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Pseudo-Differential Operators with Point Interactions

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Abstract. Point interactions for pseudo-differential operators are studied. Necessary and sufficient conditions for a pseudo-differential operator to have nontrivial point perturbations are given. The results are applied to the construction of relativistic spin zero Hamiltonians with point interactions.

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1. Introduction

The extension theory for symmetric operators has been used to study exactly solvable operators with the complicated structure of the spectrum given by 'point interactions'. Such interactions has been studied up to now only as perturbations of differential operators (see, e.g., [2, 8]). In this Letter, we consider point interactions for pseudo-differential operators. The most complete review of the modern theory of pseudo-differential operators is given by L. Hörmander [12]. The spectral theory for self-adjoint pseudo-differential operators has been developed in [19]. Perturbation theory for self-adjoint operators has been used there to study the Hamiltonian of a relativistic spin-zero particle. Self-adjoint pseudo-differential operators have also been studied as generators of symmetric Markov semigroups, see, e.g., [11, 15].

Our aim is to study pseudo-differential operators with point interactions. Pseudo-differential operators with the interaction introduced on certain manifolds in the configuration space have been already studied by B. Pavlov [17] in an application to a few-body scattering problem. We concentrate our attention to the case where the perturbation has finite rank. Moreover, we suppose that the perturbation is supported by one point. Necessary and sufficient conditions for a pseudo-differential operator to have nontrivial point interactions are given. The resolvent of the pseudo-differential operator with point interaction is calculated. We apply the methods we developed to study point interaction perturbations of a relativistic spin zero Hamiltonian which has recently been intensively studied [13, 18, 14, 3]. The spectrum and the scattering matrix for such a Hamiltonian are calculated.

2. Point Interactions for Pseudo-Differential Operators

Consider the set $S^m(\mathbb{R}^n \times \mathbb{R}^n), m \in \mathbb{R}$ of all functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying the following estimate:

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leqslant c_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}; \quad x,\xi \in \mathbb{R}^n, \tag{1}$$

(for some constants $c_{\alpha,\beta} \geqslant 0$). We call $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ a set of symbols of order m. The Fourier transform $\mathcal{U}$ is defined for every function $f \in L_1(\mathbb{R}^n)$ by the following formula:

$$(\mathcal{U}f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \lim_{R\to\infty} \int_{|x|<\mathbb{R}} f(x)\, \mathrm{e}^{-i\langle \xi, x\rangle}\, \mathrm{d}x \tag{2}$$

and can be extended to the whole $L_2(\mathbb{R}^n)$ as an unitary operator. Let $a \in S^m$, then the pseudo-differential operator A_S is defined by the following formula for all functions from the Schwartz space $\mathcal{S}$:

$$(A_S f)(x) = \frac{1}{(2\pi)^{n/2}} \int \mathrm{e}^{i\langle x,\xi\rangle} a(x,\xi) \hat{f}(\xi)\, \mathrm{d}\xi. \tag{3}$$

We are going to study pseudo-differential operators with real-valued symbols a depending only on the second variable: $a(x,\xi) \equiv a(\xi)$. The self-adjoint operator A determined by the symbol a is then defined by the formula

$$Af = \mathcal{U}^{-1}(a(\mathcal{U}f)) \tag{4}$$

on the domain

$$\mathrm{Dom}(A) = \{f \in L_2(\mathbb{R}^n) \colon a(\mathcal{U}f) \in L_2(\mathbb{R}^n)\}.$$

The operator A is then the closure of the symmetric operator A_S.

We are going to study point interactions for the operator A. An operator $\tilde{A}$ is a perturbation of the operator A at a certain point $x_0 \in \mathbb{R}^n$ if it is equal to a self-adjoint extension of the operator $A^0 = A|_{\{\psi \in C_0^\infty(\mathbb{R}^n \setminus \{x_0\})\}}$. The operator $\tilde{A}$ does not coincide with the operator A only if the operator A^0 has nontrivial deficiency indices. This occurs only if there exists a distribution $f \in S'(\mathbb{R}^n)$ with support at the origin which is a bounded linear functional on the domain of the operator A equipped with the norm equal to the graph norm of the operator A. The set of distributions with the support at one point is formed by the delta function and its derivatives. Therefore, such a distribution f exists only if the delta function is an element from the space $\mathcal{H}_{-2}(|A|)$ from the standard scale of Hilbert spaces associated with the positive self-adjoint operator $|A|$, i.e. $\delta \in \mathcal{H}_{-2}(|A|)$. The following lemma has first been proven in [17]:

LEMMA 2.1. *If*

$$\int_{\mathbb{R}^n} \frac{1}{|a(\xi)|^2 + 1} \, \mathrm{d}\xi = \infty,$$

then the deficiency indices of A^0 are equal to zero. But if

$$\int_{\mathbb{R}^n} \frac{1}{|a(\xi)|^2 + 1} \, \mathrm{d}\xi < \infty,$$

then the deficiency indices of A^0 are not trivial.

