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Zero-Range Potentials with Internal Structures and the Inverse Scattering Problem

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Abstract. The inverse scattering problem on the half-axis is solved for scattering matrices which result from scattering on a zero-range potential with an internal structure. The solutions are constructed as pointwise limits of Bargmann potentials and form a one-parameter family. We obtain a class of potentials decreasing slowly at infinity.

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1. Introduction

The method of zero-range potentials with internal structures [1, 2] presents an opportunity to construct exactly solvable model operators for different physical phenomena. Here we will investigate the connections between these model problems and usual scattering problems with nonzero-range potentials. Earlier, this problem had been investigated by comparing the resolvents of the operators (see [3] and references herein). Here, however, we will discuss this question from the scattering theory point of view, i.e. construct an ordinary potential which has the same scattering matrix as the zero-range potential with an internal structure.

This Letter is devoted to the simplest case of the scattering problem on the half-axis. The inverse scattering problem in one dimension was extensively discussed in the literature [4, 5, 6], but only for the case of rapidly decreasing potentials, such that the following condition is satisfied:

$$\int_a^\infty |V(x)|x \, dx < \infty, \quad a > 0. \tag{1}$$

The Gel'fand–Levitan–Marchenko procedure for the solution of the inverse problem is based on the analytical properties of the Jost function. Therefore, in the first step, we will generalize the notions of the Jost solution and Jost function to the case of zero-range potentials. It will be shown that the Jost function can be introduced only for the so-called regular model operators, i.e. for operators with the standard asymptotic behaviour of the scattering matrix $S(k)$ at infinity: $S(k) \to 1$, $k \to \infty$. The corresponding Jost functions have singularities on the real axis. We shall show that the inverse problem for such Jost functions can be solved in a class of square integrable potentials which violate the condition (1). The solution of the problem can be obtained as the pointwise limit of the solutions corresponding to the usual Jost functions with singularities in the lower half-plane.

The solution of the inverse problem for the investigated class of scattering matrices is not unique. The nonuniqueness of the solution of the inverse scattering problem on the half-axis, found in earlier works [4–6], is due to the existence of bound states, where to reconstruct the potential, one needs to not only know the energies of the bound states but also the normalization constants for the eigenfunctions. In our case, the nonuniqueness of the solution of the inverse problem is connected with arbitrariness in the definition of the Jost function for problems with internal structure, which have singularities on the real axis. Such singularities, situated symmetrically with respect to the origin, do not make any contribution to the scattering matrix.

A similar problem was discussed in [7, 8], where the inverse scattering problem on the whole line was considered. It was shown that substitution of the scattering matrix corresponding to the operator with the boundary condition at point zero into the Marchenko equation has a delta function as a solution. In our approach, only continuous bounded potentials are considered.

2. Schrödinger Operator with Zero-Range Potential with an Internal Structure

In this section, we present the simplest restriction-extension model of the scattering problem on the half-axis $\mathbf{R}_+ = [0, \infty)$. To construct the model, we start from the operator $\mathscr{A} = -\mathrm{d}^2/\mathrm{d}x^2$, acting in the Hilbert space $L_2(\mathbf{R}_+)$ with the Dirichlet boundary condition at the origin. In order to construct a nontrivial model, we choose some self-adjoint internal operator A_{in} acting in a finite-dimensional Hilbert space H_{in}. We define the unperturbed operator $\mathscr{L}_0$ as the orthogonal sum of the external and internal operators $\mathscr{L}_0 = \mathscr{A} \oplus A_{\text{in}}$, which acts in the orthogonal sum of the Hilbert spaces: $L_2(\mathbf{R}_+) \oplus H_{\text{in}}$. To 'switch on' the interaction between the external and internal channels, we restrict the external operator $\mathscr{A} \to \mathscr{A}_0$ onto the domain of all functions with the first derivative equal to zero at the origin:

$$\mathrm{Dom}(\mathscr{A}_0) = \{u \in W_2^2(\mathbf{R}_+), u(0) = 0, u'(0) = 0\}.$$

The adjoint operator is defined on the domain

$$\mathrm{Dom}(\mathscr{A}_0^*) = \{u \in W_2^2(\mathbf{R}_+ \backslash 0)\}.$$

We define the operator with the interaction by the formula:

$$\mathscr{L}\begin{pmatrix} u \\ u_{\text{in}} \end{pmatrix} = \begin{pmatrix} \mathscr{A}_0^* u \\ \mathscr{A}_{\text{in}} u_{\text{in}} + \left(a \left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} + bu|_{x=0} \right) \theta \end{pmatrix}, \tag{2}$$

where $\theta \in H_{\text{in}}$, $|\theta| = 1$, a and b are real numbers.

