no edges. It is clear that pruning does not change the Euler characteristic of the graph. Applying pruning several times we get a graph without vertexes of valence one. This graph to be denoted by $\ker\Gamma$ is uniquely determined by Γ and is called the **kernel of** Γ, or reduction of Γ. For trees the kernel is given by the trivial graph.

The vertexes of the kernel to which some of the deleted during pruning edges were attached will be considered as **contact vertexes**, *i.e.* the vertexes through which the graph can be investigated (see Section 4·1 for definition). Consider any deleted edge connected by one end point to the kernel. Let us identify all deleted edges connected to the other end of this edge. Continue this procedure to get a metric tree having the first edge as a root. Every such tree will be called **branch**. Hence the original graph Γ can be obtained by adding a finite number of branches to its kernel. Note that several branches may be attached to the same contact point.

Another way to see the kernel is using basic cycles. Every graph Γ can be turned to a tree by deleting several edges. Let us associate with each such edge the minimal cycle lying on the original graph. Then the kernel of Γ contains all basic cycles and all shortest paths connecting them.

All vertexes on the graph are divided into two non intersecting classes: boundary and internal vertexes. These classes of vertexes are going to play slightly different roles in our studies. The differential operator will be defined on the set of functions satisfying certain conditions at the vertexes of both types. We are going to investigate the dependence of the spectrum upon the conditions at the boundary. Conditions at the internal vertexes will be fixed. In order to underline this difference conditions at internal vertexes will be called *matching conditions*, whereas the name *boundary conditions* will be reserved for the conditions at the boundary vertexes.

The magnetic Schrödinger operator $L = L_{q,a}$ is determined by (electric) potentials q and magnetic potential a

$$q \in L_2(\Gamma), \quad a \in C(\Gamma). \tag{2·1}$$

It is defined by the differential expression

$$L_{q,a} = \left(-\frac{1}{i}\frac{d}{dx} + a(x)\right)^2 + q(x), \tag{2·2}$$

on the domain of functions ψ from the Sobolev space $W_2^2(\Gamma \setminus \mathbf{V})$ satisfying standard matching and boundary conditions at every vertex V_m, $m = 1, 2, \ldots, M$

$$\begin{cases} \psi \text{ is continuous at } V_m, \\ \sum_{x_j \in V_m} \partial\psi(x_j) = 0, \end{cases} \tag{2·3}$$

where $\partial\psi(x_j)$ is the extended derivative at the point x_j defined as follows

$$\partial\psi(x_j) = \begin{cases} \dfrac{d}{dx}u(x_j) - ia(x_j)\psi(x_j), & x_j \text{ is a left end point;} \\ -\dfrac{d}{dx}u(x_j) + ia(x_j)\psi(x_j), & x_j \text{ is a right end point.} \end{cases} \tag{2·4}$$

Note that these derivatives include the values of the magnetic field at the end points of the interval in contrast to normal derivatives which are suitable to describe matching conditions in the case of zero magnetic field (see formula 2·9 below for definition). Standard boundary conditions (*i.e.* at the boundary vertexes) imply that the corresponding extended derivatives are zero and do not provide any restriction on the function's value.

The operator so defined is self-adjoint and its spectrum is pure discrete, since this operator is a finite rank perturbation (in the resolvent sense) of the orthogonal sum of operators on separated intervals with say Dirichlet boundary conditions at the end points [**1**].

2·2. *'Elimination' of the magnetic field*

It is well known that magnetic field does not play any essential role in dimension one, *i.e.* it may be eliminated by a certain unitary transformation. Therefore consider the following unitary transformation in $L_2(\Gamma)$

$$\hat{\psi}(x) := (U\psi)(x) = \exp\left(-i\int_{x_{2n-1}}^{x} a(y)dy\right)\psi(x), \quad x \in (x_{2n-1}, x_{2n}), n = 1, 2, \ldots, N. \tag{2·5}$$

Let us calculate the differential operator obtained by the corresponding similarity transformation

$$UL_{q,a}U^{-1}. \tag{2·6}$$

Direct calculations show that the differential expression transforms as follows

$$U\left(\left(-\frac{1}{i}\frac{d}{dx} + a(x)\right)^2 + q(x)\right)U^{-1}\hat{\psi}(x) = -\frac{d^2}{dx^2}\hat{\psi}(x) + q(x)\hat{\psi}(x). \tag{2·7}$$

On the other hand we have

$$\begin{cases} (U^{-1}\hat{\psi})(x_j) = \hat{\psi}(x_j) \\ \partial(U^{-1}\hat{\psi})(x_j) = \partial_n\hat{\psi}(x_j) \end{cases}$$

if $j = 2n + 1$, *i.e.* x_j is a left end point, and

$$\begin{cases} (U^{-1}\hat{\psi})(x_j) = e^{i\phi_n}\hat{\psi}(x_j), \\ \partial(U^{-1}\hat{\psi})(x_j) = e^{i\phi_n}\partial_n\hat{\psi}(x_j), \end{cases}$$

if $j = 2n$, *i.e.* x_j is a right end point. Here we introduce the following notations:

(i) the phase ϕ_n is the integral of the magnetic field over the interval $[x_{2n-1}, x_{2n}]$

$$\phi_n = \int_{\Delta_n} a(x)\,dx; \tag{2·8}$$

(ii) ∂_n denote the normal derivatives

$$\begin{cases} \partial_n\psi(x_j) = \psi'(x_j), & j \text{ is odd, } \textit{i.e. } x_j \text{ is a left end point,} \\ \partial_n\psi(x_j) = -\psi'(x_j), & j \text{ is even, } \textit{i.e. } x_j \text{ is a right end point.} \end{cases} \tag{2·9}$$

With this notation we have that $\hat{\psi} \in \mathrm{Dom}\,(UL_{A,q,S}U^{-1})$ if and only if $\hat{\psi} \in W_2^2(\Gamma \setminus \mathbf{V})$ and satisfies the matching/boundary conditions

$$\begin{cases} \exp\left(i\dfrac{(-1)^j+1}{2}\phi_{[j/2]}\right)\hat{\psi}(x_j) = \exp\left(i\dfrac{(-1)^l+1}{2}\phi_{[l/2]}\right)\hat{\psi}(x_l), \quad x_j, x_l \in V_m, \\ \displaystyle\sum_{x_j \in V_m} \exp\left(i\frac{(-1)^j+1}{2}\phi_{[j/2]}\right)\partial_n\hat{\psi}(x_j) = 0; \end{cases} \tag{2·10}$$

at every vertex V_m, $m = 1, 2, \ldots, M$. The factors $e^{i\frac{(-1)^j+1}{2}\phi_{[j/2]}}$ appearing in the formulas are equal to one for any j odd and are equal to $e^{i\phi_{[j/2]}}$ for j even, where $[\,\cdot\,]$ denotes the integer part. For boundary vertexes these conditions coincide with conventional Neumann boundary conditions

$$\partial_n\hat{\psi}(x_j) = 0, \ x_j \in \partial\Gamma. \tag{2·11}$$

Summing up the operator $UL_{q,a}U^{-1}$ is given by the differential expression with zero magnetic field, but as a compensation the functions from the domain satisfy conditions (2·10) instead of the standard conditions (2·3), unless $\phi_n = 0|\mathrm{mod}2\pi$. We have proved that the spectrum of the operator $L_{q,a}$ does not depend on the particular form of the magnetic potential a, but is determined by its integrals $\phi_n = \int_{\Delta_n} a(x)\,dx$. It is natural to introduce the following notation

$$L_{q,\{\phi_n\}} = UL_{q,a}U^{-1}. \tag{2·12}$$

The inverse problem for the operator $L_{q,\{\phi_n\}}$ will be studied in what follows.

The unitary transformation (2·5) is defined without paying any attention to the connectivity of Γ and therefore is not optimal for certain classes of graphs. For example for trees it is natural to consider the following transformation. Let us denote by x_1 one of the boundary vertexes to be called the root. Then points on the edges can be parameterized by the distance from the root x_1. Consider the unitary transformation

$$\hat{\psi}(x) := (U_{x_1}\psi)(x) = \exp\left(-i\int_{x_1}^{x} a(y)\,dy\right)\psi(x), \tag{2·13}$$

where the integral is taken along the shortest path connecting x_1 and x. The operator $U_{x_1}L_{q,a}U_{x_1}^{-1}$ coincides with the Schrödinger operator $-d^2/dx^2 + q$ defined on the functions from $W_2^2(\Gamma \setminus \mathbf{V})$ satisfying standard matching/boundary conditions (2·3) at all vertexes. In other words the operators $L_{q,0}$ and $L_{q,a}$ are unitary equivalent and therefore all operators $L_{q,a}$ with different magnetic potentials a and the same potential q have the same spectrum. We have just proved that in the case of trees the magnetic field can be eliminated completely.

The described transformation cannot be uniquely extended for graphs with cycles and moreover is not always the most appropriate one. The reason is that the corresponding Titchmarsh–Weyl matrix function is different for the operators $L_{q,a}$ and $L_{q,\{\phi_n\}}$ (see Subsection 4·1 for the definition of Titchmarsh–Weyl function for graphs). For the studies involving Titchmarsh–Weyl function it is more convenient to consider such parameterization of Γ that all boundary vertexes are given by x_j with odd values of j. In this case transformation (2·5) preserves the Titchmarsh–Weyl matrix.

The following statement is rather straightforward (see [**42**] where similar results are discussed).

PROPOSITION 2·1. *Let Γ be a finite compact metric graph and let the potentials q and a satisfy* (2·1). *Then the spectrum of the magnetic Schrödinger operator $L_{q,a}$ is pure discrete and does not depend on the particular form of the magnetic field, but just on the fluxes Φ_j of the magnetic field through the cycles C_j*

$$\Phi_j = \int_{C_j} a(x)\, dx. \tag{2·14}$$

3. *On the inverse spectral problem for the periodic Sturm–Liouville operator*

In what follows we shall see that the spectral problems on graphs are entirely connected with the spectral problems for corresponding periodic Sturm–Liouville operators. Therefore we are going to use the language developed for such operators to study the inverse spectral problems on graphs. For the sake of simplicity we consider the interval $[0, \pi]$ in this subsection and assume that $q \in L_2([0, \pi])$. The potential q can be extended periodically with the period π to a real valued locally square integrable function on the whole axis $\mathbb{R}$. The corresponding Sturm-Liouville operator will be denoted by L_q^{per}. In what follows it will be convenient to assume that the operator L_q^{per} is nonnegative $L_q^{\mathrm{per}} \geqslant 0$, more precisely that 0 is the left end point of the spectrum.

To study the spectrum of this operator one introduces the transfer matrix $T(a, b; k)$ connecting the Cauchy data at points a and b

$$\begin{pmatrix} \psi(b) \\ \psi'(b) \end{pmatrix} = T(a, b; k) \begin{pmatrix} \psi(a) \\ \psi'(a) \end{pmatrix}, \tag{3·1}$$

where ψ is any solution to the eigenfunction equation

$$-\psi''(x) + q(x)\psi(x) = k^2\psi(x). \tag{3·2}$$

The transfer matrix has unit determinant.

The trace of the transfer matrix $\mathrm{Tr}\,(T(0, \pi; k))$, so-called *Lyapunov function*, plays a very important role in the spectral theory of the operator L_q^{per}, therefore we introduce the notations

$$\begin{aligned} u_+(k) &= (t_{11}(0, \pi; k) + t_{22}(0, \pi; k))/2, \\ u_-(k) &= (t_{11}(0, \pi; k) - t_{22}(0, \pi; k))/2, \end{aligned} \tag{3·3}$$

where t_{ij} are the entries of the matrix T. (In what follows writing the matrix T and its entries we are going to omit the first two arguments for simplicity.) The functions $u_\pm(k)$ are even entire functions of exponential type not larger than π, real on the real axis.

The operator L_q^{per} commutes with the translations over the period and therefore all its eigenfunctions are quasiperiodic

$$\psi(x + \pi) = e^{i\theta}\psi(x), \tag{3·4}$$

where $\theta \in [-\pi, \pi]$ is the quasimomentum. The spectrum of the periodic operator is pure absolutely continuous and is formed by several bands separated by a finite or infinite number of forbidden gaps. A point $\lambda = k^2$ belongs to the spectrum of L_q^{per} if and only if there exists a real θ such that $e^{i\theta}$ is an eigenvalue of the transfer matrix $T(k)$

$$\det\left(T(k) - e^{i\theta}\right) = 0 \Leftrightarrow \mathrm{Tr}\, T(k) = 2\cos\theta \Longleftrightarrow u_+(k) = \cos\theta. \tag{3·5}$$

In other words the spectrum of L_q^{per} is determined by the Lyapunov function and coincides with the set of positive λ satisfying

$$|u_+(\sqrt{\lambda})| \leqslant 1. \tag{3·6}$$

To calculate the spectrum of the operator L_q^{per} it is enough to study the spectra of different boundary problems for differential equation (3·2) on the interval $[0, \pi]$.

In the rest of section we are going to discuss different eigenvalue problems associated with (3·2). Let us denote by

$$\tilde{\mu}_0, \mu_2, \tilde{\mu}_2, \ldots, \mu_{2k}, \tilde{\mu}_{2k}, \ldots$$

and

$$\mu_1, \tilde{\mu}_1, \mu_3, \tilde{\mu}_3, \ldots, \mu_{2k+1}, \tilde{\mu}_{2k+1}, \ldots$$

the eigenvalues of the corresponding periodic

$$\begin{cases} \psi(0) = \psi(\pi), \\ \psi'(0) = \psi'(\pi); \end{cases} \tag{3·7}$$

and antiperiodic

$$\begin{cases} \psi(0) = -\psi(\pi), \\ \psi'(0) = -\psi'(\pi); \end{cases} \tag{3·8}$$

problems respectively. Then the sequence of intervals

$$[\tilde{\mu}_0, \mu_1], [\tilde{\mu}_1, \mu_2], [\tilde{\mu}_2, \mu_3], \ldots. \tag{3·9}$$

form the spectrum of the operator L_q^{per}. Note that few or even all forbidden gaps may be empty. In accordance with the assumption $\tilde{\mu}_0 = 0$.