The following lemma gives sufficient conditions for the operator A^0 to have nontrivial deficiency indices.

LEMMA 2.2. *Let R, c, m' be certain positive real constants. Let a be a real symbol of order m: $a \in S^m(\mathbb{R}^n)$, satisfying the following inequality:*

$$|a(\xi)| \geqslant c(1 + |\xi|)^{m'} \tag{5}$$

for all $|\xi| > R$. If $n < 2m'$, then the operator A^0 has nontrivial deficiency indices.

Proof. The proof presented here does not use the previous lemma. Without loss of generality, we can consider $x_0 = 0$. To prove that the operator A^0 is not essentially self-adjoint, it is enough to present at least one deficiency element g_λ corresponding to a point $\lambda, \Im\lambda \neq 0$. One of these elements is given by $g_\lambda = \mathcal{U}^{-1}(a - \lambda)^{-1}$. If $2m' > n$ then the inequality (5) implies the following estimate

$$\left|\frac{1}{a(\xi) - \lambda}\right| \leqslant \begin{cases} \dfrac{1}{|\Im\lambda|}, & |\xi| < R \\ \dfrac{1}{c}(1 + |\xi|)^{-m'}, & |\xi| \geqslant R \end{cases},$$

and it follows that $(a - \lambda)^{-1} \in L_2(\mathbb{R}^n)$. Consider an arbitrary $\varphi \in \mathrm{Dom}(A^0)$. The following calculations prove that the function g_λ is a deficiency element, since $\delta(\varphi) = 0$ for every $\varphi \in \mathrm{Dom}(A^0)$:

$$\begin{aligned} \langle A^{0*} g_\lambda, \varphi \rangle &= \langle g_\lambda, A^0 \varphi \rangle = \langle g_\lambda, \mathcal{U}^{-1}(a(\mathcal{U}\varphi)) \rangle \\ &= \left\langle \frac{a}{a - \lambda}, \mathcal{U}\varphi \right\rangle = \left\langle 1 + \frac{\lambda}{a - \lambda}, \mathcal{U}\varphi \right\rangle \\ &= \sqrt{2\pi}\,\delta(\varphi) + \langle \lambda g_\lambda, \varphi \rangle = \lambda \langle g_\lambda, \varphi \rangle. \end{aligned}$$

(A^{0*} is the distributional adjoint, or formal adjoint, of A^0.) The lemma is proven. □

We have calculated the deficiency elements for the operator A^0 but may be not all of them, since the dimension of the deficiency subspace can be larger than one. The deficiency indices of the operator A^0 can be calculated according to the following Lemma:

THEOREM 2.1. *Let $n = 1$ and let A^0 be the restriction of the self-adjoint pseudo-differential operator A determined by the symbol a to the domain $C_0^\infty(\mathbb{R}\backslash\{0\})$. The deficiency indices of the operator A^0 are equal to $(k+1, k+1)$, where $k \in \mathbb{N}$ is the maximal natural number such that the integral $\int_{\mathbb{R}} \xi^{2k}/(|a(\xi)|^2+1)\,\mathrm{d}\xi$ is finite.*

Proof. If the integral $\int_{\mathbb{R}} \xi^{2k}/(|a(\xi)|^2+1)\,\mathrm{d}\xi$ converges, then the following functions are deficiency elements for the operator A^0 at point $\lambda, \Im\lambda \neq 0$, since the symbol a is by assumption a real function (i.e. the operator A is self-adjoint): $g_\lambda^i = \mathcal{U}^{-1}\xi^i/(a(\xi)-\lambda), i = 0, 1, 2, \ldots, k$. The proof of the latter statement is similar to the one of Lemma 2.2. It is obvious that the functions g_λ^i are linear independent. Therefore, to prove that the deficiency indices of the operator A^0 are equal to $(k+1, k+1)$, we have to show that every deficiency element g_λ can be represented by a linear combination of the elements g_λ^i. Suppose that g_λ is a certain deficiency element. Then the distribution $f = \mathcal{U}^{-1}(a-\lambda)\mathcal{U}g_\lambda$ vanishes on all functions from $C_0^\infty(\mathbb{R}\backslash\{0\})$ and therefore can be represented by a linear combination of the delta function and its derivatives: $f = \Sigma_{i=0}^K f_i \delta^{(i)}$, where K is a certain natural number and $f_i \in \mathbb{C}$. This implies that

$$(\mathcal{U}g_\lambda)(\xi) = \frac{i\Sigma_{i=0}^K f_i \xi^i}{(\sqrt{2\pi})(a(\xi)-\lambda)}.$$