THEOREM 1. *The operator* $\mathscr{L}$ *defined by formula* (2) *is a self-adjoint operator on the domain of functions from* $\mathrm{Dom}(\mathscr{A}_0^*) \oplus H_{\mathrm{in}}$ *satisfying the boundary condition*

$$\left(c \frac{\mathrm{d}u}{\mathrm{d}x} + du \right) \bigg|_{x=0} = \langle u_{\mathrm{in}}, \theta \rangle \tag{3}$$

with real parameters c and d such that

$$\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -1. \tag{4}$$

Proof. The boundary form of the operator is given by the following expression:

$$\begin{aligned}
&\langle \mathscr{L}\mathscr{U}, \mathscr{V} \rangle - \langle \mathscr{U}, \mathscr{L}\mathscr{V} \rangle \\
&= \left(\frac{\mathrm{d}u}{\mathrm{d}x} \bar{v} - u \frac{\mathrm{d}\bar{v}}{\mathrm{d}x} \right) \bigg|_{x=0} + \left(a \frac{\mathrm{d}u}{\mathrm{d}x} + bu \right) \bigg|_{x=0} \langle v_{\mathrm{in}}, \theta \rangle - \\
&\quad - \langle u_{\mathrm{in}}, \theta \rangle \left(a \frac{\mathrm{d}\bar{v}}{\mathrm{d}x} + b\bar{v} \right) \bigg|_{x=0} = \left(\frac{\mathrm{d}u}{\mathrm{d}x} \bar{v} - u \frac{\mathrm{d}\bar{v}}{\mathrm{d}x} \right) \bigg|_{x=0} + \\
&\quad + \left(a \frac{\mathrm{d}u}{\mathrm{d}x} + bu \right) \bigg|_{x=0} \left(c \frac{\mathrm{d}\bar{v}}{\mathrm{d}x} + d\bar{v} \right) \bigg|_{x=0} - \\
&\quad - \left(c \frac{\mathrm{d}u}{\mathrm{d}x} + du \right) \bigg|_{x=0} \left(a \frac{\mathrm{d}\bar{v}}{\mathrm{d}x} + b\bar{v} \right) \bigg|_{x=0} \\
&= \left(1 + \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right) \left(\frac{\mathrm{d}u}{\mathrm{d}x} \bar{v} - u \frac{\mathrm{d}\bar{v}}{\mathrm{d}x} \right) \bigg|_{x=0} = 0.
\end{aligned}$$

The last equality is valid due to the condition (4) for the constants a, b, c, d. We also used the fact that these constants are real. Thus, we have proved that the operator is symmetric. Let $\mathscr{U}$ be an element from the domain of the operator, then it is easy to show that the domain of the adjoint operator will be a subset of $\mathrm{Dom}(\mathscr{A}_0^*) \oplus H_{\mathrm{in}}$. The expression $\langle \mathscr{L}\mathscr{U}, \mathscr{V} \rangle$ defines a bounded functional on $\mathscr{U}$ only if the function $\mathscr{V}$ satisfies the boundary condition (3). □

3. Scattering Matrix for the Point Interaction

The essential spectrums of the perturbed operator $\mathscr{L}$ and of the original operator $\mathscr{L}_0$ coincide. Hence, it is possible to introduce a scattering matrix which is identical to the reflection coefficient in this case. To solve the eigenfunction equation in the distributional sense

$$\mathscr{L}\mathscr{U} = \lambda \mathscr{U}, \quad \mathscr{U} = \begin{pmatrix} u \\ u_{\mathrm{in}} \end{pmatrix}, \tag{5}$$