Let us denote by $\lambda_1 = k_1^2, \lambda_2 = k_2^2, \ldots$ the eigenvalues of the Dirichlet–Dirichlet problem

$$\begin{cases} \psi(0) = 0, \\ \psi(\pi) = 0. \end{cases} \tag{3·10}$$

The numbers k_j are precisely the zeroes of the function $t_{12}(k)$. It is possible to prove that Dirichlet-Dirichlet eigenvalues are situated in the forbidden gaps

$$\mu_j \leqslant \lambda_j \leqslant \tilde{\mu}_j, \quad j = 1, 2, \ldots. \tag{3·11}$$

In the case of vanishing gap the corresponding eigenvalues of the periodic or antiperiodic and Dirichlet–Dirichlet problems coincide.

In [**56**] the following theorem is proved:

PROPOSITION 3·1 ([**56**, theorem 5·1]). *For the sequences*

$$0 = \tilde{\mu}_0 < \mu_1 \leqslant \tilde{\mu}_1 < \mu_2 \leqslant \tilde{\mu}_2 < \cdots \tag{3·12}$$

to be the spectra of periodic and antiperiodic boundary value problems generated on the interval $[0, \pi]$ *by the [differential] operator* $-y'' + q(x)y$ *with real potential* $q(x) \in \tilde{W}_2^n[0, \pi]$, *it is necessary and sufficient that there exist a sequence of real numbers* h_j $(j = 0, \pm 1, \pm 2, \ldots)$ *satisfying the conditions*

$$\sum_{j=1}^{\infty} (j^{n+1} h_j)^2 < \infty, \quad h_0 = 0, \quad h_j = h_{-j} \geqslant 0 \quad (j = 1, 2, \ldots), \tag{3·13}$$

such that

$$\sqrt{\mu_j} = z(\pi j - 0), \quad \sqrt{\tilde{\mu}_j} = z(\pi j + 0) \quad (j = 1, 2, \ldots),$$

where the function $z(\theta)$ effects a conformal mapping of the region

$$\{\theta : \operatorname{Im}\theta > 0\} \setminus \bigcup_{j=-\infty}^{+\infty} \left\{\theta : \operatorname{Re}\theta = j\pi, 0 \leqslant \operatorname{Im}\theta \leqslant h_j\right\} \tag{3·14}$$

into the upper half-plane, normalized by the conditions

$$\theta(0) = 0, \lim_{y\to\infty} (iy)^{-1}\theta(iy) = \pi.$$

We refer to [**56**] for the proof which is based on the investigation of the function

$$u_+(k) = \cos\theta(k). \tag{3·15}$$

The function $\theta(k)$ generalizes the notion of the quasimomentum introduced in (3·4). The hard analysis consists in proving that there exists a potential q such that u_+ is its Lyapunov function, *i.e.* it is related to the corresponding transfer matrix via (3·3).

It has been shown by B. M. Levitan and M. G. Gasymov [**52**] that the potential can be reconstructed from the spectra of the Dirichlet-Dirichlet (see (3·10)) and Dirichlet–Neumann

$$\begin{cases} \psi(0) = 0, \\ \psi'(\pi) = 0; \end{cases} \tag{3·16}$$

problems. The eigenvalues of the Dirichlet–Neumann problem are given by the zeroes of the entire function $t_{22}(k)$. Hence, the potential q is uniquely determined by the functions t_{22} and t_{12}.

Assume the Lyapunov function u_+ is known. The eigenvalues of the Dirichlet–Dirichlet problem are not determined by it and therefore can be chosen as parameters describing the family of potentials having the same Lyapunov function. Consider arbitrary real points $\lambda_j = k_j^2$ satisfying (3·11). At these points both the sum and the product of the functions $t_{11}(k_j)$ and $t_{22}(k_j)$ are known

$$\begin{gathered} t_{11}(k_j) + t_{22}(k_j) = 2u_+(k_j), \\ t_{11}(k_j)t_{22}(k_j) = 1. \end{gathered}$$

Then the values of the function u_- at the same points k_j can be recovered up to the sign ν_j

$$u_-(k_j) = \nu_j\sqrt{u_+^2(k_j) - 1}, \quad \nu_j = \pm 1. \tag{3·17}$$

Choosing the signs ν_j, $j = 1, 2, \ldots$ in arbitrary way we may reconstruct the function $u_-(k)$, since it is an entire function of exponential type not larger than π. Then the main result of [**56**] can also be formulated as (see also [**57**, theorem A])

PROPOSITION 3·2. *Assume that all conditions of Proposition* 3·1 *are satisfied. Then the family of periodic potentials having the spectrum*

$$[0 = \tilde{\mu}_0, \mu_1] \cup [\tilde{\mu}_1, \mu_2] \cup [\tilde{\mu}_2, \mu_3] \cup \cdots, \tag{3·18}$$

can be uniquely parameterized by the sequence of λ_j of arbitrary real numbers satisfying (3·11) *and by the sequence of signs $\nu_j = \pm 1$. The numbers λ_j coincide with the eigenvalues of the Dirichlet–Dirichlet problem and ν_j determine the signs of $u_-(k_j)$, $k_j^2 = \lambda_j$.*

This, a slightly stronger, Proposition is proved in [**56**], but the result is not formulated explicitly there. It was suggested in [**57**] to call the following two sets the *spectral data* for the Hill operator

$$-\infty < \tilde{\mu}_0 < \mu_1 \leqslant \lambda_1 \leqslant \tilde{\mu}_1 < \mu_2 \leqslant \lambda_2 \leqslant \tilde{\mu}_2 < \cdots, \tag{3·19}$$

and

$$\nu_j = \operatorname{sign} u_-(k_j),\ k_j^2 = \lambda_j, \quad j = 1, 2, \dots. \tag{3·20}$$

Note that the spectral points $\tilde{\mu}_0, \mu_j, \tilde{\mu}_j,\ j = 1, 2, \dots$, cannot be chosen arbitrarily but have to satisfy the restriction given by Proposition 3·1. (See [**27**, **28**] where this arbitrariness is discussed in details.) On the other hand the sequences $\{\lambda_j\}$ and $\{\nu_j\}$ can be chosen arbitrarily, of course subject to the condition (3·11).

In the case where a forbidden band disappears ($\tilde{\mu}_{j_0} = \mu_{j_0}$) the corresponding Dirichlet–Dirichlet eigenvalue is uniquely determined $\lambda_{j_0} = \mu_{j_0}$ and $u_-(k_{j_0}) = 0$, which implies that no extra sign parameter is needed. Hence in the case of finite-zone potentials the corresponding family is described by a finite number of parameters. (Periodic Schrödinger operators with a finite number of forbidden gaps have been studied by A. R. Its and V. B. Matveev [**38**, **39**].) Otherwise the number of parameters is infinite.

Remark 1. Proposition 3·2 implies that the potential is uniquely determined by the Lyapunov function u_+ and the Dirichlet-Dirichlet spectrum only in the case of constant potential. In the case of finite-zone potentials the number of equivalent potentials is also finite.

4. *Titchmarsh–Weyl matrix and the dynamical response operator*

To solve the inverse problem for the Schrödinger operator on a metric graph one may use the relation between the Titchmarsh–Weyl matrix and the dynamical response operator. In this section we are going to define these notions rigorously and discuss their properties.

4·1. *The Titchmarsh–Weyl matrix*

The classical Titchmarsh–Weyl function describes the relation between the boundary values for a square integrable solution to the stationary Schrödinger equation on the half-axis. It is a Nevanlinna function and it encodes all spectral properties of the differential operator. We are going to generalize this definition for the case of metric graphs. The Titchmarsh–Weyl function is nothing else than the Dirichlet-to-Neumann map proved to be an efficient tool in solving inverse problems. Assume that the graph can be investigated by looking at the values of the functions at several vertexes from a subset $\mathcal{V} \subset \mathbf{V}$. These vertexes will be called *contact vertexes* and can be seen as the vertexes through which the graph may be approached. Let us denote these vertexes by $\gamma_j,\ j = 1, 2, \dots, \mathcal{M},\ \mathcal{M} \leqslant M$. Consider the solution to the differential equation

$$\left(-\frac{1}{i}\frac{d}{dx} + a(x)\right)^2 \psi_j(x, \lambda) + q(x)\psi_j(x, \lambda) = \lambda\psi_j(x, \lambda), \quad \Im\lambda > 0, \tag{4·1}$$

satisfying the standard matching/boundary conditions (2·3) at all vertexes from $\mathbf{V} \setminus \mathcal{V}$ and the following conditions at the contact vertexes

$$\psi_j(\gamma_i, \lambda) = \delta_{ij}, \quad \gamma_i \in \mathcal{V}, \tag{4·2}$$

where δ_{ij} is Kronecker's delta. This solution exists and is unique, since the homogeneous problem (corresponding to the boundary conditions $\psi_j(\gamma_i, \lambda) = 0,\ i = 1, 2, \dots, \mathcal{M}$) has only trivial solution. Let us denote by $M_{ij}(\lambda)$ the sums of extended derivatives (given by (2·4)) of the function ψ_j at the contact vertex γ_i

$$M_{ij}(\lambda) = \sum_{x_m \in \gamma_i} \partial\psi_j(x_m), \quad \gamma_i \in \mathcal{V}. \tag{4·3}$$

We introduce the boundary maps $B_{0,1}$ defined on the functions that are continuous and are C^1 in a neighborhood of the boundary, respectively

$$\begin{aligned} B_0 &: \psi \longmapsto \left\{\psi(\gamma_j)\right\}_{j=1}^{\mathcal{M}} \in \mathbb{C}^{\mathcal{M}}, \\ B_1 &: \psi \longmapsto \left\{\sum_{x_m \in \gamma_j} \partial\psi(x_m)\right\}_{j=1}^{\mathcal{M}} \in \mathbb{C}^{\mathcal{M}}. \end{aligned} \tag{4·4}$$

The matrix $\mathbf{M}(\lambda) = \{M_{ij}(\lambda)\}_{i,j=1}^{\mathcal{M}}$ connects the boundary values of any function ψ solving (4·1) and satisfying standard matching/boundary conditions (2·3) at all vertexes from $\mathbf{V} \setminus \mathcal{V}$

$$B_1\psi = \mathbf{M}(\lambda)\,(B_0\psi)\,. \tag{4·5}$$

The matrix $\mathbf{M}(\lambda)$ is called the **Titchmarsh–Weyl matrix function** for the graph Γ, or in short **TW-function**. From the definition we see that the TW-function is nothing else than the Dirichlet-to-Neumann map for the boundary value problem associated with the operator $L_{q,a}$.

$\mathbf{M}(\lambda)$ is a Nevanlinna matrix function. To prove that it has positive imaginary part for $\Im\lambda > 0$, take any function ψ as above, then it holds

$$\begin{aligned} \lambda\,\|\psi\|^2 = \int_\Gamma (-\psi'' + q(x)\psi)\overline{\psi}dx = \langle B_1\psi,\, B_0\psi\rangle_{\mathbb{C}^{\mathcal{M}}} + \int_\Gamma |\psi'|^2 + \int_\Gamma q(x)|\psi|^2 dx \\ \Longrightarrow \Im\langle \mathbf{M}(\lambda)B_0\psi,\, B_0\psi\rangle_{\mathbb{C}^{\mathcal{M}}} = \Im\lambda\,\|\psi\|^2\,. \end{aligned}$$

The last formula implies that $\Im\mathbf{M}(\lambda) \geqslant 0$ provided $\Im\lambda > 0$, since the vector $B_0\psi$ can be chosen arbitrarily.

TW-function's dependence on the magnetic field is rather simple: it is independent of the particular form of the magnetic field, but do depend on the integrals ϕ_n given by (2·8). This can be seen by applying the unitary transform (2·5). Note that the TW-function may be changed under such transformation. If all contact vertexes belong to the boundary of the graph (and the graph contains more than one edge), then parameterizing the boundary edges in the direction inside the graph we get operator $L_{q,\{\phi_n\}}$ with precisely the same TW-function as the original operator. The TW-function for the operator $L_{q,\{\phi_n\}} = UL_{q,a}U^{-1}$ is defined analogously by considering the solutions to the equation

$$-\frac{\partial^2\psi_j}{\partial x^2}(x,\lambda) + q(x)\psi_j(x,\lambda) = \lambda\psi_j(x,\lambda), \tag{4·6}$$

satisfying matching/boundary conditions (2·10) at vertexes from $\mathbf{V} \setminus \mathcal{V}$ and condition (4·2) at the contact points. Then the matrix element $M_{ij}(\lambda)$ is given by formula similar to (4·3) with the extended derivatives substituted by the normal ones.

Every such function possesses Riesz–Herglotz representation ([**32**, theorem 5·4])

$$\mathbf{M}(\lambda) = \mathbf{C} + \mathbf{D}\lambda + \int_{\mathbb{R}} \left((t-\lambda)^{-1} - t(1+t^2)^{-1}\right) d\Omega(t), \quad \Im\lambda > 0, \tag{4·7}$$

where $\mathbf{C} = \mathbf{C}^*$, $\mathbf{D} \geqslant 0$ and Ω is a matrix valued positive measure satisfying

$$\int_{\mathbb{R}} (1+t^2)^{-1}\langle d\Omega(t)\vec{c}, \vec{c}\rangle < \infty \ \text{ for all } \vec{c} \in \mathbb{C}^{\mathcal{M}}. \tag{4·8}$$

Then the following estimate holds

$$|M_{ij}(\lambda)| \leqslant C(\epsilon)(1+|\lambda|)^2, \quad \Im\lambda \geqslant \epsilon, \tag{4·9}$$

where $C(\epsilon) \in \mathbb{R}_+$ is a certain positive constant.

