Singular cluster interactions in few-body problems [1]

S.ALBEVERIO, Institut für Angewandte Mathematik, Universität Bonn, D-53115 Bonn; SFB 256; SFB 237; BiBoS; CERFIM (Locarno); Acc.Arch., USI (Mendrisio); Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum.

P.KURASOV, Dept. of Math., Stockholm University, 10691 Stockholm, Sweden; Dept. of Math., Luleå University, S-97187 Luleå, Sweden; St. Petersburg University, 198904 St. Petersburg, Russia.

1 Introduction

The present paper is devoted to the study of the few–body quantum mechanical problem with singular finite rank cluster interactions. The corresponding Hamiltonians play an important role in mathematical physics, since few–body Hamiltonians with more regular interactions present considerable difficulties which make impossible a detailed analytic study [13]. Also a numerical study of such Hamiltonians using Faddeev equations present very hard problems, since the interaction between the particles does not vanish at large distances. On the other hand few-body problems with singular cluster interaction are useful and intensively studied in statistical physics, since the corresponding Hamiltonians can be analyzed in detail even if the number of particles is very large. Models describing one dimensional particles are of particular interest, since the eigenfunctions of the many body Hamiltonians can often be calculated using Bethe Ansatz [14]. In fact the well-known Yang-Baxter equation was first written in connection with the study of system of

[1] Published as: S.Albeverio and P.Kurasov, Singular cluster interactions in few-body problems, Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), 277–292, Lecture Notes in Pure and Appl. Math., **215**, Dekker, New York, 2001.

several one-dimensional particles with pairwise delta interactions. Similar methods were used in atomic physics to study collisions of several particles [12] (in three dimensions). Few-body systems in applications are usually investigated by combining the classical description of the dynamics of heavy atoms with the quantum description of the dynamics of electrons and other light particles. The first attempt to construct a three body Hamiltonian describing three quantum particles in $\mathbf{R}^3$ interacting through pairwise delta functional potentials is due to G.V.Skorniakov and K.A.Ter-Martirosian [29]. R.A.Minlos and L.D.Faddeev proved that the corresponding Hamiltonian is not bounded from below and therefore cannot be used in the originally intended physical applications [24, 25]. The operator was defined using the method of self–adjoint perturbations used by F.A.Berezin and L.D.Faddeev for investigating Schödinger operators with delta potential in $\mathbf{R}^3$ [10]. Almost three decades later semibounded three-body operators in dimension three with generalized two-body interactions were constructed by extending the standard Hilbert space of square integrable functions in $\mathbf{R}^9$ [17, 18, 19, 27, 30]. The most interesting model considered used the theory of generalized extensions suggested by B.Pavlov [7, 26]. Different aspects of these models were analyzed recently [8, 23, 20]. A general approach to these operators is described in [7, 22]. Let us mention that a realization of a many-body lower bounded Hamiltonian with point interactions for particles in $\mathbf{R}^3$ has been obtained in the original Hilbert space by using the theory of Dirichlet forms [2]. The few-body systems of two dimensional particles with two-body interactions was studied in [11], where it was proven that the corresponding Hamiltonian is semibounded.

It is stressed particularly in [7] that few-body Hamiltonians with delta interactions can be efficiently studied by using the theory of finite rank perturbations. Hamiltonians describing point interactions created at many centers, e.g. of importance in solid state physics, have been discussed in [1]. Rank one form bounded interactions were analyzed in detail in an abstract setting by B.Simon and F.Gesztesy [15, 28]. Perturbations in terms of quadratic form were studied in [3]. In [16] rank one form unbounded interactions are defined following the paper by F.A.Berezin and L.D.Faddeev [10] and introducing a renormalization of the coupling constant. See also [4, 5], where these interactions are defined without renormalization of the coupling constant using a certain regularization procedure. It is not hard to extend this technique to obtain few-body operators with form bounded cluster interactions (see e.g. [7]), but operators with more singular interactions need a more detailed investigation. The present paper is devoted to the study of the few–body operator with one singular cluster interaction having finite rank. It is proven that this interaction in general is described by unbounded boundary operators. This operator can serve as an elementary brick in the construction of the few–body Hamiltonian with several singular cluster interactions.

The paper is organized as follows. In Section 2 the few–body operator with finite rank cluster interaction is heuristically defined. A precise definition of this operator is given in Section 3 by separating the center of mass motion and using the extension theory of operators with finite deficiency indices. To describe the analytic properties of these operators a study of rational transformations of Stieltjes functions is given in Section 4. The resolvent of the few–body operator is calculated in Section 5. The few–body operator is described in Section 6 without separation of the center of mass motion. Then the extension theory for symmetric operators with infinite deficiency indices is used. The extension is described by certain boundary conditions involving unbounded operators.

2 Few–body operator with finite rank cluster interaction

The Schrödinger operator describing several quantum mechanical particles is characterized by the following property: the original Hilbert space $\mathcal{H}$ and the operator $\mathcal{A}$ possess several tensor decompositions

$$\begin{aligned} \mathcal{H} &= K_n \otimes H_n, \qquad\qquad n = 1, 2, \ldots, N \\ \mathcal{A} &= B_n \otimes I_{H_n} + I_{K_n} \otimes A_n; \end{aligned} \tag{1}$$

where B_n and A_n are positive self–adjoint operators acting in the Hilbert spaces K_n and H_n respectively. The index n parameterizes the cluster decompositions and the number N of such decompositions is in general different from the number of particles. The operator A_n describes the motion of the particles forming the corresponding cluster in the coordinate system associated with the center of mass of the cluster. The operator B_n describes the motion of all other particles in the same coordinate system. Each pair of operators (A_n, B_n) appearing in the tensor decomposition determines the unperturbed operator $\mathcal{A}$ uniquely, since it is essentially self–adjoint on the algebraic tensor product of the domains of the operators A_n and B_n.