the standard ansatz for the external component can be used

$$u = \mathrm{e}^{-ikx} - S(k)\, \mathrm{e}^{ikx}.$$

By excluding the internal component from the second equation (5), we get

$$\left(a\frac{\mathrm{d}u}{\mathrm{d}x} + bu\right)\bigg|_{x=0} \theta + A_{\mathrm{in}}u_{\mathrm{in}} = \lambda u_{\mathrm{in}}$$

and, with the help of the boundary conditions (3), we receive the following energy-dependent boundary conditions for the external component:

$$\frac{\mathrm{d}u/\mathrm{d}x}{u}\bigg|_{x=0} = -\frac{b\mathbf{R}(\lambda) + d}{a\mathbf{R}(\lambda) + c} \equiv -\mathscr{D}(\lambda), \quad \mathbf{R}(\lambda) = \langle (A_{\mathrm{in}} - \lambda)^{-1}\theta, \theta \rangle. \tag{6}$$

Thus, the following theorem is proven:

THEOREM 2. *The scattering matrix for the pair of the operators $\mathscr{L}_0$ and $\mathscr{L}$ is given by the formula*

$$S(k) = \frac{\mathscr{D}(\lambda) - ik}{\mathscr{D}(\lambda) + ik}. \tag{7}$$

The form of the scattering matrix shows that all the spectral information about the operator is contained in the function $\mathscr{D}$. The function $\mathscr{D}(\lambda)$, like the function $\mathbf{R}(\lambda)$, has a positive imaginary part in the upper half-plane $\Im\lambda > 0$ and is real on the real axis. These properties of the function $\mathscr{D}(\lambda)$ define the usual behaviour of the scattering matrix. It has singularities in the lower half-plane $\Im k < 0$ and on the positive part of the imaginary axis, corresponding to resonances and bound states, respectively. We shall discuss here the scattering matrices without bound states which correspond to positive internal operators. The singularities of the scattering matrix are denoted in the standard way ([6]) as follows: $S^{-1}(-ia_j) = 0$. The number M of the singularities of the S-matrix cannot be larger than $2N + 1$, where N is the dimension of the internal space. Zeroes of the S-matrix are situated at the points conjugated to the singularities.

The scattering matrix is unitary on the real axis $k \in \mathbf{R}$. Being a rational function of k, it can be represented by the product

$$S(k) = \beta \prod_j \frac{k - ia_j}{k + ia_j} \tag{8}$$

with an arbitrary constant β, $|\beta| = 1$. One can prove that the constant β is equal to ± 1 and is defined by the asymptotic behaviour of the function $\mathscr{D}(\lambda)$ at infinity. The function $\mathscr{D}(\lambda)$, as a function with a positive imaginary part in the upper half-plane, has the asymptotic representation

$$\mathscr{D}(\lambda) = \alpha_{-1}\lambda + \mathrm{O}(1), \tag{9}$$

where $\alpha_{-1} \in \mathbf{R}_+$. The linear term appears in the asymptotics of the function $\mathscr{D}$ only if the constant c is equal to zero. The corresponding scattering matrix tends to 1 at infinity. We call such problems regular and we shall restrict ourselves by these problems.

We illustrate the discussed ideas by the case of one-dimensional internal space when the internal operator is a multiplication operator by a real constant: $H_{\text{in}} = \mathbf{C}$, $\mathscr{A}_{\text{in}} = \lambda_1$. The resolvent function is unique: $R(\lambda) = (\lambda_1 - \lambda)^{-1}$. The scattering matrix in the regular case has no more than two singularities:

$$S(k) = \frac{k^2 - ia^2k + ab - \lambda_1}{k^2 + ia^2k + ab - \lambda_1}, \qquad a_{1,2} = \frac{a^2 \pm \sqrt{a^4 - 4(\lambda_1 - ba)}}{2}.$$

If $\lambda_1 = ba$, then the scattering matrix has only one singularity.