4·2. *The response operator*

Consider the solution to the wave equation on the graph Γ

$$\frac{\partial^2 w}{\partial t^2}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) + q(x)w(x,t) = 0, \quad x \in \Gamma \setminus \mathbf{V}, t \in (0,T), \ T > 0, \tag{4·10}$$

satisfying trivial initial data

$$w(x,0) = \frac{\partial w}{\partial t}(x,0) = 0, \quad x \in \Gamma \setminus \mathbf{V}, \tag{4·11}$$

standard matching/boundary conditions (2·3) at all vertexes from $\mathbf{V} \setminus \mathcal{V}$ and the Dirichlet boundary control

$$(B_0 w)(t) = \vec{f}(t), \quad t \in (0,T), \tag{4·12}$$

on $\mathcal{V}$, where $\vec{f} \in L_2((0,T), \mathbb{C}^m)$. Solution $w^{\vec{f}}(x,t)$ to (4·10), (4·11), (4·12) is unique and can be constructed by considering waves on the intervals like it has been done in [**2**]. It depends on the vector valued control function $\vec{f} \in \mathcal{F}^T = L_2((0,T), \mathbb{C}^m)$. Then the **Response operator**, or the **Dynamical Dirichlet-to-Neumann map** $R^T : \mathcal{F}^T \to \mathcal{F}^T$ is defined on the domain $\mathrm{Dom}\,(R^T) = \{\vec{f} \in C^2([0,T], \mathbb{C}^{\mathcal{M}}) : f(0) = f'(0) = 0\}$ by the formula

$$(R^T \vec{f})(t) = B_1 w^{\vec{f}}. \tag{4·13}$$

4·3. *On the relation between the Titchmarsh–Weyl matrix and the dynamical response operator*

Consider any vector valued function $\vec{f} \in C_0^\infty((0,T), \mathbb{C}^{\mathcal{M}})$ and its inverse Fourier transform

$$\vec{F}(k) = \int_0^\infty \vec{f}(t) e^{ikt} dt.$$

The function $\vec{F}$ is analytic in $\Im k > 0$ and decreases faster than any polynomial, *i.e.* the following estimate holds

$$|\vec{F}(k)| \leqslant C_\alpha (1 + |k|)^{-\alpha}, \ \Im k \geqslant 0, \tag{4·14}$$

for any $\alpha > 0$ and a certain $C_\alpha \in \mathbb{R}_+$.

Let ψ be a solution to the differential equation

$$-\frac{d^2\psi}{dx^2}(x,k) + q(x)\psi(x,k) = k^2 \psi(x,k), \quad x \in \Gamma, \tag{4·15}$$

that satisfies matching/boundary conditions (2·10) at all vertexes from $\mathbf{V} \setminus \mathcal{V}$ and conditions

$$B_0 \psi = \vec{F}(k) \tag{4·16}$$

on $\mathcal{V}$. The vector $B_1\psi(k)$ (of 'extended' derivatives) can be calculated using TW-matrix

$$B_1 \psi(k) = \mathbf{M}(k^2)\vec{F}(k).$$

On the other hand this function allows one to solve the problem (4·10), (4·11), (4·12)

$$w(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(x, k + i\nu) e^{-i(k+i\nu)t} dk, \quad \nu > 0, \tag{4·17}$$

and therefore to calculate the dynamical response operator

$$(R^T \vec{f})(t) = (B_1 w)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{M}((k+i\nu)^2)\vec{F}(k+i\nu) e^{-i(k+i\nu)t} dk, \quad \nu > 0. \tag{4·18}$$

The convergence of the integral follows from estimates (4·9) and (4·14). Hence the Titchmarsh–Weyl matrix $\mathbf{M}(\lambda)$ and the dynamical response operator R^T taken for all $T > 0$ are in one-to-one correspondence.

It is straightforward to see that the response operator may be used to reconstruct the potential on the boundary edges [**2**, **3**]. In the following section we are going to show, that it determines all branches of the graph and potential there if the set $\mathcal{V}$ includes all boundary points. In this way we shall prove that the TW-function determines the branches on the graph and essentially determines the TW-function associated with graph's kernel.

5. *Reconstruction of the branches from the response operator*

It was shown that in the case of trees, standard matching conditions and no magnetic field to reconstruct the graph and the potential q it is enough to know the TW-function reduced by one dimension [**2**]. In this section we generalize these results for the case where the magnetic field is present and the graphs may have cycles. We follow ideas from [**2**], but certain obvious modifications of the results presented there are needed. It will be shown that the knowledge of the TW-function allows one to reconstruct graph's branches and the potential q there. In addition the TW-function corresponding to the kernel of the graph can be reconstructed up to similarity transformation including a certain diagonal unitary matrix. The problem to reconstruct graph's kernel and potential on it will be discussed in the following sections.

We start by extending [**2**, lemma 3·1].

LEMMA 5·1. *Let Γ be a graph with the boundary edge $[x_1, x_2]$, where $x_1 \in \partial\Gamma$ and x_2 belongs to the vertex with valence $v > 2$ and let $L_{q,a} = (-(1/i)(d/dx) + a)^2 + q$ be a magnetic Schrödinger operator in $L_2(\Gamma)$ defined on the functions satisfying standard matching conditions* (2·3). *Then the knowledge of the corresponding diagonal element $R^T_{x_1x_1}$ of the response operator for $0 \leqslant t \leqslant T, \quad T > 2d_1$ allows one to reconstruct the length $d_1 = x_2 - x_1$, the valence v of the nearest to x_1 internal vertex and the potential q on the boundary edge $[x_1, x_2]$.*

The proof is almost identical to the proof of [**2**, lemma 3·1]. One needs just to carry out the unitary transformation (2·5) and take into account that the reflection from the vertex described by matching conditions (2·10) and standard ones are identical.

Hence the knowledge of the diagonal part of the response operator allows one to reconstruct all boundary edges, the potential q on these edges and valencies of all vertexes which are end points of the boundary edges. In the next step we need to identify which boundary edges are connected to each other forming sheafs.

LEMMA 5·2. *Let Γ be a metric graph with the boundary $\partial\Gamma$. Then the response operator associated with the boundary $\partial\Gamma$ determines which boundary edges form sheafs,* i.e. *are connected to each other (have common end points).*

Proof. Lemma 5·1 implies that the diagonal part of the response operator determines the lengths of all boundary edges and potential q on these edges. Let $\Delta_1 = [x_1, x_2]$ be the shortest boundary edge. Consider any other boundary edge, say $\Delta_2 = [x_3, x_4]$. Then the edges Δ_1 and Δ_2 have one common end point, *i.e.* x_2 and x_4 belong to the same vertex, if and only if the propagation time between x_1 and x_3 is equal to the sum of lengths of the intervals $d_1 + d_2$. The propagation time t_{x_3,x_1} can be defined by the formula

$$t_{x_3,x_1} = \inf_{f \in C_0^\infty(0,\infty)} \left\{ \inf \left\{ t : \left(R^T_{x_3x_1} f \right)(t) \neq 0 \right\} \right\}, \tag{5·1}$$

where T is taken sufficiently large. In this way we may identify the whole sheaf of boundary edges attached to the vertex containing x_2. If the propagation time is equal to d_1, then the graph consists of just one edge. If no neighboring edges are found, then the shortest edge is attached directly to the kernel of the graph Γ. The sheaf consists of just one edge in this case.

Repeating the same procedure for the remaining boundary edges we get the next set of boundary edges having the same end point. Finally all possible connections will be identified and we end up with separating all boundary edges into equivalence classes which we call sheafs.

This Lemma implies that we may determine the number of vertexes in the pruned graph Γ' obtained by cutting away all boundary edges from Γ. The vertexes in Γ' to which the sheafs of boundary edges from Γ were attached will be considered as contact vertexes, denoting the corresponding set by $\mathcal{V}'$. For sure all boundary points from $\partial\Gamma'$ belong to the set $\mathcal{V}'$, but it may happen that few of the contact vertexes are internal vertexes for Γ'. We may also determine which of the contact vertexes do not belong to $\partial\Gamma'$. This can be done by comparing the number of edges in the sheaf with the valence of the vertex to which the sheaf is attached.

Having in mind applying pruning procedure several times it is natural to use the following convention. Assume that Γ is a metric graph with the set of contact vertexes $\mathcal{V}$ which includes all boundary vertexes. Consider the pruned graph Γ'. We shall agree that the set of contact points $\mathcal{V}'$ contains all vertexes to which the cut off edges were attached as well as all internal contact points from $\mathcal{V}$. It is clear that with this convention the set of internal contact vertexes is increasing at the same time as the number of boundary points is decreasing. This shows another one time that pruning a graph sufficiently many times we shall get a graph without boundary edges, but with several internal contact vertexes.

The boundary control method allows one to reconstruct the Titchmarsh–Weyl matrix for the pruned graph Γ' but up to a certain unitary diagonal matrix.

LEMMA 5·3. *Let Γ be a metric graph with at least one boundary point. Let $L_{q,a}$ be a magnetic Schrödinger operator in $L_2(\Gamma)$, $\mathcal{V}$ be a set of contact vertexes which includes all boundary vertexes $\mathcal{V} \supset \partial\Gamma$ and $\mathbf{M}(\lambda)$ be the corresponding Titchmarsh–Weyl matrix function. Then the Titchmarsh–Weyl function for the pruned graph is uniquely determined up to similarity transformation with a certain diagonal unitary matrix.*

Proof. Without loss of generality we may assume that the sheafs of boundary edges are cut away one after another. Let us denote by $\gamma_1, \gamma_2, \ldots, \gamma_{\mathcal{M}}$ the contact vertexes in Γ. Consider the graph $\tilde{\Gamma}$ obtained from Γ by deleting the sheaf of boundary edges connected to the last l boundary points. The contact vertexes for $\tilde{\Gamma}$ are

$$\tilde{\gamma}_1 = \gamma_1, \tilde{\gamma}_2 = \gamma_2, \ldots, \tilde{\gamma}_{\tilde{\mathcal{M}}-1} = \gamma_{\tilde{\mathcal{M}}-1}, \quad \text{and} \quad \tilde{\gamma}_{\tilde{\mathcal{M}}},$$

i.e. just one contact vertex is new and $\tilde{\mathcal{M}} = \mathcal{M} - l + 1$ is the number of contact vertexes in $\tilde{\Gamma}$.

It will be convenient to use the parametrization so that the edges forming the disappearing sheaf are $\Delta_n = [x_{2n-1}, x_{2n}], n = \tilde{\mathcal{M}}, \ldots, \mathcal{M}$. The end points with odd indexes are the boundary points $\gamma_n, \tilde{\mathcal{M}} \leqslant n \leqslant \mathcal{M}$ and the end points with even indexes belong to the same vertex to be denoted by V_1, which is becoming the contact vertex $\tilde{\gamma}_{\tilde{\mathcal{M}}}$ for the graph $\tilde{\Gamma}$.

Consider the functions ψ_j and $\tilde{\psi}_j$, which are used to calculate the entries $M_{ij}(\lambda)$ and $\tilde{M}_{ij}$, respectively. These functions are solutions to equation (4·1) on Γ (respectively $\tilde{\Gamma}$) satisfying the boundary conditions $\psi_j(\gamma_i) = \delta_{ij}$ (respectively $\tilde{\psi}_j(\tilde{\gamma}_i) = \delta_{ij}$). Lemma 5·2 states that the diagonal of the Titchmarsh–Weyl function determines the potential on the boundary edges including $\Delta_n, n = \tilde{\mathcal{M}}, \dots, \mathcal{M}$. Consider the solutions u_{nj} to the second order differential equations on the boundary edges

$$-u''_{nj}(x) + q(x)u_{nj}(x) = \lambda u_{nj}(x), \quad x \in [x_{2n-1}, x_{2n}], \ n = \tilde{\mathcal{M}}, \dots, \mathcal{M}, \ j = 1, 2, \dots, \mathcal{M}$$

satisfying the same Cauchy data as the functions ψ_j

$$u_{nj}(x_{2n-1}) = \delta_{nj}, \quad \partial_n u_{nj}(x_{2n-1}) = M_{nj}(\lambda).$$

These functions allow us to calculate the value of the functions $\psi_j(V_1)$ provided at least one of the phases $\phi_n = \int_{x_{2n-1}}^{x_{2n}}$, $\tilde{\mathcal{M}} \leqslant n \leqslant \mathcal{M}$ is known, say $\phi_{\mathcal{M}}$

$$\psi_j(V_1) = e^{i\phi_{\mathcal{M}}} u_{\mathcal{M}j}(x_{2\mathcal{M}}) = e^{i\phi_n} u_{nj}(x_{2n}), \ n = \tilde{\mathcal{M}}, \dots, \mathcal{M} - 1. \tag{5·2}$$

To calculate the last column in $\tilde{\mathbf{M}}(\lambda)$ let us take into account that the function $\tilde{\psi}_{\tilde{\mathcal{M}}}$ is proportional to $\psi_{\mathcal{M}}$

$$\tilde{\psi}_{\tilde{\mathcal{M}}} = \frac{1}{e^{i\phi_{\mathcal{M}}} u_{\mathcal{M}\mathcal{M}}(x_{2\mathcal{M}})} \psi_{\mathcal{M}}|_{\tilde{\Gamma}}. \tag{5·3}$$

It follows that the last column is given by

$$\begin{aligned} \tilde{M}_{j\tilde{\mathcal{M}}}(\lambda) &= e^{-i\phi_{\mathcal{M}}} \left(u_{\mathcal{M}\mathcal{M}}(x_{2\mathcal{M}})\right)^{-1} M_{j\mathcal{M}}(\lambda), \quad j = 1, 2, \dots, \tilde{\mathcal{M}} - 1 \\ \tilde{M}_{\tilde{\mathcal{M}}\tilde{\mathcal{M}}}(\lambda) &= -\left(u_{\mathcal{M}\mathcal{M}}(x_{2\mathcal{M}})\right)^{-1} \sum_{n=\tilde{\mathcal{M}}}^{\mathcal{M}} e^{i(\phi_n - \phi_{\mathcal{M}})} \partial_n u_{n\mathcal{M}}(x_{2n}), \end{aligned} \tag{5·4}$$

where the phases $\phi_n - \phi_{\mathcal{M}}$ are given by (5·2). The second formula follows directly from the balance of extended derivatives at V_1

$$\sum_{x_m \in V_1} \partial\psi_{\mathcal{M}}(x_m) = 0 \Longrightarrow \sum_{x_m \in \tilde{\gamma}_{\tilde{\mathcal{M}}}} \partial\psi_{\mathcal{M}}(x_m) = -\sum_{n=\tilde{\mathcal{M}}}^{\mathcal{M}} \partial_n u_{n\mathcal{M}}(x_{2n}) e^{i\phi_n}$$

$$\Longrightarrow \sum_{x_m \in \tilde{\gamma}_{\tilde{\mathcal{M}}}} \partial\tilde{\psi}_{\tilde{\mathcal{M}}}(x_m) = -\left(u_{\mathcal{M}\mathcal{M}}(x_{2\mathcal{M}})\right)^{-1} \sum_{n=\tilde{\mathcal{M}}}^{\mathcal{M}} e^{i(\phi_n - \phi_{\mathcal{M}})} \partial_n u_{n\mathcal{M}}(x_{2n}).$$

Note that formula (5·4) contains one unknown phase $\phi_{\mathcal{M}}$.