The few–body operator with singular finite rank interactions can heuristically be defined by

$$\mathcal{A}_\alpha = \mathcal{A} + \sum_{n=1}^{N} \sum_{j=1}^{d_n} \alpha_{n,jk} \langle \varphi_{nj}, \cdot \rangle_{H_n} \varphi_{nk}, \tag{2}$$

where φ_{nj} are singular vectors defining the cluster interactions. In what follows we suppose that these vectors belong to the Hilbert spaces $\mathcal{H}_{-2}(A_n)$ associated with the corresponding operators A_n

$$\varphi_{nj} \in \mathcal{H}_{-2}(A_n),\ j = 1, 2, \ldots, d_n,\ n = 1, 2, \ldots, N.$$

The numbers $d_n \in \mathbf{N}$ determine the rank of the cluster interaction. The coupling constants $\alpha_{n,jk}$ form $d_n \times d_n$ Hermitian matrices. Then the perturbation term is formally symmetric.

If the vectors φ_{nj} do not belong to the spaces $\mathcal{H}_{-1}(A_n)$ then the perturbation term is not form bounded with respect to the unperturbed operator. To determine the few–body operator in this case it is necessary to carry out a special analysis including the extension theory for symmetric operators.

The aim of the current paper is to describe few–body Hamiltonians with one cluster interaction. This is the first step towards the definition of general few–body Hamiltonian with finite rank cluster interactions.

Let us consider the Hilbert space $\mathcal{H}$ and the operator $\mathcal{A}$ possessing the tensor decomposition

$$\begin{aligned} \mathcal{H} &= K \otimes H, \\ \mathcal{A} &= B \otimes I_H + I_K \otimes A \end{aligned} \tag{3}$$

where B and A are positive self–adjoint operators in the Hilbert spaces K and H respectively. Consider d singular vectors from the Hilbert space $\mathcal{H}_{-2}(A)$. The the few–body operator with single cluster interaction is heuristically described by

$$\mathcal{A}_\alpha = \mathcal{A} + \sum_{j,k=1}^{d} \alpha_{jk} \langle \varphi_j, \cdot \rangle_H \varphi_k, \tag{4}$$

where the coupling constants α_{jk} form an Hermitian matrix.

If the vectors φ_j are from the Hilbert space H, then the perturbation term is a bounded operator. The perturbed operator in this case has the same domain as the original operator $\mathcal{A}$. The problem of defining the operator with cluster interactions is trivial in this case. Therefore we are going to concentrate our attention on the case of so-called H-independent vectors.

Definition 2.1 *The set of vectors $\{\varphi_j\}_{j=1}^d \subset \mathcal{H}_{-2}(A)$ is called* H**-independent** *if and only if any nontrivial linear combination*

$$\sum_{j=1}^{d} f_j \varphi_j,\ f_j \in \mathbf{C},\ \sum_{j=1}^{d} |f_j|^2 > 0$$

does not belong to the Hilbert space H.

In what follows without loss of generality we suppose that the vectors $\{\varphi_j\}_{j=1}^d$ form an orthonormal set in $\mathcal{H}_{-2}(A)$, i.e.

$$\left\langle \frac{1}{A-i}\varphi_i, \frac{1}{A-i}\varphi_j \right\rangle_H = \delta_{ij}. \tag{5}$$

Any set of independent vectors can be orthonormalized and the new set is independent also. In particular

$$\varphi_j \in \mathcal{H}_{-2}(A) \setminus H.$$

The Hermitian coupling matrix can be diagonalized using a certain orthogonal transformation. Therefore it is enough to consider only diagonal coupling matrices. Thus the following heuristic operator will be studied in this paper

$$\mathcal{A}_\alpha = \mathcal{A} + \sum_{j=1}^{d} \alpha_j \langle \varphi_j, \cdot \rangle_H \varphi_j, \tag{6}$$

where the vectors φ_j form an orthonormal subset of $\mathcal{H}_{-2}$ and the real coupling constants are different from zero,

$$\alpha_j \in \mathbf{R},\ \alpha_j \neq 0,\ j = 1, 2, \ldots, d. \tag{7}$$

3 Cluster interaction via separation of the center of mass motion

The operator $\mathcal{A}_\alpha$ can be defined using the following formal decomposition

$$\begin{aligned}\mathcal{A}_\alpha &= B \otimes I_H + I_K \otimes A + \sum_{j=1}^{d} \alpha_j \langle \varphi_j, \cdot \rangle_H \varphi_j \\ &= B \otimes I_H + I_K \otimes \left(A + \sum_{j=1}^{d} \alpha_j \langle \varphi_j, \cdot \rangle_H \varphi_j \right).\end{aligned} \tag{8}$$

The operator

$$A_\alpha = A + \sum_{j=1}^{d} \alpha_j \langle \varphi_j, \cdot \rangle_H \varphi_j \tag{9}$$

is a finite rank perturbation of the operator A and it can easily be defined following [6]. To define the operator A_α we restrict first the operator A to the domain

$$\mathrm{Dom}\,(A^0) = \{\psi \in \mathrm{Dom}\,(A) : \langle \varphi_j, \psi \rangle_H = 0, j = 1, 2, \ldots, d\}. \tag{10}$$

The restricted operator A^0 is symmetric and densely defined. The deficiency elements at point i are given by

$$\frac{1}{A-i}\varphi_j, j = 1, 2, \ldots, d,$$

and form an orthonormal subset of the Hilbert space H. The intersection between the deficiency subspace and the domain of the original operator is trivial

$$\mathrm{Ker}\,(A^{0*} \pm i) \cap \mathrm{Dom}\,(A) = \{0\}.$$

Therefore the restricted operator is densely defined and has deficiency indices (d, d). The vectors $\frac{1}{A-i}\varphi_j$ form a basis in the deficiency subspace $\mathrm{Ker}\,(A^{0*} - i)$ denoted by M, i.e.