The function on the right-hand side of the latter equality belongs to the Hilbert space if and only if $K \leqslant k$. This implies that the distribution f is equal to the linear combination of the delta function and its first k derivatives and therefore the function g_λ is equal to the linear combination of the deficiency elements g_λ^i. The lemma is proven. □

COROLLARY 2.1. *Let R, c, C be certain positive real constants. Let a be a symbol of order m such that the following estimates hold*

$$c(1+|\xi|)^m \leqslant |a(\xi)| \leqslant C(1+|\xi|)^m$$

for all $|\xi| > R$. Then the operator A^0 has deficiency indices $(\lim_{\varepsilon\downarrow 0}[m+\frac{1}{2}-\varepsilon], \lim_{\varepsilon\downarrow 0}[m+\frac{1}{2}-\varepsilon])$ where $[\,\cdot\,]$ denotes the integer part of a real number.

In what follows, we consider the case where the restricted operator has deficiency indices (1, 1). We denote by A^{00} the restriction of the operator A to the domain

$\mathrm{Dom}(A^{00}) = \{\varphi \in \mathrm{Dom}(A) \colon \varphi(0) = 0\}$. If the operator A is self-adjoint and the operator A^0 has nontrivial deficiency indices, then the operator A^{00} is symmetric with the deficiency indices (1, 1). The deficiency elements for the operator A^{00} have been constructed in the proof of Lemma 2.2. The resolvent identity implies the following equality for the deficiency elements:

$$g_\mu - g_\lambda = (\mu - \lambda) R_\lambda g_\mu, \tag{6}$$

where R_λ is the resolvent of the original operator at the point $\lambda, \Im\lambda \neq 0, \Im\mu \neq 0$.

All self-adjoint extensions of the operator A^{00} can be described using the von Neumann extension theory for symmetric operators. The adjoint operator A^{00*} has the following domain:

$$\mathrm{Dom}(A^{00*}) = \mathrm{Dom}(A^{00}) \dotplus \{b_+ g_z + b_- g_{\bar z} \colon b_\pm \in \mathbb{C}\},$$

where z is a certain complex number with nontrivial imaginary part. Consider two functions $\psi, \varphi \in \mathrm{Dom}(A^{00*})$, possessing the standard representation

$$\varphi = \tilde\varphi + b_+(\varphi) g_z + b_-(\varphi) g_{\bar z}, \qquad \psi = \tilde\psi + b_+(\psi) g_z + b_-(\psi) g_{\bar z},$$

$\tilde\varphi, \tilde\psi \in \mathrm{Dom}(A^{00})$. The boundary form of the operator is given by

$$\begin{aligned} &\langle A^{00*}\varphi, \psi\rangle - \langle \varphi, A^{00*}\psi\rangle \\ &\quad = 2\Re z \langle g_z, g_z\rangle (b_+(\varphi)\bar b_+(\psi) - b_-(\varphi)\bar b_-(\psi)). \end{aligned}$$

The self-adjoint extensions A_θ of A^{00} are parameterized by the unimodular parameter $\mathrm{e}^{i\theta}, \theta \in [0, 2\pi]$. The domain of the operator A_θ is equal to $\mathrm{Dom}(A_\theta) = \mathrm{Dom}(A^{00}) \dotplus \{\alpha(\mathrm{e}^{i\theta} g_{\bar z} - g_z), \alpha \in \mathbb{C}\}$. The operator A_θ is self-adjoint. The operator A_0 coincides with the original operator A. The resolvent of the operator A_θ can be calculated using the Krein formula and the resolvent of the original operator A.

LEMMA 2.3. *The resolvent R_λ^θ of the operator A_θ is equal to*

$$R_\lambda^\theta f = R_\lambda f - \frac{\mathrm{e}^{i\theta} - 1}{\mathrm{e}^{i\theta}\langle g_\lambda, g_z\rangle + \langle g_\lambda, g_{\bar z}\rangle} \frac{1 - \frac{\langle g_{\bar\lambda}, g_z\rangle}{\langle g_{\bar\lambda}, g_{\bar\lambda}\rangle}}{\lambda - \bar z} \langle f, g_{\bar\lambda}\rangle g_\lambda, \tag{7}$$

where R_λ is the resolvent of the original operator A and $g_\lambda = \mathcal{U}^{-1}(a - \lambda)^{-1}$ is a deficiency element.