4. Levinson's Theorem

The calculated scattering matrix has properties different from the properties of the scattering matrices for potentials, satisfying the condition (1). We shall illustrate this fact by calculating the phase shift on the real axis for our class of scattering matrices. For a scattering matrix unitary on the real axis, the phase for the zero-range potential can be introduced in the standard way:

$$e^{2i\delta(k)} = S(k).$$

We shall prove the following theorem:

THEOREM 3. *Let $\mathscr{L}$ be the operator defined by formulas* (2) *and* (3), *then the number of bound states N_{bs} and the number of resonances N_{res} are related to the phase shift as follows*:

$$\delta(\infty) - \delta(-\infty) = \pi(N_{\text{res}} - N_{\text{bs}}) = -2\pi N_{\text{bs}} + \pi M. \tag{10}$$

Proof. The phase shift on the real axis is given by the integral

$$\delta(\infty) - \delta(-\infty) = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{S'(k)}{S(k)}\, dk. \tag{11}$$

The integral can be calculated with the help of the representation (7) for the scattering matrix

$$\delta(\infty) - \delta(-\infty) = \sum_m \int_{-\infty}^{\infty} \frac{a_m}{k^2 + a_m^2}\, dk = -\pi N_{\text{bs}} + \pi N_{\text{res}}. \tag{12}$$

To finish the proof of Theorem 3, we mention that the number M of singularities of the scattering matrix S is equal to $M = N_{\text{res}} + N_{\text{bs}}$. □

Theorem 3 shows that it is impossible to introduce Jost functions analytic in the closed upper half-plane with the unit limit at infinity for our class of scattering matrices. The number of singularities of the Jost function in the closed upper half-plane is defined by the total number of resonances and bound states:

$$N_{\Im > 0} + \tfrac{1}{2} N_{\Im = 0} = \tfrac{1}{2}(N_{\text{res}} + N_{\text{bs}}) = \tfrac{1}{2} M,$$

where $N_{\Im > 0}$ is the number of singularities inside the upper half-plane and $N_{\Im = 0}$ is the number of singularities on the real axis.

5. The Jost Function

This section is devoted to the construction of an analog of the Jost function for the case of the scattering by zero-range potential with an internal structure. The Jost function cannot be defined from the scattering data only. Singularities of the scattering matrix define zeroes of the Jost function in the investigating case. But we proved that the Jost function cannot be an analytical function in the whole upper half-plane up to the real axis. From Levinson's theorem, we know only the number of singularities of the Jost function in the closed upper half-plane. We have some arbitrariness in the calculation of the Jost function.

In the first step, we shall introduce the Jost function connected with a generalization of the notion of the Jost solution. The Jost solution can be defined on the usual way as a solution of the equation:

$$\begin{pmatrix} A_0^* f \\ \mathscr{A}_{\text{in}} f_{\text{in}} + \left(a \dfrac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=0} + bf|_{x=0} \right) \theta \end{pmatrix} = k^2 \begin{pmatrix} f \\ f_{\text{in}} \end{pmatrix}, \tag{13}$$

with the following asymptotics of the external component at infinity:

$$f(k, x) = \mathrm{e}^{ikx} + \mathrm{o}(1)\,\mathrm{e}^{-\Im kx}, \quad x \to \infty.$$

We do not assume that the boundary conditions (3) are satisfied. The usual definition of the Jost function as a value of the Jost solution at point zero cannot be used because the resulting function is identically equal to 1 and does not carry any information about the scattering problem. It is possible to define the Jost function by the equality

$$F(k) \equiv 1 - \frac{\langle f_{\text{in}}, \theta \rangle}{\left(c \dfrac{\mathrm{d}f}{\mathrm{d}x} + df \right)\Big|_{x=0}}. \tag{14}$$

With this definition, we obtain

$$F(k) = 1 + \frac{ika + b}{ikc + d} \mathbf{R}(k^2). \tag{15}$$

This Jost function retains some properties of the usual Jost function:

$$F(k) \to 1, \quad k \to \infty; \tag{16}$$

$$F(-k) = F(k). \tag{17}$$

Such functions have a finite number of singularities and zeroes. One can try to calculate the corresponding scattering matrix using the standard formula:

$$S_J(k) = \frac{F(-k)}{F(k)}. \tag{18}$$

However, $S_J(k)$ does not coincide with the correct S-matrix $S(k)$ (Equation (7)):

$$S_J(k) = \frac{F(-k)}{F(k)} = \frac{\Delta(\lambda) - ik}{\Delta(\lambda) + ik}\frac{d + cik}{d - cik} = S(k)\frac{d + cik}{d - cik}. \tag{19}$$