$$M \equiv \mathrm{Ker}\,(A^{0*} - i).$$

All self–adjoint extensions of the operator A^0 can be described using von Neumann formulas. But we prefer to define the extension using Krein's resolvent formula. Essentially all extensions of the operator A^0 can be parameterized by an Hermitian operators γ acting in M in such a way that the resolvent of the corresponding self–adjoint operator A^γ is given by [2]

$$\frac{1}{A^\gamma - z} = \frac{1}{A-z} - \frac{A+i}{A-z}\frac{1}{\gamma + q(z)} P_M \frac{A-i}{A-z}, \tag{11}$$

[2] The self–adjoint extensions that would not be described by this formula would have the following property: the operator A^0 is not the largest common symmetric restriction of the perturbed and unperturbed operators. But this implies that at least one of the coupling constants is equal to zero, which is impossible due to our assumption (7).

where "Krein's Q-operator" q can be chosen as follows

$$q(z) = P_M \frac{1+zA}{A-z}|_M.$$

q is a $d \times d$ matrix Nevanlinna function, i.e. its imaginary part is positive in the upper half plane $\Im z > 0$. Our choice of q is determined by the normalization condition

$$q(i) = i|_M. \tag{12}$$

If all vectors φ_j are from the space $\mathcal{H}_{-1}(A)$ then the relation between the coupling parameters $\alpha_1, \alpha_2, \ldots, \alpha_d$ and the Hermitian operator γ is given by

$$\gamma = \alpha^{(-1)} + P_M A|_M, \tag{13}$$

where α is the following coupling operator defined in M

$$\alpha = \sum_{j=1}^{d} \alpha_j \langle \frac{1}{A-i}\varphi_j, \cdot \rangle \frac{1}{A-i}\varphi_j.$$

The second term in (11) is well defined. In fact using the orthonormal basis $\{ \frac{1}{A-i}\varphi_j$ in M we have

$$P_M A P_M = \sum_{j,k=1}^{d} \frac{1}{A-i}\varphi_k \left\langle \frac{1}{A-i}\varphi_j, A\frac{1}{A-i}\varphi_k \right\rangle_H \left\langle \frac{1}{A-i}\varphi_j, \cdot \right\rangle_H, \tag{14}$$

since the deficiency subspace M is a subset of $\mathcal{H}_{-1}(A)$. (We remark that the scalar products $\left\langle \frac{1}{A-i}\varphi_j, A\frac{1}{A-i}\varphi_k \right\rangle_H \equiv \left\langle \varphi_j, \frac{A}{A^2+1}\varphi_k \right\rangle_H$ are well defined.)

In the case where some of the φ_j are not from $\mathcal{H}_{-1}(A)$ the operator γ cannot be calculated from the coupling parameters without using additional assumptions. Only some partial information concerning the operator γ can be recovered. The formal expression (9) does not define a unique operator in this case, but a certain family of self–adjoint operators described by several parameters. The number of free real parameters can be different from d^2 (the number of real parameters in von Neumann formulas), since the operator γ should satisfy some admissibility conditions if the vectors φ_j are not $\mathcal{H}_{-1}$-independent. Suppose that a certain linear combination of the vectors φ_j belongs to the space $\mathcal{H}_{-1}$

$$\phi = \sum_{j=1}^{d} a_j \varphi_j \in \mathcal{H}_{-1}(A), \quad a_j \in \mathbf{C}$$

then the scalar product $\left\langle \phi, \frac{A}{A^2+1}\phi \right\rangle_H$ is well defined. Therefore the operator γ necessarily satisfies the admissibility condition

$$\left\langle \sum_{j=1}^{d} a_j \varphi_j, \frac{A}{A^2+1} \sum_{k=1}^{d} a_k \varphi_k \right\rangle_H = \sum_{j,k=1}^{d} \overline{a_j} a_k \left(\gamma_{jk} - \alpha_j^{-1} \delta_{jk} \right),$$

where γ_{ik} are the coefficients of the operator γ in the chosen orthonormal basis

$$\gamma_{jk} = \left\langle \frac{1}{A-i}\varphi_j, \gamma \frac{1}{A-i}\varphi_k \right\rangle_H .$$

The operators satisfying all admissibility conditions are called *admissible*. The family of admissible operators γ was described in [6]. If in addition the operator A and all vectors φ_j are homogeneous with respect to a certain group of unitary transformations, then the operator γ could be determined uniquely by requiring additional natural homogeneity properties of the perturbed operator. This approach has been developed in [4, 5] for perturbations of rank one.

In what follows we suppose that the extension of the operator A^0 is determined by a certain admissible Hermitian operator γ, which is compatible with the heuristic expression (9). This extension will be denoted by A^γ. In what follows the operator A^γ will be substituted for the operator A_α. The corresponding few-body operator with single cluster interaction is defined by

$$\mathcal{A}^\gamma = B \otimes I_H + I_K \otimes A^\gamma. \tag{15}$$

The last formula determines the operator $\mathcal{A}^\gamma$ uniquely, since $B \otimes I_H + I_K \otimes A^\gamma$ is essentially self–adjoint on the algebraic tensor product of the domains of the operators B and A^γ. Hence

$$\mathrm{Dom}\,(\mathcal{A}^\gamma) = \overline{\mathrm{Dom}\,(B) \times \mathrm{Dom}\,(A^\gamma)}. \tag{16}$$

In order to simplify our presentation let us restrict ourselves to the case of perturbations of rank one ($d = 1$). We are going to drop the lower index of the coupling constant and singular vector

$$\alpha \equiv \alpha_1, \quad \varphi \equiv \varphi_1.$$

To calculate the resolvent of the operator $\mathcal{A}^\gamma$ we need to study the properties of the Nevanlinna functions describing the interaction.

4 Rational transformations of Stieltjes functions

Let us prove first some facts concerning Nevanlinna functions F which belong to the **Stieltjes class**, i.e. possess the representation

$$F(z) = \int_A^\infty \frac{1+z\lambda}{\lambda - z} d\rho(\lambda), \tag{17}$$

where the real measure ρ is finite $\int_A^\infty d\rho(\lambda) < \infty$.

Lemma 4.1 *Let F be a Stieltjes function. Then for any real y and any positive $\epsilon > 0$ there exists a certain $b = b(y, \epsilon) > 0$ such that the following estimate holds*

$$|F(x+iy)| < \epsilon |x| + b \tag{18}$$

for all $x < A$.