To calculate the other entries in $\tilde{\mathbf{M}}(\lambda)$ consider the functions ψ_j, $j = 1, 2, \dots, \tilde{\mathcal{M}} - 1$ and $\psi_{\mathcal{M}}$. The solution $\tilde{\psi}_j$ is given by

$$\tilde{\psi}_j = \psi_j - \frac{u_{\mathcal{M}j}(x_{2\mathcal{M}})}{u_{\mathcal{M}\mathcal{M}}(x_{2\mathcal{M}})} \psi_{\mathcal{M}}(x)|_{\tilde{\Gamma}}. \tag{5·5}$$

To verify this equality one should take into account that the function on the right hand side is equal to zero at $\tilde{\gamma}_{\tilde{\mathcal{M}}}$ and satisfies the necessary conditions at all other contact vertexes of $\tilde{\Gamma}$. The remaining entries of $\tilde{\mathbf{M}}(\lambda)$ can be determined now:

$$\begin{aligned} \tilde{M}_{ij}(\lambda) &= M_{ij}(\lambda) - \tfrac{u_{\mathcal{M}j}(x_{2\mathcal{M}})}{u_{\mathcal{M}\mathcal{M}}(x_{2\mathcal{M}})} M_{i\mathcal{M}}(\lambda), \quad i = 1, 2, \dots, \tilde{\mathcal{M}} - 1; \\ \tilde{M}_{\tilde{\mathcal{M}}j} &= -e^{i\phi_{\mathcal{M}}} \sum_{n=\tilde{\mathcal{M}}}^{\mathcal{M}} e^{i(\phi_n - \phi_{\mathcal{M}})} \left(\partial_n u_{nj}(x_{2n}) - \tfrac{u_{\mathcal{M}j}(x_{2\mathcal{M}})}{u_{\mathcal{M}\mathcal{M}}(x_{2\mathcal{M}})} \partial_n u_{n\mathcal{M}}(x_{2n})\right). \end{aligned} \tag{5·6}$$

The TW-function $\tilde{\mathbf{M}}(\lambda)$ is not reconstructed uniquely but rather depends on the phase $\phi_{\mathcal{M}}$

$$\tilde{\mathbf{M}}(\lambda) = U\tilde{\mathbf{M}}^0(\lambda)U^{-1}, \tag{5·7}$$

where $\tilde{\mathbf{M}}^0(\lambda)$ is the Titchmarsh–Weyl function reconstructed assuming that $\phi_{\mathcal{M}} = 0$ and U is the diagonal unitary matrix

$$U = \operatorname{diag}(1, 1, \ldots, 1, e^{i\phi_{\mathcal{M}}}).$$

We are going to prove now the main result of this section.

THEOREM 5·4. *Let Γ be a finite metric graph wit at least one boundary edge, let $L_{q,a}$ be a magnetic Schrödinger operator in $L_2(\Gamma)$ and $\mathbf{M}(\lambda)$ be the corresponding Titchmarsh–Weyl matrix function associated with all boundary points of Γ. Suppose that $\mathbf{M}(\lambda)$ is known. Then the branches of Γ and the potential q on the branches are uniquely determined. The Titchmarsh–Weyl matrix function for the kernel of Γ, associated with all vertexes to which the branches of Γ are attached, is determined up to the similarity transformation which includes a certain diagonal unitary matrix.*

Proof. The theorem is already proved in the case the kernel of Γ is obtained by pruning Γ just one time. Let us show how to calculate the Titchmarsh–Weyl function for the graph $\hat{\Gamma}$ obtained by pruning Γ twice. Every contact vertex $\hat{\gamma}_j \in \hat{\Gamma}$ is either an internal contact vertex for $\tilde{\Gamma}$ or can be reached from the boundary of Γ following along certain two edges. Assume that these edges are $[x_1, x_2]$ and $[x_3, x_4]$, so that $x_1 \in \partial\Gamma$, x_2 and x_3 belong to the same vertex and $x_4 \in \hat{\gamma}_j$. The method of proving Lemma 5·3 is based on calculating explicitly the values of the functions ψ_j at the new contact vertexes. The same method allows us to calculate the value of ψ_j and of its extended derivative at the point x_3 up to the unimodular factor $e^{i\phi_1}$, where $\phi_1 = \int_{x_1}^{x_2} a(y)dy$. Then the value of $\psi_j(x_4)$ can also be calculated using the same method and includes the factor $e^{i\phi_1}e^{i\phi_2} = e^{i(\phi_1+\phi_2)}$ with $\phi_1 + \phi_2 = \int_{x_1}^{x_4} a(y)dy$. In other words the value $\psi_j(x_4)$ is uniquely determined up to multiplication by the factor including the integral of the magnetic potential along the path connecting the contact vertex $\hat{\gamma}_j$ to the boundary $\partial\Gamma$. Hence the corresponding TW-function $\hat{\mathbf{M}}(\lambda)$ is calculated up to similarity transformation including unitary diagonal matrix, as it has been shown for graphs pruned just one time.

Generalization of this method to the case where a finite number of prunings is needed is straightforward.

Remark 2. The theorem holds also in the case where the contact vertexes for Γ include certain internal vertexes.

The theorem implies that operators on trees can be reconstructed from the reduced TW-matrix obtained from $\mathbf{M}(\lambda)$ by deleting say the last column and the last row associated with root of the tree. The reconstruction includes just one unknown parameter equal to the integral of the magnetic potential between the root and any other boundary vertex.

Remark 3. The theorem not only states that the TW-function for the kernel can be calculated up to the described similarity transformation, but it also shows that this non-uniqueness is intrinsic in the following sense: for any nontrivial graph Γ there exist two different operators $L_{q,a}$ and $L_{q',a'}$ leading to the same TW-function $\mathbf{M}(\lambda)$ and two different TW-functions for the kernel.

6. *On the inverse problem for elementary graphs with cycles*

In this section we are going to study the problem to reconstruct the potential function q in the magnetic Schrödinger operator from the described data measured for different values

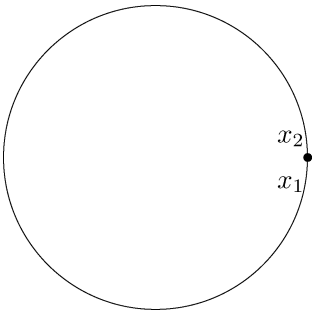

Fig. 1. Ring graph Γ_1.

of the magnetic potential. Here we study just two graphs with cycles: ring and lasso graphs. We show that in general potential can not be reconstructed neither from the spectrum of the ring graph nor from the TW-function for the lasso graph.

6·1. *Ring graph*

Consider the graph Γ_1 formed by the interval $[x_1, x_2]$ with the end points identified. Without loss of generality we assume that the length is equal to π. This graph can be viewed as a ring (see Fig. 1).

Let $L_{q,a}$ be a magnetic Schrödinger operator in $L_2(\Gamma_1)$. As always we assume that the potentials are real and satisfy (2·1) and the functions from the domain of $L_{q,a}$ satisfy standard matching conditions at the unique vertex.

As we have already shown the spectrum of the operator $L_{q,a}$ is pure discrete and does not depend on the form of the magnetic potential a, but rather on the total flux $\Phi \equiv \phi_1 = \int_{x_1}^{x_2} a(y)dy$ through the ring. Therefore in what follows we are going to study the family of operators $L_{q,\Phi}$

$$\mathrm{Dom}\,(L_{q,\Phi}) = \left\{ \psi \in W_2^2(x_1, x_2) : \begin{pmatrix} \psi(x_1) \\ \psi'(x_1) \end{pmatrix} = e^{i\Phi} \begin{pmatrix} \psi(x_2) \\ \psi'(x_2) \end{pmatrix} \right\}$$

$$L_{q,\Phi}\psi = -\frac{d^2}{dx^2}\psi + q\psi.$$

The spectrum of the operator $L_{q,\Phi}$ is pure discrete and tends to $+\infty$. As for the periodic Schrödinger operator, the spectrum of $L_{q,\Phi}$ can be calculated using the corresponding transfer matrix $T(k) = T(x_1, x_2; k)$: a point $E = k^2$ is an eigenvalue of the operator $L_{q,\Phi}$ if and only if the transfer matrix has eigenvalue $e^{-i\Phi}$

$$\det\,(e^{i\Phi}T(k) - 1) = 0 \Rightarrow \mathrm{Tr}\,T(k) = 2\cos\,\Phi. \tag{6·1}$$

In other words, a point E is an eigenvalue of $L_{q,\Phi}$ if and only if it belongs to the interval of the absolutely continuous spectrum of the operator L_q^{per} (studied in Section 3) and the corresponding quasimomentum θ is equal to Φ (compare formulas (3·5) and (6·1)).

For sake of completeness let us discuss the inverse (spectral) problem to reconstruct the potential q and the flux Φ from the set of eigenvalues $\{E_n(\Phi)\}_{n=0}^{\infty}$. It is clear that this problem does not have a unique solution, since rotating the potential q one gets an operator with precisely the same spectrum. (This transformation corresponds to the translation of the potential in the periodic Schrödinger operator.) Therefore it is natural to ask the question whether it is possible to reconstruct the potential q up to such rotations. The answer to this

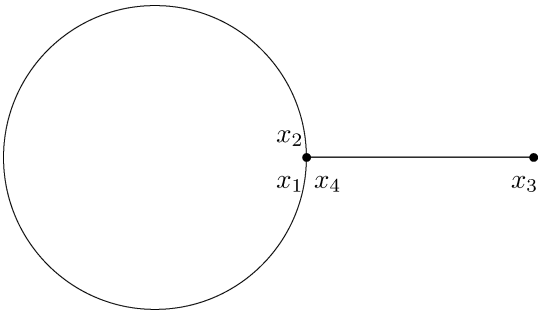

Fig. 2. Lasso graph Γ_2.

question in general is 'No'. We are going to show that in general even knowing the set of eigenvalues for all possible values of Φ the potential cannot be reconstructed.

THEOREM 6·1. *Let $L_{q,a}(\Gamma_1)$ be a magnetic Schrödinger operator on the ring Γ_1 with real square integrable potential $q \in L_2(\Gamma_1)$ and continuous real magnetic potential $a \in C(\Gamma_1)$. Then the knowledge of the spectrum $\{E_n(\Phi)\}$ as a function of the total flux $\Phi = \int_{\Gamma_1} a(y)dy$ in general is not enough to reconstruct the potential q.*

Proof. Assume that the functions $E_n(\Phi)$, $n = 0, 1, 2, \ldots$, are known for all real values of the parameter $\Phi \in [0, 2\pi)$. Then the Lyapunov function $u_+(k) = \operatorname{Tr} T(k)/2$ can be reconstructed (see (3·3). It was shown in Section 3 (following [**56**]), that to reconstruct the potential one needs to know in addition the spectrum of the corresponding Dirichlet problem. This spectrum is lying arbitrarily inside the forbidden gaps and is uniquely determined by the function $\operatorname{Tr} T(k)$ if and only if all forbidden gaps are degenerated, *i.e.* the potential q is a constant function.

Proposition 3·2 not only implies that if q is not a constant function, then it cannot be reconstructed uniquely, but an explicit parametrization of all possible potentials is given using the spectrum of the Dirichlet–Dirichlet problem and the corresponding sequence of signs. The Dirichlet–Dirichlet eigenvalues are situated arbitrarily inside the forbidden gaps for the periodic operator. Therefore this reconstruction in general contains an infinite number of real parameters. Only in the case of finite gap potentials the number of parameters is also finite. We note that such potentials have been studied in connection with periodic solutions of the KdV equation [**38**].

6·2. *Lasso graph (ring with one wire)*

In this section we consider the graph Γ_2 formed by a ring and one finite interval attached to it (see Fig. 2). (Similar graphs have been considered for example in [**69**].) This graph can be defined as a union of two intervals $[x_1, x_2]$ and $[x_3, x_4]$ with the endpoints x_1, x_2, and x_4 identified. Hence the graph has two vertexes $V_1 = \{x_1, x_2, x_4\}$ and $V_2 = \{x_3\}$. The vertex V_2 is the only boundary vertex. We assume that the length of the interval $[x_1, x_2]$ is again equal to π.

Consider the magnetic Schrödinger operator $L_{q,a}$ defined on the functions from $W_2^2(\Gamma_2 \setminus V_1)$ and satisfying standard matching conditions at the vertex V_1. We are going to study the possibility to reconstruct potential q from the knowledge of the TW-function $\mathbf{M}_\Phi(\lambda, \Gamma_2)$ associated with the boundary vertex V_2. Here we indicate the dependence on the magnetic flux by the lower index.