The additional term disappears when the constant c is equal to zero, i.e. in the regular case. The Jost function in this case is

$$F(k) = 1 - ika^2\mathbf{R}(\lambda) - ab\mathbf{R}(\lambda). \tag{20}$$

Zeroes of the Jost function coincide with the singularities of the scattering matrix. The singularities of the Jost function are situated symmetrically at the points of discrete spectrum of the internal operator $k = -ib_j$, $-b_j^2 \in \sigma_{\text{disc}}(\mathscr{A}_{\text{in}})$. Since we restricted ourselves to the case of the positive internal operator, then all the singularities of the Jost function are situated on the real axis. One can try to solve the system of Equations (5) at these points. But the second equation cannot be solved for arbitrary boundary values of the external component because the matrix $\mathscr{A}_{\text{in}} - \lambda$ has a zero eigenvalue and a solution exists only if the element θ is orthogonal to the corresponding eigenvector. The Jost function in our case is a rational function and can be written in the form

$$F(k) = \prod_j \frac{k + ia_j}{k + ib_j}, \quad \Re b_j = 0. \tag{21}$$

One can change the positions of the singularities of the Jost function and calculate another Jost function with the same scattering matrix. This corresponds to the arbitrary choice of the constants b_j with zero real part in representation (21). Singularities of the Jost function situated symmetrically over the origin, do not make any contribution to the scattering matrix. For odd constants M, one singularity is situated at the origin.

6. The Inverse Scattering Problem for the Zero-Range Potential Scattering Matrix

In this section, we discuss the inverse scattering problem on the half-axis for the scattering matrices obtained for the interaction with internal structure. The scattering matrix can be defined for the operators

$$\mathscr{A} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{22}$$

on the half-axis with the Dirichlet boundary condition at the origin for the smooth square integrable potentials V. An algorithm to solve the inverse problem for the scattering matrices of the class defined by Equation (7) will be presented.

To solve the problem, the Jost function representation of the scattering matrix will be used.

The inverse problem on the half-axis was solved for the rational Jost functions of the type:

$$F(k) = \prod_{j=1}^{M} \frac{k + ia_j}{k + ib_j}, \tag{23}$$

where the constants $a_j, b_j, j = 1, \ldots, M$ are supposed to have positive real parts. By solving the Gel'fand–Levitan–Marchenko equation, analytic formulas for the potential $V(x)$ and the corresponding regular solution $\varphi(k, x)$ can be obtained [6].

The Jost function corresponding to the scattering matrix for the problem with internal structure has a similar form, but the singularities are situated on the real axis. For such Jost functions, approximate Bargmann-type Jost functions can be introduced:

$$F_\varepsilon = \prod_{j=1}^{M} \frac{k + ia_j}{k + i(b_j + \varepsilon)}, \quad \varepsilon > 0 \tag{24}$$

Then one can restore the Bargmann potential $V_\varepsilon(x)$ and the regular solution $\varphi_\varepsilon(x)$ corresponding to the Jost function $F_\varepsilon(k)$. The calculated potential exponentially decreases at infinity like $\exp(-\varepsilon x)$. Then one takes the pointwise limits of the potential and regular solution when $\varepsilon \to 0$. The limit potential does not exponentially decrease at infinity. Moreover, it violates the condition (1).

The following conjecture will be formulated for a general number of singularities M, but we are able to prove it only for $M = 1, 2$.

CONJECTURE 1. *The limit of the regular solution* $\varphi_0(x) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(x)$, $x \in [0, \infty)$ *is a regular solution for the Schrödinger equation with the limit potential*: $V_0(x) = \lim_{\varepsilon \to 0} V_\varepsilon(x)$, $x \in [0, \infty)$. *The scattering matrix for the limit potential is given by the formula*

$$S(k) = \prod_{j=1}^{M} \frac{k - ia_j}{k + ia_j}.$$

Proof for $M = 1$. We assume that the original scattering matrix has only one singularity:

$$S(k) = \frac{k - ia_1}{k + ia_1}. \tag{25}$$

The Jost function and approximate Jost functions are introduced as follows:

$$F(k) = \frac{k + ia_1}{k} \Rightarrow F_\varepsilon(k) = \frac{k + ia_1}{k + i\varepsilon}. \tag{26}$$