Proof. We are going to use the Krein formula for the resolvent of the perturbed operator $R_\lambda^\theta f = R_\lambda f - p(\lambda)\langle f, g_{\bar\lambda}\rangle g_\lambda$. The function $p(\lambda)$ can be calculated in the following way. The Hilbert space possesses the following orthogonal decomposition $\mathcal{R}(A^{00} - \lambda I) \oplus \mathcal{L}(g_{\bar\lambda})$ ($\mathcal{R}$ denoting the range and $\mathcal{L}$ denoting the linear hull). Therefore the projection of $R_\lambda f$ on the deficiency element g_z depends only on the projection $\langle f, g_{\bar\lambda}\rangle$:

$$\begin{aligned}\langle R_\lambda f, g_z\rangle &= \frac{\langle f, g_{\bar\lambda}\rangle}{\langle g_{\bar\lambda}, g_{\bar\lambda}\rangle}\langle R_\lambda g_{\bar\lambda}, g_z\rangle \\ &= \frac{\langle f, g_{\bar\lambda}\rangle}{\lambda - \bar z}\left(1 - \frac{\langle g_{\bar\lambda}, g_z\rangle}{\langle g_{\bar\lambda}, g_{\bar\lambda}\rangle}\right).\end{aligned}$$

The resolvent identity (6) has been used to derive the latter formula. Projecting $R_\lambda^\theta f$ onto the deficiency elements $g_z, g_{\bar z}$ we get the following equality:

$$\begin{aligned}&\mathrm{e}^{i\theta}(\langle R_\lambda f, g_z\rangle - p(\lambda)\langle f, g_{\bar\lambda}\rangle\langle g_\lambda, g_z\rangle) \\ &\quad = -\langle R_\lambda f, g_{\bar z}\rangle + p(\lambda)\langle f, g_{\bar\lambda}\rangle\langle g_\lambda, g_{\bar z}\rangle.\end{aligned}$$

It follows that

$$p(\lambda) = \frac{\mathrm{e}^{i\theta} - 1}{\mathrm{e}^{i\theta}\langle g_\lambda, g_z\rangle + \langle g_\lambda, g_{\bar z}\rangle}\frac{1 - \frac{\langle g_{\bar\lambda}, g_z\rangle}{\langle g_{\bar\lambda}, g_{\bar\lambda}\rangle}}{\lambda - \bar z}.$$

Finally, we get the Krein formula (7). □

The singularities of the perturbed resolvent coincide with the singularities of the free resolvent and the zeroes of the denominator of $p(\lambda)$. The perturbed resolvent has a removable singularity at the point $\lambda = \bar z$, where both the nominator and denominator vanish. The discrete spectrum of the perturbed operator can be situated only at the points where

$$\mathrm{e}^{i\theta}\langle g_\lambda, g_z\rangle + \langle g_\lambda, g_{\bar z}\rangle = 0 \Rightarrow \mathrm{e}^{i\theta} = -\frac{\langle g_\lambda, g_{\bar z}\rangle}{\langle g_\lambda, g_z\rangle}. \tag{8}$$

This condition means that the element g_λ belongs to the domain of the perturbed operator: $g_\lambda \in \mathrm{Dom}(A_\theta)$. We introduce a new real parameter $\gamma(\theta, z)$ related to the parameters θ and z as follows:

$$\gamma(\theta, z) = \frac{z - \mathrm{e}^{i\theta}\bar z}{1 + \mathrm{e}^{i\theta}}.$$

Then using the equality $\langle R_\lambda g_z, g_z\rangle = \langle R_\lambda g_{\bar z}, g_{\bar z}\rangle$, the condition (8) can be written as

$$\langle R_\lambda g_z, g_z\rangle = \frac{1}{\gamma - \lambda}\langle g_z, g_z\rangle. \tag{9}$$

The dispersion equation (9) has only real solutions. Consider the solutions of the dispersion equation (9), which are situated in the gaps of the spectrum of the original operator. The deficiency element considered at these points belongs to the Hilbert space. Moreover, these functions are elements from the domain of the perturbed operator. Therefore, such points λ are points of the discrete spectrum of the operator A_θ. Each gap in the spectrum of the original operator contains not more than one solution of the dispersion equation as can be seen using methods similar to the ones used in [5] for the corresponding problem for Schrödinger operators.

Remark. The self-adjoint operators in $L_2(\mathbb{R}^n)$ which are generators of symmetric Markov semigroups and are given by symbols $a(\xi)$ as in Corollary 2.1 belong to the Lévy–Khintchine class (this class has been studied, e.g., in [11]). They include A^0 of the form $A^0 = (-\Delta)^\beta$ for $0 < \beta < 1$. In the latter case, we have nonzero deficiency indices for $n = 1$.