The kernel of the graph Γ_2 is the ring Γ_1 with one contact vertex $\gamma_1 = \{x_1, x_2\}$. (We keep the same notation Γ_1 hoping that no misunderstanding will arise. The graph Γ_1 considered in the previous subsection has no contact vertexes, while the graph Γ_1 considered here has one contact vertex.) We have proved (Theorem 6·1), that the knowledge of the TW-matrix $\mathbf{M}_\Phi(\lambda, \Gamma_2)$ allows one to calculate the TW-matrix for the kernel up to similarity transformation, but in the case under consideration the TW-function is scalar and therefore is uniquely determined. In other words $\mathbf{M}_\Phi(\lambda, \Gamma_2)$ determines $\mathbf{M}_\Phi(\lambda, \Gamma_1)$ independently of the value of the phase $\phi_2 = \int_{x_3}^{x_4} a(y)dy$.

Let us calculate the TW-function $\mathbf{M}_\Phi(\lambda, \Gamma_1)$ in terms of the transfer matrix for zero magnetic field. The easiest way is to carry out the unitary transformation (2·5), in order to get the same TW-function we need to modify the formula (see (6·4) below) in a way similar to how the boundary conditions are modified. The transfer matrix $T(k) = \left(\begin{smallmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{smallmatrix}\right)$ for the interval $[x_1, x_2]$ gives us the following relations

$$\begin{cases} \hat\psi(x_2) = t_{11}\hat\psi(x_1) + t_{12}\hat\psi'(x_1), \\ \hat\psi'(x_2) = t_{21}\hat\psi(x_1) + t_{22}\hat\psi'(x_1). \end{cases} \tag{6·2}$$

The first condition in (2·10) gives

$$e^{i\phi_1}\hat\psi(x_2) = \hat\psi(x_1). \tag{6·3}$$

Solving the linear system we get

$$\mathbf{M}_\Phi(\lambda, \Gamma_1) := \frac{\psi'(x_1) - \psi'(x_2)}{\psi(x_1)} = \frac{\hat\psi'(x_1) - e^{i\phi_1}\hat\psi'(x_2)}{\hat\psi(x_1)} = \frac{2\cos\,\Phi - \operatorname{Tr} T(k)}{t_{12}(k)}, \tag{6·4}$$

where Φ is the flux of the magnetic field through the cycle.

We are ready now to prove that the potential in general is not determined by $\mathbf{M}_\Phi(\lambda, \Gamma_2)$, or $\mathbf{M}_\Phi(\lambda, \Gamma_1)$, even known for all values of the flux Φ.

THEOREM 6·2. *Let $L_{q,a}(\Gamma_2)$ be a magnetic Schrödinger operator on the graph Γ_2 with the real square integrable potential $q \in L_2(\Gamma_2)$ and the continuous real magnetic potential $a \in C(\Gamma_2)$. Then the knowledge of the TW-function $\mathbf{M}_\Phi(\lambda, \Gamma_2)$ as a function of the total flux of the magnetic field though the ring $\Phi = \int_{x_1}^{x_2} a(y)dy$ in general is not enough to reconstruct the potential q.*

Proof. We have already shown that the TW-functions $\mathbf{M}_\Phi(\lambda, \Gamma_2)$ and $\mathbf{M}_\Phi(\lambda, \Gamma_1)$ (given by (6·4)) are in one-to-one correspondence. The function $\mathbf{M}_\Phi(\lambda, \Gamma_1)$ depends only on the functions $u_+ = \operatorname{Tr} T/2$ and t_{12}. Proposition 3·2 implies that the knowledge of these two functions in general is not enough to calculate the potential q.

In fact Proposition 3·2 allows not only to prove that reconstruction of the potential q in general is not unique, but characterize the corresponding arbitrariness. Assume that $\mathbf{M}_{\phi_1}(\lambda, \Gamma_1)$ is known for two different values of the flux, say $\Phi = 0, \pi$. Then formula (6·4) implies

$$\mathbf{M}_0(\lambda, \Gamma_2) = \frac{2 - \operatorname{Tr} T(k)}{t_{12}(k)}, \quad \mathbf{M}_\pi(\lambda, \Gamma_2) = \frac{-2 - \operatorname{Tr} T(k)}{t_{12}(k)},$$

and the functions $u_+ = \operatorname{Tr} T/2$ and t_{12} can be calculated

$$t_{12}(k) = \frac{4}{\mathbf{M}_0(\lambda, \Gamma_2) - \mathbf{M}_\pi(\lambda, \Gamma_2)}, \quad u_+(k) = 2\frac{\mathbf{M}_\pi(\lambda, \Gamma_2) + \mathbf{M}_0(\lambda, \Gamma_2)}{\mathbf{M}_\pi(\lambda, \Gamma_2) - \mathbf{M}_0(\lambda, \Gamma_2)}. \tag{6·5}$$

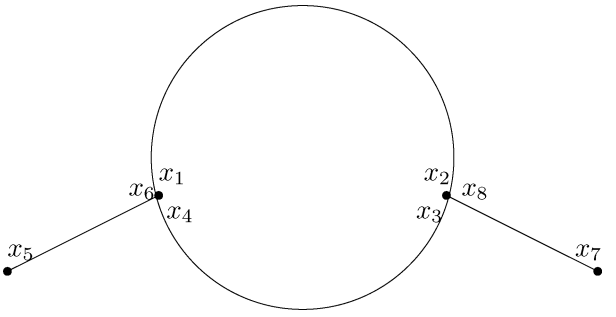

Fig. 3. Ring with two wires. Graph Γ_3.

To calculate the potential in the general case one needs to know the function u_-, or the sequence of signs already mentioned in Proposition 3·2.

If the TW-function is known for just one value of the magnetic flux, say $\Phi = 0$, then even the function u_+ in general is not determined.

7. *Ring with two wires*

In this section we are going to study the possibility to calculate the potential q from the TW-function for the graph formed by ring and two wires attached to it. Transfer properties of such graphs have been considered in [**17**, **19**, **20**, **72**], but the corresponding inverse problem has not been investigated.

7·1. *Notations and reduction to the kernel*

Consider the graph Γ_3 shown in Figure 3 which is formed by four intervals $[x_1, x_2]$, $[x_3, x_4]$, $[x_5, x_6]$, and $[x_7, x_8]$ with four vertexes $V_1 = \{x_1, x_4, x_6\}$, $V_2 = \{x_2, x_3, x_8\}$, $V_3 = \{x_5\}$, $V_4 = \{x_7\}$. The vertexes $\gamma_1 = V_3$, $\gamma_2 = V_4$ form the boundary of Γ_3. The kernel of Γ_3 is the graph Γ_4 formed by the intervals $[x_1, x_2]$ and $[x_3, x_4]$ with the end points building the contact vertexes $\tilde{\gamma}_1 = \{x_1, x_4\}$ and $\tilde{\gamma}_2 = \{x_2, x_3\}$.

Let us calculate the TW-function for the kernel Γ_4 in terms of the transfer matrices T^1 and T^2 for the intervals $[x_1, x_2]$ and $[x_3, x_4]$ respectively. We eliminate the magnetic field using transformation (2·5). The values of the function $\hat{\psi}$ are related as follows

$$\begin{cases} \hat{\psi}(x_2) = t_{11}^1 \hat{\psi}(x_1) + t_{12}^1 \hat{\psi}'(x_1), \\ \hat{\psi}'(x_2) = t_{21}^1 \hat{\psi}(x_1) + t_{22}^1 \hat{\psi}'(x_1), \end{cases} \quad \begin{cases} \hat{\psi}(x_4) = t_{11}^2 \hat{\psi}(x_3) + t_{12}^2 \hat{\psi}'(x_3), \\ \hat{\psi}'(x_4) = t_{21}^2 \hat{\psi}(x_3) + t_{22}^2 \hat{\psi}'(x_3). \end{cases} \tag{7·1}$$

In addition (2·10) gives

$$\begin{cases} \hat{\psi}(x_1) = e^{i\phi_2} \hat{\psi}(x_4), \\ \hat{\psi}(x_3) = e^{i\phi_1} \hat{\psi}(x_2), \end{cases} \tag{7·2}$$

where $\phi_1 = \int_{x_1}^{x_2} a(y)dy$, $\phi_2 = \int_{x_3}^{x_4} a(y)dy$. Solving this system of six linear equations we get the TW-function

$$\begin{pmatrix} \psi'(x_1) - e^{i\phi_2}\psi'(x_4) \\ \psi'(x_2) - e^{i\phi_1}\psi'(x_2) \end{pmatrix} = \mathbf{M}(\lambda, \Gamma_4) \begin{pmatrix} \psi(x_1) \\ \psi(x_3) \end{pmatrix},$$

where

$$\mathbf{M}(\lambda, \Gamma_4) = \frac{1}{t_{12}^1 t_{12}^2} \begin{pmatrix} -(T^1T^2)_{12} & t_{12}^1 e^{i\phi_2} + t_{12}^2 e^{-i\phi_1} \\ t_{12}^2 e^{i\phi_1} + t_{12}^1 e^{-i\phi_2} & -(T^2T^1)_{12} \end{pmatrix}. \tag{7·3}$$

As we have already discussed in Section 5 the knowledge of the TW-function $\mathbf{M}(\lambda, \Gamma_3)$ allows one to reconstruct the TW-function for the kernel, *i.e.* the function $\mathbf{M}(\lambda, \Gamma_4)$, up to the similarity transformation with a diagonal unitary matrix. Let us denote the corresponding matrix by $\mathbf{M}_\Phi(\lambda)$, where the lower index shows the dependence of the matrix upon the total magnetic flux $\Phi = \phi_1 + \phi_2$

$$\mathbf{M}_\Phi(\lambda) = \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{i\phi_4} \end{pmatrix} \mathbf{M}(\lambda, \Gamma_4) \begin{pmatrix} e^{-i\phi_3} & 0 \\ 0 & e^{-i\phi_4} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{(T^1T^2)_{12}}{t^1_{12}t^2_{12}} & \left(\frac{1}{t^2_{12}} + \frac{1}{t^1_{12}}e^{-i\Phi}\right) e^{i(\phi_2+\phi_3-\phi_4)} \\ \left(\frac{1}{t^2_{12}} + \frac{1}{t^1_{12}}e^{i\Phi}\right) e^{-i(\phi_2+\phi_3-\phi_4)} & -\frac{(T^2T^1)_{12}}{t^1_{12}t^2_{12}} \end{pmatrix}. \tag{7·4}$$

The phases appearing in the similarity transformation are equal to the integrals of the magnetic field along the boundary edges $\phi_3 = \int_{x_5}^{x_6} a(y)dy$, $\phi_4 = \int_{x_7}^{x_8} a(y)dy$. It is clear that the intervals $[x_1, x_2]$ and $[x_3, x_4]$ can be exchanged without changing the TW-function. Therefore it is reasonable to study the possibility to reconstruct the graph and the real potential q up to such transformation. We are going to show that the knowledge of the TW-function for different values of the total flux allows one to reconstruct the graph and the potential q. Therefore we are going to indicate this dependence using the lover index, ignoring the dependence of $\mathbf{M}_\Phi$ upon the other phases.

7·2. *On the resonance condition*

Since the potential on the branches can be reconstructed with the help of Boundary Control method using the response operator, the restriction of the differential operator to the kernel of the graph is going to play central role in the solution of the inverse problem. In what follows we are going to introduce a certain new no-resonance condition which is sufficient for the solvability of the inverse problem. We are going to formulate this condition for the graph Γ_3 here, but it will be generalized later for arbitrary graphs with Euler characteristic zero. It will be shown that this condition is deeply connected with the special properties of the differential operators on graphs already discussed in the introduction.

Let $L_{q,a}$ be a magnetic Schrödinger operator in $L_2(\Gamma_3)$. Let us denote by $L_{q,a}|_{\ker\Gamma_3}$ the differential operator determined in $L_2(\ker\Gamma_3) = L_2(\Gamma_4)$ by the same differential expression as the operator $L_{q,a}$ on the domain of functions from $W_2^2([x_1, x_2] \cup [x_3, x_4])$ satisfying Dirichlet boundary conditions at the contact vertexes. Note that the operator $L_{q,a}|_{\ker\Gamma_3}$ is not a restriction of the operator $L_{q,a}$ in the standard linear operator sense.

Observe that the operator $L_{q,a}|_{\ker\Gamma_3}$ can be seen as the orthogonal sum of two operators in $L_2([x_1, x_2])$ and $L_2([x_3, x_4])$ with Dirichlet boundary conditions at the end points. The magnetic potential can be eliminated, *i.e.* the operator $L_{q,a}|_{\ker\Gamma_3}$ is unitary equivalent to the operator $L_q|_{\ker\Gamma_3} = L_{q,0}|_{\ker\Gamma_3}$. The spectra of the two operators appearing in the orthogonal decomposition are pure discrete and are given by eigenvalues of multiplicity one. In what follows we are going to use the following

No-resonance condition 1. We say that the **no-resonance condition is satisfied** if and only if the spectrum of the self-adjoint operator $L_{q,a}|_{\ker\Gamma_3}$ is simple, *i.e.* no multiple eigenvalue occurs.

In other words the no-resonance conditions is satisfied if and only if the spectra of the two Schrödinger operators on the intervals $[x_1, x_2]$ and $[x_3, x_4]$ do not intersect. If the

no-resonance condition is violated for $\lambda = E$, *i.e.* both operators have eigenvalue E, then we are going to say that the operator $L_{q,a}$ **has a resonance** with the energy E.

It is clear that generically the no-resonance condition is satisfied. The following two lemmas give the corresponding necessary and sufficient conditions.