The potential and the regular solution corresponding to the approximate Jost function are

$$V_\varepsilon(x) = \frac{8\varepsilon^2}{a_1^2 - \varepsilon^2}\left(\frac{e^{\varepsilon x}}{a_1 - \varepsilon} - \frac{e^{-\varepsilon x}}{a_1 + \varepsilon}\right)^{-2},$$

$$\varphi_\varepsilon(k, x) = \frac{\sin kx}{k} + \frac{\varepsilon^2 - a_1^2}{2k(k^2 + \varepsilon^2)} \times \frac{k(e^{\varepsilon x} - e^{-\varepsilon x})(e^{ikx} + e^{-ikx}) + i\varepsilon(e^{\varepsilon x} + e^{-\varepsilon x})(e^{ikx} - e^{-ikx})}{\varepsilon(e^{\varepsilon x} + e^{-\varepsilon x}) + a_1(e^{\varepsilon x} - e^{-\varepsilon x})}. \tag{27}$$

The limits of the potential and regular solution for $\varepsilon \to 0$ are

$$V_0(x) = \frac{2a_1^2}{(1 + a_1 x)^2},$$

$$\varphi_0(k, x) = \frac{\sin kx}{k} - \frac{a_1^2}{2ik^3} \frac{e^{ikx}(ikx - 1) + e^{-ikx}(ikx + 1)}{1 + a_1 x}. \tag{28}$$

By direct calculation, one can prove that the limit of the regular solution is a regular solution for the limit potential. The asymptotics of the regular solution for $x \to \infty$ is

$$\varphi_0(k, x) \sim \frac{\sin kx}{k} - \frac{a_1 \cos kx}{k^2} = \frac{ik - a_1}{2k^2}\left(e^{-ikx} - \frac{k - ia}{k + ia_1} e^{ikx}\right) \Rightarrow S_0(k) = \frac{k - ia_1}{k + ia_1}. \tag{29}$$

This completes the proof for the case $M = 1$. □

Proof for $M = 2$. The original scattering matrix is given by the expression

$$S(k) = \frac{k - ia_1}{k + ia_1} \frac{k - ia_2}{k + ia_2}, \quad a_1 = \bar{a}_2.$$

There is arbitrariness in the definition of the Jost function in this case. The family of the possible Jost functions depends on the real parameter b_0:

$$F_{b_0}(k) = \frac{k + ia_1}{k - b_0} \frac{k + ia_2}{k + b_0}. \tag{30}$$

Approximate Jost functions are introduced in different ways for $b_0 \neq 0$ and for $b_0 = 0$. In the first case, the family of approximate Jost functions is one-parameter:

$$k_{b_0, \varepsilon}(k) = \frac{k + ia_1}{k - b_0 + i\varepsilon} \frac{k + ia_2}{k + b_0 + i\varepsilon}, \quad \varepsilon > 0. \tag{31}$$

For $b_0 = 0$, there is a two-parameter family of approximate Jost functions:

$$F_{0,\varepsilon_1,\varepsilon_2}(k) = \frac{k+ia_1}{k+i\varepsilon_1}\frac{k+ia_2}{k+i\varepsilon_2}, \quad \varepsilon_1, \varepsilon_2 > 0. \tag{32}$$

We shall discuss here only the first family of the Jost functions. The limit potential when $\varepsilon \to 0$ is

$$V_{a_1,a_2,b_0}(x) = 16b_0^2\frac{1-(b_0x+B)\sin 2(b_0x+\delta(b_0)) - \cos 2(b_0x+\delta(b_0))}{(2b_0x+2B-\sin 2(b_0x+\delta(b_0)))^2}, \tag{33}$$

where we used the following notations

$$B = B(b_0, a_1, a_2) = \frac{b_0(a_1+a_2)(a_1a_2+b_0^2)}{(a_1^2+b_0^2)(a_2^2+b_0^2)},$$
$$e^{2i\delta(b_0)} = S(b_0). \tag{34}$$