Proof Consider the real and imaginary parts of the function

$$F(x+iy) = \int_A^\infty \frac{1+(x+iy)\lambda}{\lambda - x - iy} d\rho(\lambda).$$

The imaginary part

$$\Im F(x+iy) = y \int_A^\infty \frac{\lambda^2+1}{(\lambda-x)^2+y^2} d\rho(\lambda)$$

is uniformly bounded for all $x < A$. The real part is given by the sum of two integrals as follows:

$$\Re F(x+iy) = \int_A^\infty \frac{\lambda(1-y^2)-x}{(\lambda-x)^2+y^2} d\rho(\lambda) + \int_A^\infty \frac{x\lambda(\lambda-x)}{(\lambda-x)^2+y^2} d\rho(\lambda).$$

The first integral is uniformly bounded. The second integral can be estimated as follows

$$\left| \int_A^\infty \frac{x\lambda(\lambda-x)}{(\lambda-x)^2+y^2} d\rho(\lambda) \right| \leq \int_A^\infty \frac{|x||\lambda|}{\lambda - x} d\rho(\lambda)$$

for all $x < A-1$. To estimate the latter integral we choose $C > A$ such that $\int_c^\infty d\rho(\lambda) < \epsilon/2$ and we get

$$\begin{aligned} \int_A^\infty \frac{|x|\,|\lambda|}{\lambda-x} d\rho(\lambda) \leq & \quad |x| \frac{C}{C-x} \int_A^\infty d\rho(\lambda) + |x| \int_C^\infty d\rho(\lambda) \\ \leq & \quad \left(\frac{C}{C-x} \int_A^\infty d\rho(\lambda) + \epsilon/2 \right) |x|. \end{aligned}$$

For all $x \leq x_0 = C(1 - \frac{1}{\epsilon}\int_A^\infty d\rho(\lambda))$ the last expression is estimated by $\epsilon|x|$. The function $F(x+iy)$ is continuous on the bounded interval $x_0 \leq x \leq A$ and is therefore uniformly bounded on this interval (which is empty if $x_0 > A$). The lemma is proven.

□

The following lemma describes rational transformations of Stieltjes functions.

Lemma 4.2 *Let F be a Stieltjes function. Let a, b, c, d be real numbers such that*

$$ad - bc = 1. \tag{19}$$

Then there exists a real number A_1, such that for any real y and any positive $\epsilon > 0$ there exists $b_1 = b_1(y, \epsilon)$ such that the function

$$G(z) = \frac{aF(z)+b}{cF(z)+d} \tag{20}$$

possesses the representation

$$G(z) = \beta z + g(z),\ \beta > 0, \tag{21}$$

where

$$|g(x+iy)| < \epsilon|x| + b_1 \tag{22}$$

for all $x < A_1$.

Proof Condition (19) guarantees that the function G is a Nevanlinna function and possesses the representation

$$G(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \frac{1 + \lambda z}{\lambda - z} d\rho_1(\lambda),$$

where $\int_{-\infty}^{\infty} d\rho_1(\lambda) < \infty,\ \alpha, \beta \in \mathbf{R},\ \beta > 0$. The support of the measure ρ_1 coincides with the set of real points z where the boundary values $G(z + i0)$ are not real or do not exist. The function G is real on the interval $(-\infty, A)$ outside the points where $cF(z) + d = 0$. The derivative

$$\frac{dF(z)}{dz} = \int_{A}^{\infty} \frac{\lambda^2 + 1}{(\lambda - z)^2} d\rho_1(\lambda)$$

is positive for all $z < A$. It follows that there exists at most one point where $F(z) = -\frac{d}{c}$. Therefore the support of the measure ρ_1 is bounded from below. Lemma 4.1 implies then estimate (22). The lemma is proven.

□

We remark that the constant β appearing in (22) is different from zero only if the function F has a finite limit $F(\infty)$ at infinity and $F(\infty) = -\frac{d}{c}$.

5 Krein's resolvent formula

To calculate the resolvent of the operator $\mathcal{A}^\gamma$ we will need the following corollary of the two previous Lemmas.

Lemma 5.1 *Let y be an arbitrary positive real number. Consider the Nevanlinna function*

$$G(\lambda) = \frac{1}{\gamma - q(\lambda)} = \frac{1}{\gamma - \langle \varphi, \frac{1+\lambda A}{A-\lambda} \frac{1}{A^2+1} \varphi \rangle_H}.$$

If $\varphi \in \mathcal{H}_{-1}(A)$, then the function satisfies the estimate

$$|G(x + iy)| \leq C_1(y)(1 + |x|) \tag{23}$$

for all negative $x < 0$ and a certain $C_1(y) > 0$. If $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$, then the function can be estimated by

$$|G(x + iy)| \leq C_2(y) \tag{24}$$

for all negative $x < 0$ and a certain positive $C_2(y) > 0$.

Proof The function q is a Stieltjes function, since the operator A is positive. Lemma 4.1 implies that the estimate (23) holds for all $\varphi \in \mathcal{H}_{-2}(A)$.

Consider the case $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$. We are going to prove that the real part of $q(x + iy)$ tends to minus infinity when $x \to -\infty$. In fact the function q can be presented by the following integral

$$q(\lambda) = \int_0^\infty \frac{1 + \mu\lambda}{\mu - \lambda} d\rho(\lambda),$$

where the measure ρ is finite, i.e. $\int_0^\infty d\rho(\mu) < \infty$, but the integral $\int_0^\infty \mu d\rho(\mu) = \infty$ diverges.