LEMMA 7·1. *Assume that the operator $L_{q,a}$ in $L_2(\Gamma_3)$ has an eigenvalue with the corresponding eigenfunction having support separated from the boundary of the graph, then the no-resonance condition is not satisfied.*

Proof. Assume that conditions of the lemma are satisfied. Every eigenfunction having support separated from the boundary is identically equal to zero on the two boundary edges. It follows that every such eigenfunction is supported by the kernel $\ker \Gamma_3 = \Gamma_4$ and satisfies Dirichlet boundary conditions at the contact vertexes. This implies directly that the operator $L_{q,a}|_{\ker \Gamma_3}$ has a double eigenvalue, *i.e.* the no-resonance condition is violated.

Note that the no-resonance condition may be violated even if $L_{q,a}$ has no eigenfunction supported by the kernel of the graph. Such eigenfunction exists for a suitable magnetic flux if and only if the following two conditions are satisfied:

$$t_{12}^1(E) = t_{12}^2(E) = 0 \tag{7·5}$$

and

$$|t_{22}^1(E) t_{22}^2(E)| = 1. \tag{7·6}$$

The first condition implies that the operator $L_{q,a}|_{\ker \Gamma_3}$ has a double eigenvalue, but the second condition provides additional restrictions on the potential q and does not follow from the first one.

It appears that sufficient condition for the resonance can be formulated conveniently using the scattering matrix. The scattering matrix associated with a compact graph Γ with contact vertexes γ_j, $j = 1, 2, \dots, \mathcal{M}$ can be defined as follows. Consider the extended non-compact graph Γ^{ext} obtained from Γ by attaching semiaxes $[\gamma_j, +\infty)$, $j = 1, 2, \dots, \mathcal{M}$ to all contact vertexes and extend the potentials q and a to the whole Γ^{ext} by putting $q_{\Gamma^{\text{ext}} \setminus \Gamma} = a_{\Gamma^{\text{ext}} \setminus \Gamma} = 0$. Then the corresponding magnetic Schrödinger operator $L_{q,a}(\Gamma^{\text{ext}})$ is determined by the standard boundary conditions at all vertexes. Its spectrum contains the branch of absolutely continuous spectrum $[0, \infty)$ with the multiplicity equal to the number $\mathcal{M}$ of contact vertexes in Γ. Consider solutions $\psi^l(\lambda, x)$, $l = 1, 2, \dots, \mathcal{M}$ to the differential equation

$$\left(\left(-\frac{1}{i} \frac{d}{dx} + a(x) \right)^2 + q(x) \right) \psi^l(\lambda, x) = \lambda \psi^l(\lambda, x) \tag{7·7}$$

satisfying standard boundary conditions at all vertexes and having the following asymptotics

$$\psi^l(\lambda, x) = \begin{cases} s_{jl}(\lambda) e^{ik(x-\gamma_j)}, & x \in [\gamma_j, +\infty), \\ e^{-ik(x-\gamma_l)} + s_{ll} e^{ik(x-\gamma_l)}, & x \in [\gamma_l, +\infty). \end{cases} \tag{7·8}$$

The corresponding matrix $\mathbf{S}(\lambda)$ is a unitary $\mathcal{M} \times \mathcal{M}$ matrix and is precisely the scattering matrix for the operator $L_{q,a}(\Gamma^{\text{ext}})$ and the operator $L_{q,a}^0(\Gamma^{\text{ext}})$ given by the same differential expression and Neumann boundary conditions at all contact vertexes γ_j and standard boundary conditions at all other vertexes.

The scattering matrix $\mathbf{S}$ and TW-matrix $\mathbf{M}$ are connected by the rational transformation

$$\mathbf{S}(\lambda) = \frac{ik - \mathbf{M}(\lambda)}{ik + \mathbf{M}(\lambda)}.$$

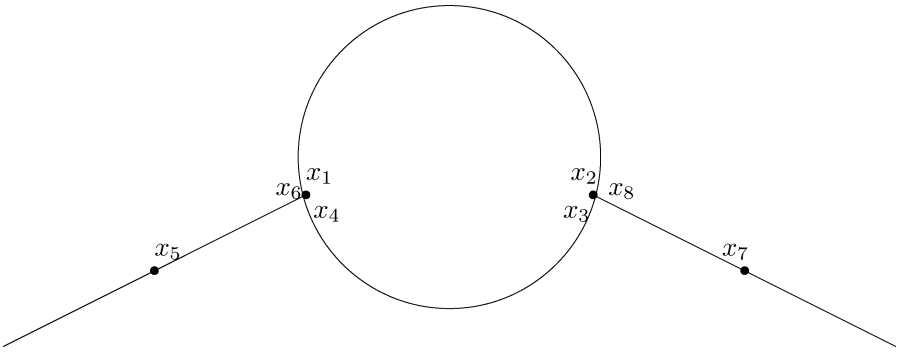

Fig. 4. Ring with two infinite wires. Graph Γ_3^{ext}.

For example the 2×2 scattering matrix for the graph Γ_4 (with two contact vertexes) is given by

$$\begin{aligned} S_{11} = S_{22} &= -\frac{2\cos(\Phi) - k^2 t_{12}^1 t_{12}^2 + ik((T^1T^2)_{12} - (T^2T^1)_{12}) - \mathrm{Tr}\,(T^1T^2)}{2\cos(\Phi) + k^2 t_{12}^1 t_{12}^2 + ik((T^1T^2)_{12} + (T^2T^1)_{12}) - \mathrm{Tr}\,(T^1T^2)}, \\ S_{12} &= \frac{2ik(t_{12}^1 + t_{12}^2 e^{-i\Phi})e^{i(\phi_2+\phi_3-\phi_4)}}{2\cos(\Phi) + k^2 t_{12}^1 t_{12}^2 + ik((T^1T^2)_{12} + (T^2T^1)_{12}) - \mathrm{Tr}\,(T^1T^2)}, \\ S_{21} &= \frac{2ik(t_{12}^1 + t_{12}^2 e^{i\Phi})e^{-i(\phi_2+\phi_3-\phi_4)}}{2\cos(\phi_1) + k^2 t_{12}^1 t_{12}^2 + ik((T^1T^2)_{12} + (T^2T^1)_{12}) - \mathrm{Tr}\,(T^1T^2)}, \end{aligned} \tag{7·9}$$

where we used the following identities

$$t_{12}^1 t_{12}^2 \mathrm{Tr}\,(T^1T^2) = (T^2T^1)_{12}(T^1T^2)_{12} - (t_{12}^1)^2 - (t_{12}^2)^2.$$

LEMMA 7·2. *Assume that the operator $L_{q,a}$ has a resonance with the energy E, then either the operator $L_{q,a}$ has an eigenfunction with the same energy supported by the kernel of the graph (equivalently, has support separated from the boundary) or the corresponding scattering matrix $\mathbf{S}(E)$ is diagonal.*

Proof. The graph Γ_3 has two contact vertexes $\gamma_1 = \{x_5\}$ and $\gamma_2 = \{x_7\}$ and two internal vertexes $V_1 = \{x_1, x_4, x_6\}$ and $V_2 = \{x_2, x_3, x_8\}$. The extended graph Γ_3^{ext} is presented in Figure 4.

Consider the case of zero magnetic flux. Assume that there is a resonance with the energy $k^2 = E$ and consider the function $\psi^1(k, x)$ having asymptotics (7·8). This function is equal to zero, or different from zero at the internal vertexes V_1 and V_2 simultaneously.

Assume first that ψ^1 is different from zero at the internal vertexes. Taking into account that in the case of resonance (7·5) is satisfied we get

$$\psi^1(k, V_2) = t_{11}^1(k)\psi^1(k, V_1) \quad \text{and} \quad \psi^1(k, V_1) = t_{11}^2(k)\psi^1(k, V_2)$$

and hence

$$t_{11}^1(k)t_{11}^2(k) = 1.$$

On the other hand using (7·5) once more time we have

$$t_{11}^1(k)t_{22}^1(k) = 1 \quad \text{and} \quad t_{11}^2(k)t_{22}^2(k) = 1,$$

which allows us to conclude that

$$t_{22}^1(k)t_{22}^2(k) = 1. \tag{7·10}$$

This is precisely the condition, which guarantees that the operator $L_{q,a}(\Gamma)$ has an eigenfunction supported by the kernel of Γ.

It remains to study the case where $\psi^1(k, V_1) = \psi^1(k, V_2) = 0$. The function ψ^1 restricted to the semiinfinite edge $[x_6, +\infty) = [x_6, \gamma_1] \bigcup [\gamma_1, +\infty)$ is a solution to equation (7·7) satisfying Dirichlet boundary condition at the left end point of the interval. Hence the scattering coefficient $s_{11}(E)$ coincides with the reflection coefficient for the differential operator $-d^2/dx^2 + q(x)$ in $L_2[x_6, \infty)$ defined on the functions satisfying Dirichlet conditions at x_6. It follows that $|s_{11}(E)| = 1$ and hence the unitary 2×2 scattering matrix $\mathbf{S}(E)$ is diagonal.

The case of non-zero magnetic potential can be treated similarly. The only difference is that instead of condition (7·10) we have

$$t^1_{22}(k)t^2_{22}(k) = e^{-i\Phi},$$

which implies (7·6).

Note that reformulation of this lemma in terms of the TW-function is not straightforward, since this matrix function is singular in the case of the resonance.

We observe here an interesting phenomena typical for embedded eigenvalues, which are usually extremely unstable (compare for example with the case of Wigner-von Neumann potential in [**66**]). The eigenfunction supported by the cycle is also unstable in the sense that it may exist for just one value of the magnetic flux, while for all other values the scattering matrix is diagonal. It might be interesting to compare the our results with [**18**], where bound states for Aharonov–Bohm rings are studied.

7·3. *Solution to the inverse problem*

We are ready to prove the main result in this section and may be the central result in the whole article:

THEOREM 7·3. *Let $L_{q,a}(\Gamma_3)$ be a magnetic Schrödinger operator on the graph Γ_3 with the real square integrable potential $q \in L_2(\Gamma_3)$ and the real continuous magnetic potential $a \in C(\Gamma_3)$. Assume that the No-resonance condition 1 is satisfied, then the potential q on Γ_3 is uniquely determined by the TW-function $\mathbf{M}_\Phi(\lambda, \Gamma_3)$ known for $\Phi = 0, \pi$, where Φ is the total flux of the magnetic field though the ring $\Phi = \int_{[x_1,x_2]\cup[x_3,x_4]} a(y)dy$.*

Proof. We have already discussed that the potential on the boundary edges is uniquely determined by the TW-function. Therefore it remains to prove, that the potential on the kernel Γ_4 can be recovered.

Consider the matrix $\mathbf{M}_\Phi(\lambda)$ defined by (7·4), where the subindex Φ indicates again the value of the total flux through the ring. The entry (12) for this matrix taken for $\Phi = 0, \pi$ implies the following equalities

$$\begin{cases} |(M_0(\lambda))_{12}| = \left|\dfrac{1}{t^1_{12}} + \dfrac{1}{t^2_{12}}\right|, \\ \dfrac{1}{4}\left(|(M_0(\lambda))_{12}|^2 - |(M_\pi(\lambda))_{12}|^2\right) = \dfrac{1}{t^1_{12}}\dfrac{1}{t^2_{12}}. \end{cases} \tag{7·11}$$

Hence the analytic functions $t^1_{12}(k)$ and $t^2_{12}(k)$ can be calculated up to multiplication by ± 1 and the exchange $t^1_{12} \leftrightarrow t^2_{12}$. The sign can be determined by comparing the asymptotics of the calculated function as $k \to \infty$ with the standard one $\sin kd_n/k$, $d_n = x_{2n} - x_{2n-1}$, $n = 1, 2$.

The exchange corresponds to the exchange of the intervals and therefore we conclude that the functions $t_{12}^{1,2}(k)$ are uniquely determined by $\mathbf{M}_0(\lambda)$ and $\mathbf{M}_\pi(\lambda)$. (Note that the functions $t_{12}^{1,2}$ may coincide only if the two intervals $[x_1, x_2]$ and $[x_3, x_4]$ have the same length, *i.e.* if $d_1 = d_2$.)

Consider now the entry (11) for the matrix $\mathbf{M}_\Phi(\lambda)$ given by (7·4). Since this entry is independent of Φ, we choose $\Phi = 0$ to get the relation

$$\left(T^1(k)T^2(k)\right)_{12} = -t_{12}^1(k)t_{12}^2(k)(M_0(\lambda))_{11}, \tag{7·12}$$

i.e. the analytic function $\left(T^1(k)T^2(k)\right)_{12}$ is uniquely determined by $\mathbf{M}_0(\lambda)$. On the other hand the same function is equal to

$$\left(T^1(k)T^2(k)\right)_{12} = t_{11}^1(k)t_{12}^2(k) + t_{12}^1(k)t_{22}^2(k).$$

Consider the zeroes k_j^1 of the function t_{12}^1. These points form precisely the Dirichlet-Dirichlet spectrum for the interval $[x_1, x_2]$. Observe that the function t_{12}^2 is different from zero at these points $t_{12}^2(k_j^1) \neq 0$, otherwise the operator $L_{q,a}$ would have a resonance. It follows that the values $t_{11}^1(k_j^1)$ can be calculated

$$t_{11}^1(k_j^1) = \left(T^1(k_j^1)T^2(k_j^1)\right)_{12} / t_{12}^2(k_j^1). \tag{7·13}$$

Then the function t_{11}^1 is uniquely determined as an entire function of exponential type not larger than $d_1 = x_2 - x_1$, since the points k_j^1 form precisely the Dirichlet-Dirichlet spectrum. The function t_{22}^1 can be reconstructed in a similar way taking into account that $t_{22}^1(k_j^1) = 1/t_{11}^1(k_j^1)$. Then Proposition 3·2 implies that the potential on the interval $[x_1, x_2]$ is uniquely determined. The potential on the interval $[x_3, x_4]$ is reconstructed in a similar way.

Note that the reconstruction has been carried out without knowledge of the phases ϕ_3 and ϕ_4. This Theorem can be proved even if the TW-function is known for some other than 0 and π values of the total flux. In the following section we are going to generalize this result for arbitrary graphs having Euler characteristic 0.