3. Relativistic Spin Zero Hamiltonian

The approach we described in Section 2 will be applied in this section to the construction of a relativistic spin zero Hamiltonian with point interaction. The original operator H is defined by the following symbol of order 1:

$$a(\xi) = \sqrt{\xi^2 + M^2},$$

where $M > 0$ is the mass of the particle moving in $\mathbb{R}^n$, $\xi \in \mathbb{R}^n$. The domain of the original operator coincides with the Sobolev space $W_2^1(\mathbb{R}^n)$:

$$\mathrm{Dom}(H) = \{\psi \in L_2(\mathbb{R}^n)\colon \sqrt{\xi^2 + M^2}\hat{\psi}(\xi) \in L_2(\mathbb{R}^n)\} = W_2^1(\mathbb{R}^n).$$

Lemma 2.2 and the Sobolev imbedding theorem imply that the delta function belongs to the Hilbert space $\mathcal{H}_{-2}(H)$ if and only if $n = 1$. All functions from the domain of the operator are continuous in this case. Theorem 2.1 implies that point interactions for the operator H can be constructed if and only if the configuration space has dimension one, thus we consider only the case $n = 1$ in what follows. For simplicity only point interactions with support at the origin are considered.

The resolvent of the original operator can be calculated using the Fourier transform. The kernel $r_z(x-y)$ of the resolvent R_z is given by the following formula

$$r_z(x-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\xi(x-y)}}{\sqrt{\xi^2 + M^2} - z}\,\mathrm{d}\xi, \quad \Im z > 0.$$

The singularities of the resolvent are situated on the interval $z \in [M, +\infty)$. The following Theorem has been proven in [19].

THEOREM 3.1. *The free Hamiltonian, H, is self-adjoint; its restriction to $D_0 = C_0^\infty(\mathbb{R})$ is essentially self-adjoint and $\sigma_e(H) = \sigma(H) = [M, +\infty)$.*

The aim of this Letter is to construct all self-adjoint point interaction perturbations of the original operator H. Consider the operator H^0-restriction of the operator H to the domain $\mathrm{Dom}(H^0) = \{\varphi \in C_0^\infty(\mathbb{R}), \varphi(0) = 0\}$. The restriction is well defined, since every function from the domain of the original operator is continuous at the origin.

LEMMA 3.1. *The operator H^0 is symmetric with the deficiency indices $(1,1)$. The deficiency element g_z for every $z, \Im z > 0$ is given by*

$$g_z(x) = \frac{1}{\pi}\int_M^\infty \mathrm{d}t\, \mathrm{e}^{-t|x|}\frac{\sqrt{t^2-M^2}}{t^2-M^2+z^2}, \tag{10}$$

if $\Re z < 0$*;*

$$g_z(x) = i\frac{\mathrm{e}^{i\sqrt{z^2-M^2}x}}{\sqrt{1-\frac{M^2}{z^2}}} + \frac{1}{\pi}\int_M^\infty \mathrm{d}t\, \mathrm{e}^{-t|x|}\frac{\sqrt{t^2-M^2}}{t^2-M^2+z^2}, \tag{11}$$

if $\Re z > 0$.

Proof. Theorem 2.1 implies that the operator H^0 is a symmetric operator with the deficiency indices (1, 1) and the deficiency element is given by the following formula

$$g_z(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\mathrm{e}^{i\xi x}}{\sqrt{\xi^2+M^2}-z}\,\mathrm{d}\xi. \tag{12}$$

The integral representation (10) can be obtained in the following way. The square root $\sqrt{\xi^2+M^2}$ is uniquely defined on the complex plane with the cuts $(-i\infty, -iM] \cup [iM, +i\infty)$ by the condition $\Re\sqrt{\xi^2+M^2} \geqslant 0$. If we suppose that $\Re z < 0$, then the integrated function is meromorphic on the complex plane with the mentioned cuts. Let $x > 0$ then the integrated function vanishes exponentially in the upper half plane $\Im\xi > 0$. Then

$$\begin{aligned} g_z(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{i\xi x}}{\sqrt{\xi^2+M^2}-z}\,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\left(\int_{i\infty-0}^{iM} + \int_{iM}^{i\infty+0}\right)\frac{\mathrm{e}^{i\xi x}}{\sqrt{\xi^2+M^2}-z}\,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_M^\infty \mathrm{e}^{-tx}\frac{2\sqrt{t^2-M^2}}{t^2-M^2+z^2}\,\mathrm{d}t. \end{aligned}$$

Formula (10) follows now from the fact that the deficiency element $g_z(x)$ is an even function.

Consider now the case $\Re z > 0$. The function $1/(\sqrt{\xi^2+M^2}-z)$ has simple poles at the points $\xi = \pm\sqrt{z^2-M^2}$. To obtain the representation (11) we have to take into account the residue at one of the singular points. Since the residue is equal to $i(\mathrm{e}^{i\sqrt{z^2-M^2}x})/(\sqrt{1-(M^2/z^2)})$, we obtain formula (11). □

The latter lemma shows that the deficiency elements are exponentially decreasing functions if $\Im z \neq 0$. The asymptotics of the limit of the deficiency element on the interval $[M,\infty)$ contains the outgoing plane wave

$$g_{E+i0}(x) = \frac{i}{\sqrt{1-\frac{M^2}{E^2}}}\,\mathrm{e}^{i\sqrt{E^2-M^2}|x|} + \mathrm{O}(\mathrm{e}^{-M|x|}), \quad E \in (M,\infty). \tag{13}$$

The deficiency elements are continuous functions outside the origin with a possible singularity there.