The real part of $q(x+iy)$ is given by

$$\Re q(x+iy) = \int_0^\infty \frac{(\mu - x) - \mu y^2}{(\mu - x)^2 + y^2} d\rho(\mu) + \int_0^\infty \frac{x\mu(\mu - x)}{(\mu - x)^2 + y^2} d\rho(\mu). \tag{25}$$

The first integral in the last formula is bounded for negative values of x :

$$\begin{aligned} \left| \int_0^\infty \frac{(\mu - x) - \mu y^2}{(\mu - x)^2 + y^2} d\rho(\mu) \right| &\leq \int_0^\infty \frac{\mu - x}{(\mu - x)^2 + y^2} d\rho(\mu) + \int_0^\infty \frac{\mu y^2}{\mu^2 + y^2} d\rho(\mu) \\ &\leq \int_0^\infty \frac{1}{2y} d\rho(\mu) + \int_0^\infty \frac{y}{2} d\rho(\mu). \end{aligned}$$

The second integral in (25) is negative and can be estimated as

$$\begin{aligned} \left| \int_0^\infty \frac{x\mu(\mu - x)}{(\mu - x)^2 + y^2} d\rho(\mu) \right| &\geq \frac{x^2}{x^2 + y^2} \int_0^\infty \frac{|x|\mu}{\mu - x} d\rho(\mu) \\ &\geq \frac{x^2}{x^2 + y^2} \frac{1}{2} \int_0^{|x|} \mu d\rho(\mu). \end{aligned}$$

The last integral tends to infinity when $x \to -\infty$. It follows that

$$\lim_{x \to -\infty} G(x+iy) = \lim_{x \to -\infty} \frac{1}{\gamma - q(x+iy)} = 0.$$

Therefore the continuous function $G(x+iy)$ is uniformly bounded on the interval $x < 0$, i.e. the estimate (24) holds. The lemma is proven.

$\square$

We remark that if $y = 0$ then the estimates (23) and (24) hold for $x \in (-\infty, A_1)$, where A_1 is a certain real constant.

Theorem 5.1 *The resolvent of the operator $\mathcal{A}^\gamma = B \otimes I_H + I_K \otimes A^\gamma$ at a certain point $\lambda, \Im\lambda \neq 0$, is given by the formula*

$$\frac{1}{\mathcal{A}^\gamma - \lambda} = \frac{1}{\mathcal{A} - \lambda} - \frac{1}{\mathcal{A} - \lambda} \left(\frac{1}{\gamma + q(\lambda - B)} \langle \frac{1}{\mathcal{A} - \bar{\lambda}} \varphi, \cdot \rangle_H \otimes \varphi \right) \tag{26}$$

where $q(\lambda - B) = \langle \varphi, \dfrac{1 + (\lambda - B) \otimes A}{\mathcal{A} - \lambda} \dfrac{1}{A^2 + 1} \varphi \rangle_h$.

Comment Let us discuss formula (26) first. Let $\varphi \in \mathcal{H}_{-1}(a)$. Then for any $f \in \mathcal{H}$ the following inclusion holds

$$\langle \frac{1}{\mathcal{A} - \bar{\lambda}} \varphi, f \rangle_h = \langle \frac{1}{\sqrt{|a| + 1}} \varphi, \frac{\sqrt{|a| + 1}}{\mathcal{A} - \lambda} f \rangle_h \in \mathcal{H}_1(B).$$

The function $\frac{1}{\gamma - q(x+iy)}$ satisfies the estimate (23) and it follows that the operator

$$\frac{1}{\gamma - q(\lambda - B)}$$

maps $\mathcal{H}_1(B)$ onto $\mathcal{H}_{-1}(B)$. This implies that

$$\frac{1}{\gamma - q(\lambda - B)} \left\langle \frac{1}{\mathcal{A} - \bar{\lambda}} \varphi, f \right\rangle_h \otimes \varphi \in \mathcal{H}_{-2}(A). \tag{27}$$

This means that formula (26) defines a bounded operator in the Hilbert space $\mathcal{H}$ for $\varphi \in \mathcal{H}_{-1}(a)$.

Consider now the case $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$. For any $f \in \mathcal{H}$ the vector $\langle \frac{1}{\mathcal{A}-\bar{\lambda}} \varphi, f\rangle_h = \langle \frac{1}{A+i}\varphi, (A+i)\frac{1}{\mathcal{A}-\lambda} f\rangle_H$ belongs to the space K. The function $\frac{1}{\gamma - q(x+iy)}$ is bounded for negative x (see (24)) and the operator $\frac{1}{\gamma - q(\lambda - B)}$ is bounded in K. This implies that condition (27) holds. Therefore formula (26) defines a bounded operator acting in the Hilbert space for any $\varphi \in \mathcal{H}_{-2}(A)$.

Proof of Theorem 5.1 Let us denote by $\mathcal{F}_B$ the operator of spectral transformation for B – the linear operator which maps the operator B into the operator of multiplication by the independent real variable x. Then the resolvent on a dense set can easily be calculated as follows

$$\frac{1}{\mathcal{A}^\gamma - \lambda} f = \psi$$

$$\begin{aligned}
\Rightarrow f &= (\mathcal{A}^\gamma - \lambda)\psi \\
\Rightarrow (\mathcal{F}_B f)(x) &= (x + A^\gamma - \lambda)(\mathcal{F}_B \psi)(x) \\
\Rightarrow (\mathcal{F}_B \psi)(x) &= \frac{1}{A^\gamma - (\lambda - x)} (\mathcal{F}_B f)(x) \\
&= \frac{1}{A - (\lambda - x)} (\mathcal{F}_B f)(x)
\end{aligned}$$

$$- \left(\frac{1}{\gamma + q(\lambda - x)} \left\langle \frac{1}{A - (\bar{\lambda} - x)} \varphi, (\mathcal{F}_B f)(x) \right\rangle_H \right)$$
$$\otimes \frac{1}{A - (\lambda - x)} \varphi$$

$$\Rightarrow \psi = \frac{1}{\mathcal{A} - \lambda} f - \frac{1}{\mathcal{A} - \lambda} \left(\left\{ \frac{1}{\gamma + q(\lambda - B)} \langle \frac{1}{A - \bar{\lambda}} \varphi, f\rangle_H \right\} \otimes \varphi \right).$$

We have supposed that $\psi \in \mathcal{L}^\gamma$, where $\mathcal{L}^\gamma$ is an algebraic tensor product of Dom (B) and Dom (a^γ). The operator $\mathcal{A}^\gamma$ is essentially self–adjoint on this domain and this completes the proof of the theorem.