It is natural to ask the question whether the knowledge of the TW-function for just one value of the magnetic flux is enough to reconstruct the potential. The first problem that we face is how to calculate the functions $t_{12}^{1,2}$ from the function $1/t_{12}^1 + 1/t_{12}^2$. This problem does not always have a unique solution.

7·4. *Important clarifying example*

Theorem 7·3 states that the potential on the inner part of the graph Γ_3 can be reconstructed if there is no resonance. It is not *apriori* clear whether this additional condition is necessary or it is just technical. In this subsection we are going to construct a counterexample showing that not all potentials can be recovered from the TW-function, that is a certain restricting condition (like the no-resonance condition formulated above) is actually needed.

Consider the case where the intervals $[x_1, x_2]$ and $[x_3, x_4]$ have equal lengths and the corresponding potentials $q^{1,2}$ extended periodically lead to Schrödinger operators with equal band spectra having nontrivial forbidden gaps. Under these conditions the functions u_+^1 and u_+^2 are equal. In addition we assume that the coefficients of the transfer matrices t_{12}^1 and t_{12}^2 are also equal. It is natural to introduce the following notations:

$$u_+ := u_+^1 = u_+^2,$$
$$t_{12} := t_{12}^1 = t_{12}^2.$$

Under these assumptions the TW-function for the kernel of the graph takes the following form

$$\begin{aligned}\mathbf{M}(\lambda, \Gamma_4) &= \frac{1}{t_{12}} \begin{pmatrix} -(t^1_{11} + t^2_{22}) & (e^{i\Phi} + 1)e^{-i\phi_1} \\ (e^{-i\Phi} + 1)e^{i\phi_1} & -(t^1_{11} + t^2_{22}) \end{pmatrix} \\ &= \frac{1}{t_{12}} \begin{pmatrix} -(u^1_- - u^2_- + 2u_+) & (e^{i\Phi} + 1)e^{-i\phi_1} \\ (e^{-i\Phi} + 1)e^{i\phi_1} & -(u^1_- - u^2_- + 2u_+) \end{pmatrix}. \end{aligned} \tag{7·14}$$

Provided that the functions u_+ and t_{12} are known, the TW-function $\mathbf{M}(\lambda, \Gamma_4)$ allows one to calculate just the combination $u^1_- - u^2_-$. We have already seen that the functions $u^{1,2}_-(k)$ are uniquely determined by their values at the points k_j being zeroes of t_{12}. The absolute values $|u^{1,2}_-(k_j)|$ are determined by $u_+(k_j)$ (see (3·17)), but the signs can be chosen arbitrarily. It follows that $\mathbf{M}(\lambda, \Gamma_4)$ determines the values $u^{1,2}_-(k_j)$ only at the points where the functions have opposite signs. On the other hand if the signs of $u^1_-(k_j)$ and $u^2_-(k_j)$ are equal, then $u^1_-(k_j) - u^2_-(k_j) = 0$ independently of the values of these signs.

We are ready to present our counterexample. Under the conditions formulated above consider two potentials $q^{1,2}$ corresponding to two may be different sequences of sings $\{\nu^{1,2}_j\}$ (see (3·20)). Let us change these sequences at several points where $\nu^1_j = \nu^2_j$ and denote the new sequences by $\{\tilde{\nu}^{1,2}_j\}$. In accordance to Proposition 3·2 there exist two (different) potentials $\tilde{q}^{1,2}$ corresponding to these two new sets of spectral data. Thus we have proved the following

LEMMA 7·4. *Let $L_{q,a}(\Gamma_3)$ be a magnetic Schrödinger operator on the graph Γ_3 with real square integrable potential $q \in L_2(\Gamma_3)$ and real continuous magnetic potential $a \in C(\Gamma_3)$. Then the knowledge of the TW-function for all values of the magnetic flux in general is not enough to reconstruct the potential q.*

The counterexample presented here can be extended to the case where the intervals $[x_1, x_2]$ and $[x_3, x_4]$ have different lengths. Such examples can be constructed using the interplay between the two quasimomenta. On the other hand our example shows clearly that the resonance does not guarantee that there exist two different potentials leading to the same TW-function.

8. *On the inverse problem for arbitrary graphs with Euler characteristic zero*

In this section we are going to generalize the results obtained in the previous section to cover the case of arbitrary graphs, but so far having Euler characteristic equal to zero. Every such graph can be seen as one cycle (the kernel) with several branches attached to it, since the case of loops will be excluded from our considerations. We have already seen that the potential on the branches can be reconstructed, but in order to guarantee the possibility to reconstruct the potential on the kernel we need a certain generalization of the No-resonance condition 1.

Let $L_{q,a}$ be a magnetic Schrödinger operator on a graph Γ with the Euler characteristic equal to zero. Consider the kernel $\ker\Gamma$, which is a cycle divided by $\tilde{\mathcal{M}}$ contact points $\tilde{\gamma}_j$, $j = 1, 2, \ldots, \tilde{\mathcal{M}}$ into $\tilde{\mathcal{M}}$ intervals $[x_{2j-1}, x_{2j}]$. Let us denote by $L_{q,a}|^{j,k}_{\ker\Gamma}$, $j \neq k$ the self-adjoint operator determined by the differential expression $L_{q,a}$ on the domain of functions from $\oplus \sum_{j=1}^{\tilde{\mathcal{M}}} W^2_2([x_{2j-1}x_{2j}])$ satisfying Dirichlet boundary conditions at the contact vertexes $\tilde{\gamma}_j$ and $\tilde{\gamma}_k$ and the standard boundary conditions at all other contact points. Every operator $L_{q,a}|^{j,k}_{\ker\Gamma}$ has pure discrete spectrum and can be decomposed into the orthogonal sum of two differential operators.

No-resonance condition 2. We say that the **no-resonance condition is satisfied** if and and only if the spectrum of at least one of the self-adjoint operators $L_{q,a}|^{j,k}_{\ker\Gamma}$ is simple, *i.e.* no multiple eigenvalue occurs.

Generically the operator $L_{q,a}$ has no resonance. It is clear that the second no-resonance condition is a generalization of the first one and therefore it is straightforward to obtain generalizations of Lemmas 7·1 and 7·2, but we prefer to turn directly to our final result in this article.

THEOREM 8·1. *Let* Γ *be a compact loop free finite metric graph with one cycle,* i.e. *with the Euler characteristic zero. Let* $L_{q,a}$ *be the magnetic Schrödinger operator in* $L_2(\Gamma)$*, where* q *is a square integrable real potential on* Γ *and* a *is a real continuous magnetic potential. Let* Φ *be the total flux through the cycle. Then the Titchmarsh–Weyl matrix function* $\mathbf{M}_\Phi(\lambda)$ *associated with all boundary vertexes, known for* $\Phi = 0, \pi$ *determines the graph* Γ *and the potential* q*, provided that the No-resonance condition 2 is satisfied.*

Proof. Theorem 5·4 implies that the knowledge of $\mathbf{M}_\Phi(\lambda)$ allows one not only to restore the branches and the potential q there, but to calculate the TW-function associated with the kernel of the graph up to the similarity transformation involving unitary diagonal matrix. Hence it remains to solve the inverse problem for the kernel.

The problem is already solved in the case where the kernel of the graph has just two contact vertexes (see Theorem 7·3). In this case No-resonance conditions 1 and 2 just coincide.

Consider the case of more than two contact vertexes. The spectrum of the operator $L_{q,a}$ restricted to the set of functions from the space $\oplus \sum_{j=1}^{\tilde{\mathcal{M}}} W_2^2(x_{2j-1}, x_{2j})$ satisfying standard matching conditions at all contact points can be obtained as the set of solutions to the equation $\det \mathbf{M}_\Phi(\lambda, \ker\Gamma) = 0$. This spectrum satisfies Weyl's asymptotics and therefore the length of the cycle is uniquely determined. Introducing the dynamical response operator we may determine the travelling times between different contact points and therefore their positions on the cycle. Let us comment how to solve this problem if there are just three contact points with travelling times d_{ij} between the points $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$, $i.j = 1, 2, 3$. Assume without loss of generality that d_{12} is the shortest distance. Then the point $\tilde{\gamma}_2$ is situated between the points $\tilde{\gamma}_1$ and $\tilde{\gamma}_3$ if and only if $d_{12} + d_{23} = d_{13}$. Otherwise the points $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ are situated on different sides of the point $\tilde{\gamma}_1$.

Let us pick up any two such points, say $\tilde{\gamma}_1 = \{x_1, x_{\tilde{\mathcal{M}}}\}$ and $\tilde{\gamma}_2 = \{x_2, x_3\}$, so that the spectrum of the operator $L_{q,a}|^{12}_{\ker\Gamma}$ is simple. (To check that the spectrum is simple we need to calculate the functions t_{12} for the two intervals and see that they do not have common zeroes. Take another contact points if this is not the case.) Let us denote by T^1 and T^2 the transfer matrices corresponding to the intervals $[x_1, x_2]$ and $[x_2, x_{\tilde{\mathcal{M}}}]$ respectively. The 2×2 TW-matrix function corresponding to the same potential and just two contact points $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ may be obtained from the $\tilde{\mathcal{M}} \times \tilde{\mathcal{M}}$ TW-function $\mathbf{M}_\Phi(\lambda, \ker\Gamma)$ using the following simple method. For any λ, $\Im\lambda > 0$ the function $\mathbf{M}_\Phi$ is invertible, hence consider the matrix function $\mathbf{M}_\Phi^{-1}(\lambda, \ker\Gamma)$. Then the main 2×2 block of this matrix is nothing else than the inverse TW-function associated with the kernel of Γ and just two contact points $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. It follows that since the spectrum of $L_{q,a}|^{12}_{\ker\Gamma}$ is simple the potential on the cycle can be reconstructed uniquely.

It remains to describe how all other contacts points should be placed on the cycle. If the points $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ do not divide the cycle into two intervals of equal lengths, then the other contact points can be uniquely placed on the cycle using just the distances between

them. Consider the case where $\tilde\gamma_1$ and $\tilde\gamma_2$ do divide the cycle into two equal intervals. Then checking the distances we find precisely two possible positions for the point $\tilde\gamma_3$. In what follows we are going to prove that the TW-matrices corresponding to different choices of $\tilde\gamma_3$ are different and therefore comparing them with the given TW-function we may identify the correct position of the point.

To prove that the TW-functions corresponding to different positions of the contact point $\tilde\gamma_3$ cannot coincide is the same as to prove that the corresponding scattering matrices have to be different. It is clear that it is enough to solve the problem how to place just one further contact point. Let us denote by x_0' and x_0'' two possible positions of the contact point (so that $x_0' \in [x_1, x_2],\ x_0'' \in [x_3, x_4]$). Let us build two extended graphs $\Gamma^{\mathrm{ext}'}$ and $\Gamma^{\mathrm{ext}''}$ by attaching semiaxes to corresponding triples of contact points. Choose any zero of the function t_{12}^2 : $t_{12}^2(k_0) = 0$, such that x_0'' is not a zero of the corresponding eigenfunction satisfying Dirichlet conditions at the end points of the interval $[x_3, x_4]$. Then there exists a scattered wave on the graph $\Gamma^{\mathrm{ext}'}$ with the following properties:

$$\psi|_{[x_1,x_2]} \equiv 0. \tag{8·1}$$

Such function can be calculated by 'gluing' together

(i) the Dirichlet–Dirichlet eigenfunction on the interval $[x_3, x_4]$;
(ii) the scattered waves on the semiaxes $[\tilde\gamma_j, +\infty)$, $j = 1, 2$ satisfying Dirichlet condition at the contacts points $\tilde\gamma_j$, $j = 1, 2$.

In particular (8·1) implies that $\psi(\tilde\gamma_1) = \psi(\tilde\gamma_2) = \psi(x_0') = 0$, in other words that the corresponding scattering matrix has eigenvalue -1 for $k = k_0$.

Assume that the scattering matrices for the graphs $\Gamma^{\mathrm{ext}'}$ and $\Gamma^{\mathrm{ext}''}$ are equal, then for $k = k_0$ there exists a scattered wave φ on $\Gamma^{\mathrm{ext}''}$ having similar properties: $\varphi(\tilde\gamma_1) = \varphi(\tilde\gamma_2) = \varphi(x_0'') = 0$. Since x_0'' is not a zero of the corresponding Dirichlet–Dirichlet eigenfunction it holds $\varphi|_{[x_3,x_4]} = 0$. Then the function $\varphi|_{[x_1,x_2]}$ is not identically equal to zero and therefore is a Dirichlet–Dirichlet eigenfunction on the interval $[x_1, x_2]$, which implies that $t_{12}^1(k_0) = 0$. We get contradiction with the no-resonance conditions. It follows that the TW-functions corresponding to two possible positions of the contact point $\tilde\gamma_3$ are indeed different.

It would be interesting to study possible generalizations of the above theorem for the case of graphs with several cycles (other values of the Euler characteristic). Most probably one should consider TW-functions corresponding to different values of the magnetic flux through each cycle. It is also possible to weaken the No-resonance condition formulating it in geometric terms, but it is not clear whether this formulation will be as simple as it is now. We are planning to return to this problem in one of the future publications.

Acknowledgements. First of all the author would like to thank professor Ricardo Weder with whom the idea to use magnetic fields to solve the inverse problem on graphs was discussed in March 2007. Fruitful discussions with J. E. Björk, J. Boman, A. Luger and A. Shkalikov helped the completion of the manuscript.

REFERENCES

[**1**] S. ALBEVERIO and P. KURASOV. *Singular perturbations of differential operators. Solvable Schrödinger type operators*. London Mathematical Society Lecture Note Series **271** (Cambridge University Press, 2000).

[**2**] S. AVDONIN and P. KURASOV. Inverse problems for quantum trees. *Inverse Probl. Imaging* **2** (2008), 1–21.