LEMMA 3.2. *Let* $\psi \in \mathrm{Dom}(H^{0*})$, *then* ψ *is continuous outside the origin with the following asymptotic representation at the origin*

$$\psi(x) =_{x\to 0} \psi_- \ln x + \psi_0 + \mathrm{o}(1), \tag{14}$$

where $\psi_-, \psi_0 \in \mathbb{C}$.

Proof. The operator H is positive, thus every element $\psi \in \mathrm{Dom}(H^{0*})$ possesses the following representation:

$$\psi = \breve{\psi} + b(\psi)g_{-1}, \tag{15}$$

where $\breve{\psi} \in \mathrm{Dom}(H_0)$ and $b(\psi) \in \mathbb{C}$. Every fuction $\breve{\psi}$ is an element from the Sobolev space $W_2^1(\mathbb{R})$ and satisfies the asymptotic representation (14). To prove the lemma it is enough to show that the deficiency element g_{-1} has the asymptotic representation (14). The deficiency element is equal to the sum of two integrals

$$\begin{aligned} g_z(x) &= 2\int_M^\infty \mathrm{d}t\,\mathrm{e}^{-t|x|}\frac{\sqrt{t^2-M^2}}{t^2-M^2+z^2} \\ &= 2\int_M^\infty \mathrm{d}t\,\mathrm{e}^{-t|x|}\left(\frac{\sqrt{t^2-M^2}}{t^2-M^2+z^2}-\frac{1}{t}\right) + 2\int_M^\infty \mathrm{d}t\,\mathrm{e}^{-t|x|}\frac{1}{t} \\ &= J_1(x) + J_2(x). \end{aligned}$$

The first integral and its first derivative are uniformly bounded

$$|J_1(x)| \leqslant 2\int_M^\infty \mathrm{d}t\left|\frac{\sqrt{t^2-M^2}}{t^2-M^2+z^2}-\frac{1}{t}\right| \leqslant C\int_M^\infty \frac{1}{t^3} < \infty;$$

$$\left|\frac{\mathrm{d}J_1}{\mathrm{d}x}\right| \leqslant 2\int_m^\infty t\,\mathrm{d}t\left|\frac{\sqrt{t^2-m^2}}{t^2-m^2+z^2}-\frac{1}{t}\right| \leqslant C\int_m^\infty \frac{1}{t^2} < \infty,$$

where C is a certain positive constant. It follows that $J_1(x)$ satisfies the representation (14). The second integral can be estimated as follows

$$\begin{aligned} J_2(x) &= 2\int_M^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-t|x|}}{t} = 2\int_{M|x|}^\infty \mathrm{d}\tau\,\frac{\mathrm{e}^{-\tau}}{\tau} \\ &= 2\int_{M|x|}^1 \mathrm{d}\tau\,\frac{\mathrm{e}^{-\tau}}{\tau} + 2\int_1^\infty \mathrm{d}\tau\,\frac{\mathrm{e}^{-\tau}}{\tau} \\ &= -2\ln(M|x|)\,\mathrm{e}^{-M|x|} + 2\int_{M|x|}^1 \mathrm{d}\tau\,\ln\tau\,\mathrm{e}^{-\tau} + 2\int_1^\infty \frac{\mathrm{e}^{-\tau}}{\tau}\,\mathrm{d}\tau \\ &= -2\ln|x| + C(M) + \mathrm{o}(1), \end{aligned}$$

where $C(M)$ is a certain real constant. □

The original operator H is positive. Using the Birman–Krein–Vishik theory ([4]), all the self-adjoint extensions of the operator H^0 can be described as follows. Every element ψ from the domain $\mathrm{Dom}(H^{0*})$ possesses the representation (15). The family of self-adjoint extensions of the symmetric operator H^0 coincides with the family of operators $H_\beta, \beta \in \mathbb{R}$. The operator H_β is the restriction of the operator H^{0*} to the domain of functions

$$\mathrm{Dom}(H_\beta) = \{\psi = \breve{\psi} + b(\psi)g_{-1} \in \mathrm{Dom}(H^{0*})\colon b(\psi) = \beta\breve{\psi}(0)\}.$$

The real parameter β is related to the parameters θ, z used in the previous section via the following formulas

$$\beta = \frac{\mathrm{e}^{i\theta} - 1}{\mathrm{e}^{i\theta} f(\bar{z}) - f(z)}; \qquad \mathrm{e}^{i\theta} = \frac{1 - \beta f(z)}{1 - \beta f(\bar{z})},$$

where

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{z+1}{(\sqrt{\xi^2 + M^2} - z)(\sqrt{\xi^2 + M^2} + 1)}\,\mathrm{d}\xi$$

is a function with positive imaginary part in the upper half-plane. The function f is Krein's Q-function corresponding to the self-adjoint operator H and its symmetric restriction H^0. It is equal to the value of the difference of the deficiency elements g_z and g_{-1} calculated at point zero: $f(z) = (g_z(x) - g_{-1}(x))|_{x=0}$.