□

6 Cluster interaction without separation of the center of mass motion

The resolvent of the operator $\mathcal{A}^\gamma$ has been calculated using the tensor decomposition. The operator $\mathcal{A}^\gamma$ is a self–adjoint extension of the symmetric operator

$$\mathcal{A}^0 = B \otimes I_h + I_K \otimes A^0$$

with infinite deficiency indices. Consider the annulating set $\mathbf{\Phi}_{\mathrm{reg}}$ of regular functionals for the operator $\mathcal{A}^0$ defined as follows:

- if $\varphi \in \mathcal{H}_{-1}(A)$ then
$$\mathbf{\Phi}_{\mathrm{reg}} = \{\Phi : \Phi = \rho(\Phi) \otimes \varphi,\ \rho(\Phi) \in \mathcal{H}_{-1}(B)\},$$
- if $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$ then
$$\mathbf{\Phi}_{\mathrm{reg}} = \{\Phi : \Phi = \rho(\Phi) \otimes \varphi,\ \rho(\Phi) \in K = \mathcal{H}_0(B)\}.$$

Let us consider the corresponding subspace of regular elements from the domain of the adjoint operator $\mathcal{A}^{0*}$:

- if $\varphi \in \mathcal{H}_{-1}(A)$ then
$\mathrm{Dom}_{\mathrm{reg}}(\mathcal{A}^{0*}) = \{\psi : \psi = \tilde{\psi} + \frac{\mathcal{A}}{\mathcal{A}^2+1}\rho(\psi) \otimes \varphi, \tilde{\psi} \in \mathrm{Dom}\,(\mathcal{A}), \rho(\psi) \in \mathcal{H}_{-1}(B)\}$;
- if $\varphi \in \mathcal{H}_{-2}(A) \setminus \mathcal{H}_{-1}(A)$ then
$\mathrm{Dom}_{\mathrm{reg}}(\mathcal{A}^{0*}) = \{\psi : \psi = \tilde{\psi} + \frac{\mathcal{A}}{\mathcal{A}^2+1}\rho(\psi) \otimes \varphi, \tilde{\psi} \in \mathrm{Dom}\,(\mathcal{A}), \rho(\psi) \in K = \mathcal{H}_0(B)\}$.

The boundary form of the adjoint operator calculated on the regular elements is given by

$U, V \in \mathrm{Dom}_{\mathrm{reg}}(\mathcal{A}^{0*}) \Rightarrow$
$$\begin{aligned} \langle U, \mathcal{A}^{0*}V\rangle - \langle \mathcal{A}^{0*}U, V\rangle &= \langle \rho(U) \otimes \varphi, \tilde{V}\rangle - \langle \tilde{U}, \rho(V) \otimes \varphi\rangle \\ &= \Big\langle \rho(U), \langle \varphi, \tilde{V}\rangle_h \Big\rangle_K - \Big\langle \langle \varphi, \tilde{U}\rangle_h, \rho(V) \Big\rangle_K . \end{aligned} \tag{28}$$

A symmetric extension of the operator $\mathcal{A}^0$ can be defined in terms of any symmetric operator Γ by restricting the operator $\mathcal{A}^{0*}$ to the domain of functions from $\mathrm{Dom}_{\mathrm{reg}}(\mathcal{A}^{0*})$ satisfying the boundary condition
$$-\langle \varphi, \hat{U}\rangle = \Gamma \rho(U). \tag{29}$$

In order to obtain the perturbed operator possessing the tensor decomposition (8) let us consider the symmetric operator $\mathcal{A}^\Gamma$ determined by the following boundary operator
$$\Gamma = \gamma - B \left\langle \varphi, \frac{A\mathcal{A} - 1}{(\mathcal{A}^2+1)(A^2+1)}\varphi \right\rangle_H . \tag{30}$$
The operator $\langle \varphi, \frac{A\mathcal{A}-1}{(\mathcal{A}^2+1)(A^2+1)}\varphi\rangle_H$ is a bounded self–adjoint operator in K commuting with the operator B. The norm of this operator is less than or equal to 1. Therefore the operator Γ is essentially self–adjoint on the domain $\mathrm{Dom}\,(B)$ of the operator B. We are going to keep the same notation Γ for the corresponding self–adjoint operator.

Let us find an expression for the resolvent of the operator $\mathcal{A}^\Gamma$. Consider an arbitrary $f \in \mathcal{H}$ and suppose that $\psi = \tilde{\psi} + \frac{\mathcal{A}}{\mathcal{A}^2+1}\rho(\psi) \otimes \varphi = \frac{1}{\mathcal{A}^\Gamma - \lambda} f$. Then the function ψ satisfies the following equation
$$(\mathcal{A} - \lambda)\tilde{\psi} - \frac{1 + \lambda\mathcal{A}}{\mathcal{A}^2+1}\left(\rho(\psi) \otimes \varphi\right) = f$$

and the boundary conditions (29). Applying the resolvent of the original operator $\mathcal{A}$ to the previous equation we get

$$\tilde{\psi} - \frac{1+\lambda\mathcal{A}}{\mathcal{A}-\lambda}\frac{1}{\mathcal{A}^2+1}\rho(\psi)\otimes\varphi = \frac{1}{\mathcal{A}-\lambda}f$$

$$\Rightarrow \left(\Gamma + \langle\varphi, \frac{1+\lambda\mathcal{A}}{\mathcal{A}-\lambda}\frac{1}{\mathcal{A}^2+1}\varphi\rangle_H\right)\rho(\psi) = -\langle\varphi, \frac{1}{\mathcal{A}-\lambda}f\rangle_H. \tag{31}$$