The corresponding regular solution for $k \neq \pm b_0$ is

$$\varphi(k,x) = \frac{\sin kx}{k} + \frac{1}{2ik}\times$$
$$\times\frac{e^{ikx}(-w(b_0,-ik,a_2)+w(b_0,-ik,a_1)) + e^{-ikx}(w(b_0,ik,a_2)-w(b_0,ik,a_1))}{w(b_0,a_1,a_2)}, \tag{35}$$

where the function w is

$$w(b_0,a_1,a_2) = \frac{a_2-a_1}{(a_1^2+b_0^2)(a_2^2+b_0^2)}(2b_0x+2B(b_0,a_1,a_2)-\sin 2(b_0x+\delta(b_0,a_1,a_2))). \tag{36}$$

The asymptotics of the solution for $x \to \infty$ is

$$\varphi(k,x) \sim \frac{e^{ikx}}{2ik}\frac{(k-ia_1)(k-ia_2)}{k^2-b_0^2} - \frac{e^{-ikx}}{2ik}\frac{(k+ia_1)(k+ia_2)}{k^2-b_0^2} \tag{37}$$

and the scattering matrix can be easily calculated:

$$S(k) = \frac{k-ia_1}{k+ia_1}\frac{k-ia_2}{k+ia_2} \tag{38}$$

for all $k \neq \pm b_0$. The potential decreases at infinity, like

$$V_{a_1,a_2,b_0}(x) \sim -4b_0\frac{\sin 2\,b_0x}{x}.$$

The usual scattering theory can be introduced for such potentials [9, 10].

We have now finished the proof of Conjecture 1 for $M = 1, 2$. □

The proof of Conjecture 1 shows that, in the case where $M = 1$, the unique potential corresponding to the internal structure scattering matrix was calculated. The arbitrariness of the solution of the inverse problem in the second case is connected with the arbitrariness of the definition of the Jost function. Our result does not contradict the general theory [4–6]. The inverse problem in one dimension was solved there for the potentials satisfying the condition (1) only. One can easily check that the calculated potentials violate this condition. It shows that the solution for the general inverse problem on the half-axis is not unique. Earlier, this fact was connected with the presence of bound states and when restoring the potential, one not only needs to know the energies of the bound states but also the normalization constants for the eigenfunctions.

Similar potentials, decreasing at infinity, like x^{-2}, have been extensively studied during the last years and some examples of the potentials with the same scattering matrix on the whole axis have been constructed [11–16]. Potentials given here represent a new class of potentials with the same scattering matrix, but for the case of the scattering problem on the half-axis.

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References

1. Pavlov, B. S., *Russian Math. Surveys* **42**(6), 127 (1987).
2. Makarov, K. A., Preprint FUB/HEP-88-3, Berlin, 1988.
3. Albeverio, S., Gesztesy, F., Høegh-Krohn, R., and Holden, H., *Solvable Models in Quantum Mechanics*, Springer-Verlag, Berlin, 1988.
4. Faddeev, L. D., *Uspekhi Mat. Nauk* **14**, 57 (1959); English translation: *J. Math. Phys.* **4**, 73 (1963).
5. Deift, P. and Trubowitz, E., *Comm. Pure Appl. Math.* **32**, 121 (1979).
6. Chadan K. and Sabatier, P. C., *Inverse Problems in Quantum Scattering Theory*, 2nd edn, Springer-Verlag, Berlin, 1989.
7. Atkinson, D. and Geschiere, J., *Inverse Problems* **2**, L51–L53 (1986).
8. Atkinson, D., Meromorphic reflection coefficient, in P. C. Sabatier (ed.), *Some Topics on Inverse Problems*, World Scientific, Singapore, 1988, pp. 175–181.
9. Reed, M. and Simon, B., *Methods of Modern Mathematical Physics vol. III Scattering Theory*, Academic Press, New York, 1979, pp. 155–168.
10. Matveev, V. B. and Skriganov, M. M., *Dokl. Acad. Nauk SSSR* **202**, 755–758 (1972).
11. Abraham, P. B., De Facio, B., and Moses, H. E., *Phys. Rev. Lett.* **46**, 1657 (1981).
12. Brownstein, K. R., *Phys. Rev. D* **25**, 2704 (1982).
13. Moses, H. E., *Phys. Rev. A* **27**, 2220 (1983).
14. Aktosun, T. and Newton, R. G., *Inverse Problems* **1**, 291 (1985).
15. Degasperis, A. and Sabatier, P. C., *Inverse Problems* **3**, 73 (1987).
16. Aktosun, T., *Inverse Problems* **4**, 347 (1988).