Consider now the scattering problem for the perturbed operator. The scattering matrix is not trivial in the symmetric channel only, since every continuous antisymmetric function on the line is equal to zero at the origin. Every symmetric eigenfunction $\psi(E, x)$ of the perturbed operator has the following asymptotics for $|x| \to \infty$:

$$\psi(E, x) \sim_{|x|\to\infty} \mathrm{e}^{-i\sqrt{E^2 - M^2}x} + S(E)\,\mathrm{e}^{i\sqrt{E^2 - M^2}x}. \tag{16}$$

The coefficient $S(E)$ introduced in the latter formula will be called the stationary scattering matrix for the operators H_β and H.

THEOREM 3.2. *The stationary scattering matrix for the operators H_β and H is equal to*

$$S(E) = \frac{1-\beta\overline{f(E+i0)}}{1-\beta f(E+i0)}, \tag{17}$$

where

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{z+1}{(\sqrt{\xi^2+M^2}-z)(\sqrt{\xi^2+M^2}+1)}\,\mathrm{d}\xi.$$

Proof. The eigenfunction of the perturbed operator can be calculated using the following Ansatz:

$$\psi(E,x) = \mathrm{e}^{i\sqrt{E^2-M^2}x} + \mathrm{e}^{-i\sqrt{E^2-M^2}x} + \rho(E)g_{E+i0}(x). \tag{18}$$

The scattering amplitude $\rho(E)$ is a solution to the following equation:

$$\rho(E) = \beta(2+\rho(E)f(E+i0)) \Rightarrow \rho(E) = \frac{2\beta}{1-\beta f(E+i0)}. \tag{19}$$

The scattering matrix can be calculated now from Equations (13), (18), (19)

$$S(E) = 1+\rho(E)\frac{iE}{\sqrt{E^2-M^2}} = \frac{1-\beta\left(f(E+i0)-2\frac{iE}{\sqrt{E^2-M^2}}\right)}{1-\beta f(E+i0)}. \tag{20}$$

Let us calculate the imaginary part of the limit of the function $f(z)$ from the upper halfplane on the interval (M,∞):

$$\Im f(E+i0) = \lim_{b\to 0}\frac{b}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(\sqrt{\xi^2+M^2}-E)^2+b^2}\,\mathrm{d}\xi.$$

The integral over the real axis can be transformed to the integral over the cut $[iM, i\infty)$ taking into account the residue at the point $\xi = \sqrt{(E+ib)^2-M^2}$:

$$\begin{aligned}
&\Im f(E+i0)\\
&\quad = \lim_{b\to 0}\frac{b}{2\pi}\Bigg(\frac{\pi}{b}\frac{E+ib}{\sqrt{(E+ib)^2-M^2}} + \frac{\pi}{b} - \frac{E-ib}{\sqrt{(E-ib)^2-M^2}} + \\
&\qquad + \int_M^{\infty}\left[\frac{1}{(i\sqrt{t^2-M^2}-E)^2+b^2} - \frac{1}{(-i\sqrt{t^2-M^2}-E)^2+b^2}\right] i\,\mathrm{d}t\Bigg).
\end{aligned}$$

The integral in the latter formula is uniformly bounded and does not make any contribution to the limit. Therefore, we have proven that

$$\Im f(E+i0) = \frac{E}{\sqrt{E^2-M^2}}.$$

The latter formula together with (20) imply (17). The theorem is proven. □

We can easily see that the scattering matrix is a unitary function. The scattering matrix can be continued analytically to the complex plane. The singularities of the scattering matrix coincide with the solutions of the dispersion equation $1-\beta f(E+i0)=0$ which is just Equation (9).