This equation can be solved and the function $\rho(\psi)$ can be calculated if the operator $\Gamma + \langle\varphi, \frac{1+\lambda\mathcal{A}}{\mathcal{A}-\lambda}\frac{1}{\mathcal{A}^2+1}\varphi\rangle_H$ is invertible. The operator can be simplified as follows taking into account equality (30)

$$\begin{aligned}
&\Gamma + \langle\varphi, \frac{1+\lambda\mathcal{A}}{\mathcal{A}-\lambda}\frac{1}{\mathcal{A}^2+1}\varphi\rangle_h \\
&= \gamma - B\left\langle\varphi, \frac{A\mathcal{A}-1}{(\mathcal{A}^2+1)(A^2+1)}\varphi\right\rangle_h - \left\langle\varphi, \frac{1+\lambda\mathcal{A}}{\mathcal{A}-\lambda}\frac{1}{\mathcal{A}^2+1}\varphi\right\rangle_H \\
&= \gamma + \left\langle\varphi, \frac{1+(\lambda-B)\otimes A}{\mathcal{A}-\lambda}\frac{1}{A^2+1}\varphi\right\rangle_H,
\end{aligned} \tag{32}$$

which holds on functions from $\mathrm{Dom}\, B$. Since $\varphi \in \mathcal{H}_{-2}(A)$ and the operators A and B are positive the following inclusion holds $\langle\varphi, \frac{1}{\mathcal{A}-\lambda}f\rangle_H \in K$. The comment after Theorem 5.1 shows that the operator $\Gamma + \langle\varphi, \frac{1+\lambda\mathcal{A}}{\mathcal{A}-\lambda}\frac{1}{\mathcal{A}^2+1}\varphi\rangle_H$ is invertible in K. Therefore there exists $\rho(\psi) \in \mathrm{Dom}\,(\Gamma) \subset K$ which satisfies equation (31).

The component $\tilde{\psi}$ of the function ψ can be calculated using the formula

$$\tilde{\psi} = \frac{1}{\mathcal{A}-\lambda}f + \frac{1}{\mathcal{A}-\lambda}\frac{1+\lambda\mathcal{A}}{\mathcal{A}^2+1}\rho(\psi)\otimes\varphi.$$

Thus the function ψ is given by

$$\psi = \frac{1}{\mathcal{A}-\lambda}f - \frac{1}{\mathcal{A}-\lambda}\left(\left\{\frac{1}{\Gamma + \langle\varphi, \frac{1+\lambda\mathcal{A}}{\mathcal{A}-\lambda}\frac{1}{\mathcal{A}^2+1}\varphi\rangle_h}\left\langle\varphi, \frac{1}{\mathcal{A}-\lambda}f\right\rangle_H\right\}\otimes\varphi\right).$$

The resolvent of the operator $\mathcal{A}^\Gamma$ coincides with the resolvent of the self–adjoint operator $\mathcal{A}^\gamma$. This implies that the operator $\mathcal{A}^\Gamma$ is in fact self–adjoint even if it has been first defined only on the regular elements.

Thus the following theorem has been proven.

Theorem 6.1 *The operator $\mathcal{A}^\Gamma$ which is the restriction of the operator $\mathcal{A}^{0*}$ to the set of regular elements*

$$\psi = \tilde{\psi} + \frac{\mathcal{A}}{\mathcal{A}^2+1}\left(\rho(\psi)\otimes\varphi\right) \in \mathrm{Dom}_{\mathrm{reg}}(\mathcal{A}^{0*}) \tag{33}$$

satisfying the boundary condition

$$\langle\varphi, \tilde{\psi}\rangle_H = \Gamma\rho(\psi) \tag{34}$$

is self–adjoint and its resolvent is given by

$$\frac{1}{\mathcal{A}^\Gamma-\lambda}=\frac{1}{\mathcal{A}-\lambda}-\frac{1}{\mathcal{A}-\lambda}\left(\left\{\frac{1}{\gamma+\langle\varphi,\frac{1+(\lambda-B)\otimes a}{\mathcal{A}-\lambda}\frac{1}{a^2+1}\varphi\rangle_H}\langle\varphi,\frac{1}{\mathcal{A}-\lambda}\cdot\rangle_h\right\}\otimes\varphi\right) \tag{35}$$

for any λ; $\Im\lambda\neq 0$.

Comment In the course of the proof of the previous theorem we have shown that the density $\rho(\psi)$ is an element from the domain of the operator Γ. It is possible to prove that the restriction of the operator $\mathcal{A}^\Gamma$ to the domain of functions possessing the representation (33), boundary conditions (34) and having $\rho(\psi)\in \mathrm{Dom}\,(B)$ is essentially self–adjoint.

Suppose that $\varphi\in\mathcal{H}_{-1}(A)$. Then the boundary conditions (34) can be simplified as follows. Consider the scalar product $\langle\varphi,\psi\rangle_H$, where ψ is any function from the domain of the operator $\mathcal{A}^\Gamma$. Then the following equalities hold

$$\begin{aligned}
\langle\varphi,\psi\rangle_H &= \left\langle\varphi,\tilde\psi\right\rangle_H+\left\langle\varphi,\frac{\mathcal{A}}{\mathcal{A}^2+1}\rho\otimes\varphi\right\rangle_H \\
&= -\Gamma\rho(\psi)+\left\langle\varphi,\frac{\mathcal{A}}{\mathcal{A}^2+1}\rho\otimes\varphi\right\rangle_H \\
&= -\gamma\rho(\psi)+\left\langle\varphi,\left(\frac{B(\mathcal{A}A-1)}{(\mathcal{A}^2+1)(A^2+1)}+\frac{\mathcal{A}}{\mathcal{A}^2+1}\right)\rho\otimes\varphi\right\rangle_h \\
&= -\gamma\rho(\psi)+\left\langle\varphi,\frac{A}{A^2+1}\varphi\right\rangle_h\rho(\psi) \\
&= (-\gamma+c)\,\rho(\psi),
\end{aligned} \tag{36}$$

where we have used the fact that the function ψ satisfies boundary condition (34). Taking into account (13) the latter condition can be written as

$$-\alpha\langle\varphi,\psi\rangle_H=\rho(\psi).$$

One can define the operator $\mathcal{A}^\Gamma$ using this boundary condition, but this condition cannot be generalized to the case of $\mathcal{H}_{-2}$ interactions, since the scalar product $\left\langle\varphi,\frac{\mathcal{A}}{\mathcal{A}^2+1}\rho\otimes\varphi\right\rangle_H$ does not necessarily define a function from K in this case.