[3] S. AVDONIN, V. MIKHAYLOV and A. RYBKIN. The boundary control approach to the Titchmarsh–Weyl m-function. I. The response operator and A-amplitude. *Comm. Math. Phys.* **275** (2005), 791–803.

[4] J. E. AVRON, A. RAVEH and B. ZUR. Adiabatic quantum transport in multiply connected systems. *Rev. Modern Phys.* **60** (1988), 873–915.

[5] J. E. AVRON and L. SADUN. Chern numbers and adiabatic transport in networks with leads. *Phys. Rev. Lett.* **62** (1989), 3082–3084.

[6] J. E. AVRON and L. SADUN. Adiabatic quantum transport in networks with macroscopic components. *Ann. Phys.* **206** (1991), 440–493.

[7] M. I. BELISHEV. Boundary spectral inverse problem on a class of graphs (trees) by the BC method. *Inverse Problems* **20** (2004), 647–672.

[8] M. I. BELISHEV. On the boundary controllability of a dynamical system described by the wave equation on a class of graphs (on trees). *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **308** (2004), Mat. Vopr. Teor. Rasprostr. Voln. 33, 23–47, 252 (English trans: *J. Math. Sci. (N. Y.)* **132** (2006), 11–25).

[9] M. I. BELISHEV. Recent progress in the boundary control method. *Inverse Problems* **23** (2007), R1–R67.

[10] M. I. BELISHEV and A. F. VAKULENKO. Inverse problems on graphs: recovering the tree of strings by the BC-method. *J. Inverse Ill-Posed Probl.* **14** (2006), 29–46.

[11] J. VON BELOW. Can one hear the shape of a network? *Partial differential equations on multistructures* (Luminy, 1999), 19–36, Lecture Notes in Pure and Appl. Math. **219** (Dekker, 2001).

[12] G. BERKOLAIKO, R. CARLSON, S. A. FULLING and P. KUCHMENT (Editors). Quantum graphs and their applications. Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference held in Snowbird, UT, June 19–23, 2005. Contemporary Mathematics **415** (American Mathematical Society, 2006).

[13] J. BOMAN and P. KURASOV. Symmetries of quantum graphs and the inverse scattering problem. *Adv. Appl. Math.* **35** (2005), 58–70.

[14] G. BORG. Eine Umlehrung der Sturm-Liouvilleschen Eigenwertaufgabe. *Acta Math.* **78** (1946), 1–96.

[15] G. BORG. Uniqueness theorems in the spectral theory of $y" + (\lambda - q(x))y = 0$. Den 11te Skandinaviske Matematikerkongress (Trondheim), 1949, 276–287.

[16] B. M. BROWN and R. WEIKARD. A Borg-Levinson theorem for trees. *Proc. Royal Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **461** (2005), 3231–3243.

[17] J. BRÜNING, L. A. CHERNOZATONSKII, V. V. DEMIDOV and V. A. GEYLER. Transport properties of two-arc Aharonov-Bohm interferometers with scattering centers. *Russ. J. Math. Phys.* **14** (2007), 417–422.

[18] E. N. BULGAKOV, K. N. PICHUGIN, A. F. SADREEV and I. ROTTER. Bound states in the continuum in open Aharonov-Bohm rings. *JETP Letters*, **84** (2006), 430–435.

[19] M. BÜTTIKER, Y. IMRY and M. YA. AZBEL. Quantum oscillations in one-dimensional normal-metal rings. *Phys. Rev A* **30** (1984), 1982–1989.

[20] M. BÜTTIKER, Y. IMRY, R. LANDAUER and S. PINHAS. Generalized many-channel conductance formula with application to small rings. *Phys. Rev. B* **31** (1985), 6207–6215.

[21] R. CARLSON. Inverse eigenvalue problems on directed graphs. *Trans. Amer. Math. Soc.* **351** (1999), 4069–4088.

[22] Y. COLIN DE VERDIÈRE. Spectres de variétés riemanniennes et spectres de Graphes (French) [Spectra of Riemannian manifolds and spectra of graphs], Proc. ICM 1986, Vol. 1, 2, 522–530 (Amer. Math. Soc., 1987).

[23] Y. COLIN DE VERDIÈRE. *Spectres de Graphes* (Société Mathématiques de France, 1998).

[24] P. EXNER and P. ŠEBA. Free quantum motion on a branching graph. *Rep. Math. Phys.* **28** (1989), 7–26.

[25] L. FRIEDLANDER. Genericity of simple eigenvalues for a metric graph. *Israel J. Math.* **146** (2005), 149–156.

[26] S. A. FULLING, P. KUCHMENT and J. H. WILSON. Index theorems for quantum graphs. *J. Phys. A: Math. Theor.* **40** (2007), 14165–14180.

[27] J. GARNETT and E. TRUBOWITZ. Gaps and bands of one-dimensional periodic Schrödinger operators. *Comment. Math. Helv.* **59** (1984), 258–312.

[28] J. GARNETT and E. TRUBOWITZ. Gaps and bands of one-dimensional periodic Schrödinger operators. II. *Comment. Math. Helv.* **62** (1987), 18–37.

[29] N. I. GERASIMENKO and B. S. PAVLOV. Scattering problems on noncompact graphs. *Teoret. Mat. Fiz.* **74** (1988), 345–359 (English trans: *Theoret. and Math. Phys.* **74** (1988), 230–240).

[**30**] N. I. GERASIMENKO. Inverse scattering problem on a noncompact graph. *Teoret. Mat. Fiz.* **75** (1988), 187–200 (Englsih trans: *Theoret. and Math. Phys.* **75** (1988), 460–470).
[**31**] F. GESZTESY and B. SIMON. A new approach to inverse spectral theory. II. General real potentials and the connection to the spectral measure. *Ann. of Math.* (2) **152** (2000), 593–643.
[**32**] F. GESZTESY and E. TSEKANOVSKII. On matrix-valued Herglotz functions. *Math. Nachr.* **218** (2000), 61–138.
[**33**] B. GUTKIN and U. SMILANSKY. Can one hear the shape of a graph? *J. Phys. A* **34** (2001), 6061–6068.
[**34**] B. HABIB, E. TATIC and M. SHAYEGAN. Strong Aharonov-Bohm oscillations in GaAs two dimensional holes. arXiv:cond-mat/0612638v2, 7 Mar 2007.
[**35**] M. HARMER. Hermitian symplectic geometry and the factorization of the scattering matrix on graphs. *J. Phys. A* **33** (2000), 9015–9032.
[**36**] M. HARMER. Inverse scattering for the matrix Schrödinger operator and Schrödinger operator on graphs with general self-adjoint boundary conditions. Kruskal, 2000 (Adelaide). *ANZIAM J.* **44** (2002), 161–168.
[**37**] M. HARMER. Inverse scattering on matrices with boundary conditions. *J. Phys. A* **38** (2005), 4875–4885.
[**38**] A. R. ITS and V. B. MATVEEV. Hill operators with a finite number of lacunae (Russian). *Funksional. Anal. i Priložen.* **9** (1975), 69–70.
[**39**] A. R. ITS and V. B. MATVEEV. Schrödinger operators with the finite-band spectrum and the N-soliton solutions of the Korteweg-de Vries equation (Russian). *Teoret. Mat. Fiz.* **23** (1975), 51–68.
[**40**] V. KOSTRYKIN and R. SCHRADER. Kirchoff's rule for quantum wires. *J. Phys. A* **32** (1999), 595–630.
[**41**] V. KOSTRYKIN and R. SCHRADER. Kirchhoff's rule for quantum wires. II. The inverse problem with possible applications to quantum computers. *Fortschr. Phys.* **48** (2000), 703–716.
[**42**] V. KOSTRYKIN and R. SCHRADER. Quantum wires with magnetic fluxes. *Comm. Math. Phys.* **237** (2003), 161–179.
[**43**] V. KOSTRYKIN and R. SCHRADER. The inverse scattering problem for metric graphs and the travelling salesman problem. arXiv:math-ph/0603010.
[**44**] T. KOTTOS and U. SMILANSKY. Periodic orbit theory and spectral statistics for quantum graphs. *Ann. Physics* **274** (1999), 76–124.
[**45**] P. KUCHMENT. Quantum graphs. I. Some basic structures. Special section on quantum graphs. *Waves Random Media* **14** (2004), S107–S128.
[**46**] P. KUCHMENT. Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs. *J. Phys. A* **38** (2005), 4887–4900.
[**47**] P. KURASOV. Graph Laplacians and topology. *Arkiv för Matematik* **46** (2008), 95–111.
[**48**] P. KURASOV. Schrödinger operators on graphs and geometry. I. Essentially bounded potentials. *J. Funct. Anal.* **254** (2008), 934–953.
[**49**] P. KURASOV and M. NOWACZYK. Inverse spectral problem for quantum graphs. *J. Phys. A* **38** (2005), 4901–4915.
[**50**] P. KURASOV and F. STENBERG. On the inverse scattering problem on branching graphs. *J. Phys. A* **35** (2002) 101–121.
[**51**] N. LEVINSON. On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase. *Danske Vid. Selsk. Mat.-Fys. Medd.* **25** (1949), no. 9, 29 pp.
[**52**] B. M. LEVITAN and M. G. GASYMOV. Determination of a differential equation by two spectra (Russian). *Uspehi Mat. Nauk* **19** (1964), 3–63.
[**53**] YU. I. LYUBARSKIĬ and V. A. MARCHENKO. Direct and inverse problems of multichannel scattering (Russian). *Funktsional. Anal. i Prilozhen.* **41** (2007), 58–77.
[**54**] V. A. MARCHENKO. Some questions of the theory of one-dimensional linear differential operators of the second order. I (Russian). *Trudy Moskov. Mat. Obšč.* **1** (1952), 327–420.
[**55**] V. A. MARCHENKO. Sturm–Liouville operators and applications. *Oper. Theory Adv. Appl.* **22** (1986).
[**56**] V. A. MARCHENKO and I. V. OSTROVSKY. A characterization of the spectrum of the Hill operator (Russian). *Mat. Sb.* (N.S.) **97** (139) (1975), 540–606.
[**57**] V. A. MARCHENKO and I. V. OSTROVSKY. Approximations of Periodic by Finite-zone potentials. *Selecta Matematica Sovietica* **6** (1987), 101–136 (Originally published: *Vestnik Khar'kovsk. Univ.*, no. 205; *Prikl. Mat. i Mekh.* **45** (1980), 4–40).
[**58**] S. NOVIKOV. Discrete Schrödinger operators and topology. *Asian J. Math.* **2** (1998), 921–934.
[**59**] M. NOWACZYK. Inverse spectral problems for quantum graphs with rationally dependent edges. *Oper. Theory Adv. Appl.* **174** (2007), 105–116.
[**60**] M. NOWACZYK. Inverse Spectral Problems for Quantum Graphs, Licentiate Thesis (Lund University, 2005).

[**61**] M. NOWACZYK. Inverse Problems for Graph Laplacians, PhD Thesis (Lund University, 2008).
[**62**] V. PIVOVARCHIK. Scattering in a loop-shaped waveguide. *Oper. Theory Adv. Appl.* **124** (2001), 527–543.
[**63**] J. PÖSCHEL and E. TRUBOWITZ. *Inverse Spectral Theory* (Academic Press, 1987).
[**64**] YU. V. POKORNYI, O. M. PENKIN, V. L. PRYADIEV, A.V. BOROVSKIKH, K. P. LAZAREV and S. A. SHABROV. *Differential equations on geometric graphs* (in Russian) (Fiziko-Matematicheskaya Literatura, Moscow, 2005).
[**65**] A. RAMM and B. SIMON. A new approach to inverse spectral theory. III. Short-range potentials. *J. Anal. Math.* **80** (2000), 319–334.
[**66**] M. REED and B. SIMON. *Methods of Modern Mathematical Physics. I-IV* (Academic Press, 1972–1979).
[**67**] J.-P. ROTH. Le spectre du laplacien sur un graphe (French) [The spectrum of the Laplacian on a graph]. Théorie du potentiel (Orsay, 1983), 521–539, Lecture Notes in Math. **1096** (Springer, 1984).
[**68**] S. RUSSO, J. B. OOSTINGA, D. WEHENKEL, H. B. HEERSCHE, SAMIRA SHAMS SOBHANI, L. M. K. VANDERSYPEN and A. F. MORPURGO. Observation of Aharonov–Bohm conductance oscillations in a graphene ring. *Phys. Rev B* **77** (2008), 085413.
[**69**] I. A. SHELYKH, N. G. GALKIN and N. T. BAGRAEV. Conductance of a gated Aharonov-Bohm ring touching a quantum wire. *Phys. Rev. B* **74** (2006), 165331.
[**70**] B. SIMON. A new approach to inverse spectral theory. I Fundamental formalism. *Ann. of Math.* **150** (1999), 1029–1057.
[**71**] CH. TEXIER and M. BÜTTIKER. Local Friedel sum rule on graphs. *Phys. Rev. B* **67** (2003), 245410.
[**72**] A. TOKUNO, M. ASHIKAWA and E. DEMLER. Dynamics of of one-dimensional Bose liquids: Andreev-like reflection at Y-junctions and absence of the Aharonov–Bohm effect. ArXiv:cond.mat/0703610v3, 14 Apr 2008.
[**73**] A. L. YEYATI and M. BÜTTIKER. Aharonov-Bohm oscillations in a mesoscopic ring with a quantum dot. *Phys. Rev. B* **52** (1995), 14360–14363.
[**74**] V. YURKO. On the reconstruction of Sturm-Liouville operators on graphs (Russian). *Mat. Zametki* **79** (2006), 619–630 (English trans: *Math. Notes* **79** (2006), 572–582).
[**75**] V. YURKO. Inverse spectral problems for Sturm-Liouville operators on graphs. *Inverse Problems* **21** (2005), 1075–1086.
[**76**] V. YURKO. Recovering differential pencils on compact graphs. *J. Differential Equations* **244** (2008), 431–443.