The approach we developed can be applied to study the point interactions for the pseudo-differential operator in one dimension determined by the symbol $a(\xi) = |\xi|$. The scaling properties of the function $|\xi|$ can be used to simplify the calculations. The deficiency elements are given by the formulas (10) and (11) with $M = 0$. The integrals which appear in these formulas possess the following asymptotic representation

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t|x|}\frac{t}{t^2+z^2} \\
&\quad = \frac{1}{\pi}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t|x|}\frac{t}{z^2} + \frac{1}{\pi}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t|x|}\frac{-t^3}{z^2(t^2+z^2)} \\
&\quad = \frac{1}{\pi}\frac{1}{(zx)^2} + \mathrm{O}\left(\frac{1}{x^4}\right).
\end{aligned}$$

It follows that the deficiency elements and the kernel of the resolvent decay polynomially in the case $M = 0$. The Q-function f can be calculated explicitly in this case: $f(z) = -(\ln z/\pi) + i$, where the branch of the logarithm is fixed using the condition $z \in \mathbb{R}_+ \Rightarrow \ln z \in \mathbb{R}$. The function f is real on the negative part of the real axis. The corresponding free operator has been studied recently using probabilistic methods [9]. The theory of stochastic processes has been used to obtain the decay estimates for the eigenfunctions also for other pseudo-differential operators. In fact, the pseudo-differential operators determined by the symbols given by Levy–Khintchine formula

$$a(\xi) = d + ib\cdot\xi + \xi\cdot C\xi - \int_{\mathbb{R}^n}(\mathrm{e}^{i\xi x} - 1 - i\xi h(x))v(\mathrm{d}x)$$

were investigated in detail. The following parameters are used in the latter formula: a real constant d, a vector $b\in\mathbb{R}^n$, a nonnegative definite matrix C, a nonnegative measure ν satisfying $\int_{\mathbb{R}^n}\min(1,|x|^2)v(\mathrm{d}x) < \infty$, a cuttoff function h. The Levy–Khintchine family includes the symbols

$$a(\xi) = \sqrt{\xi^2+M^2} - M \quad \text{and} \quad a(\xi) = |\xi|^\alpha, \quad 0<\alpha<2.$$

The decay properties of the eigenfunctions for such Hamiltonians (for $\alpha = 1$) have been studied in [6].

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Note added in proof. The following theorem is an easy extension of Theorem 2.1 and Corollary 2.1:

THEOREM. *Let A be the self-adjoint pseudo-differential operator determined by the symbol a in $L_2(\mathbb{R}^n)$. Let A^0 be the restriction of A to the domain $C_0^\infty(\mathbb{R}^n \setminus \{0\})$. Suppose that the symbol a has order m and satisfies the estimate $c(1 + |\xi|)^m \leqslant |a(\xi)| \leqslant C(1 + |\xi|)^m$ for certain $c, C \in \mathbb{R}_+$ and all $\xi > R > 0$. Then the functions*

$$g_\lambda^{\mathbf{i}} = \mathcal{U}^{-1} \frac{\xi_1^{i_1} \xi_2^{i_2} \cdots \xi_n^{i_n}}{a(\xi) - \lambda},$$

$\mathbf{i} = (i_1, i_2, \ldots, i_n), i_j \in \mathbb{N}, i_1 + i_2 + \cdots + i_n < m - (n/2)$ *form a basis in the deficiency subspace for the operator* A^0 *at point* $\lambda, \Im\lambda \neq 0$.

Proof. The symbol a is a real-valued function, since the operator A is self-adjoint. Following the main ideas used in the proof of Lemma 2.2, we can prove that the elements $g_\lambda^{\mathbf{i}}$ are deficiency elements for the operator A^0 at point λ if $\Im\lambda \neq 0$. The elements $g_\lambda^{\mathbf{i}}$ are linearly independent. Therefore, to prove the theorem it is enough to show that every deficiency element g_λ can be represented by a linear combination of the elements $g_\lambda^{\mathbf{i}}$. Suppose that g_λ is a deficiency element. Then the distribution $f = \mathcal{U}^{-1}(a - \lambda)\mathcal{U}g_\lambda$ vanishes on all functions from $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and therefore can be represented by a linear combination of the data function and its partial derivatives: $f = \sum_{\mathbf{i}} f_{\mathbf{i}} \partial_{\xi_1}^{i_1} \partial_{\xi_2}^{i_2} \ldots \partial_{\xi_n}^{i_n} \delta$, where the sum is finite and $f_{\mathbf{i}} \in \mathbb{C}$. This implies that

$$(\mathcal{U}g_\lambda)(\xi) = \frac{1}{(\sqrt{2\pi})^n} \frac{\sum_{\mathbf{i}} i^{i_1+i_2+\cdots+i_n} f_{\mathbf{i}} \xi_1^{i_1} \xi_2^{i_2} \ldots \xi_n^{i_n}}{a(\xi) - \lambda}.$$

This function is an element from $L_2(\mathbb{R}^n)$ if and only if $i_1 + i_2 + \cdots + i_n < m - (n/2)$. Thus the element g_λ is equal to a linear combination of the deficiency elements $g_\lambda^{\mathbf{i}}$. $\square$