References

[1] S.Albeverio, F.Gesztesy, R.Høegh-Krohn, and H.Holden, *Solvable problems in quantum mechanics*, Springer, Berlin, 1988 (Russian edition, Mir, Moscow, 1990).

[2] S.Albeverio, R.Høegh-Krohn, and L.Streit, Energy forms, Hamiltonians, and distorted Brownian paths, *J. Math. Phys.*, 18:907–917, (1977).

[3] S.Albeverio and V.Koshmanenko, Singular rank one perturbations of selfadjoint operators and Krein theory of selfadjoint extensions, to appear in *Potential Analysis.*

[4] S.Albeverio and P.Kurasov, Rank one perturbations, approximations, and self–adjoint extensions, *J.Funct.Anal.*, 148:152–169 (1997).

[5] S.Albeverio and P.Kurasov, Rank one perturbations of not semibounded operators, *Int. Eq. Oper. Theory*, 27:379–400 (1997).

[6] S.Albeverio and P.Kurasov, Finite rank perturbations and distribution theory, *Proc. Amer. Math. Soc.*, 127:1151–1161 (1999).

[7] S.Albeverio and P.Kurasov, *Singular perturbations of differential operators and exactly solvable Schrödinger type operators*, Cambridge Univ. Press, to appear in 1999.

[8] S.Albeverio and K.Makarov, Attractors in a model related to the three body quantum problem, *C.R.Acad. Sci. Paris Ser. I Math.*, 323:693–698 (1996).

[9] S.Albeverio and K.Makarov, Nontrivial attractors in a model related to the three-body quantum problem, *Acta Appl. Math.*, 48:113–184 (1997).

[10] F.A.Berezin and L.D.Faddeev, Remark on the Schrödinger equation with singular potential, *Dokl. Akad. NAUK SSSR*, 137:1011-1014 (1961).

[11] G.F. Dell'Antonio, R.Figari and A.Teta, Hamiltonians for systems of N particles interacting through point interactions, *Ann. Inst. H.Poincaré, Phys. Theor.*, 60:253–290 (1994).

[12] Yu.N.Demkov and V.N.Ostrovsky, *Zero-range Potentials and their Applications in Atomic Physics*, Plenum, New York, 1988.

[13] L.D.Faddeev and S.P.Merkuriev, *Quantum Scattering theory for Several Particle Systems*, Kluwer, Dordrecht, 1993.

[14] M.Gaudin, *La Fonction d'onde de Bethe*, Masson, 1983.

[15] F.Gesztesy and B.Simon, Rank one perturbations at infinite coupling, *J.Funct. Anal.*, 128:245–252 (1995).

[16] A.Kiselev and B.Simon, Rank one perturbations with infinitesimal coupling, *J. Funct. Anal.*, 130:345–356 (1995).

[17] Yu.A.Kuperin, K.A.Makarov, S.P.Merkuriev, A.K.Motovilov, and B.S.Pavlov, The quantum problem of several particles with internal structure. I. The two–body problem, *Theor. Mat. Fiz.*, 75:431–444, 1988.

[18] Yu.A.Kuperin, K.A.Makarov, S.P.Merkuriev, A.K.Motovilov, and B.S.Pavlov, The quantum problem of several particles with internal structure. II. The three–body problem, *Theor. Mat. Fiz.*, 76:242–260, 1988.

[19] Yu.A.Kuperin, K.A.Makarov, S.P.Merkuriev, A.K.Motovilov, and B.S.Pavlov, Extended Hilbert space approach to few–body problems, *J. Math. Phys.*, 31:1681–1690, 1990.

[20] P.Kurasov, Energy dependent boundary conditions and the few–body scattering problem, *Rev. Math. Phys.*, 9:853–906 (1997).

[21] P.Kurasov and J.Boman, Finite rank singular perturbations and distributions with discontinuous test functions, *Proc. Amer. Math. Soc.*, 126:1673–1683 (1998).

[22] P.Kurasov and B.Pavlov, Few-body Krein's formula, to be published in Proc. of Krein's conference, Birkhauser.

[23] K.Makarov, Semiboundedness of the energy operator of a system of three particles with paired interactions of δ-function type, *Algebra i Analiz*, 4:155–171 (1992).

[24] R.A.Minlos and L.D.Faddeev, Comment on the problem of three particles with point interactions, *Soviet Physics JEPT*, 14:1315–1316 (1962).

[25] R.A.Minlos and L.D.Faddeev, On the point interaction for a three-particle system in quantum mechanics, *Soviet Physics Dokl.*, 6:1072–1074 (1962).

[26] B.S.Pavlov, The theory of extensions and explicitly solvable models, *Uspekhi Mat. Nauk*, 136:99–131 (1987).

[27] B.S.Pavlov, Boundary conditions on thin manifolds and the semiboundedness of the three–body Schrödinger operator with point interactions, *Mat. Sb. (N.S.)*, 136:163–177 (1988).

[28] B.Simon, Spectral analysis of rank one perturbations and applications, in *Mathematical Quantum Theory. II. Schrödinger Operators (Vancouver, BC, 1993)*, volume 8 of CRM Proc. Lecture Notes, Pages 109–149, Amer. Math. Soc., Providence, RI, 1995.

[29] G.V.Skorniakov and K.A.Ter-Martirosian, Three body problem for short range forces. I. Scattering of low energy neutrons by deutrons, *Soviet Phys. JEPT*, 4:648–661 (1957).

[30] L.E.Thomas, Multiparticle Schrödinger Hamiltonians with point interactions, *Phys. Rev. D*, 30:1233–1237 (1984).