Letters in Mathematical Physics (2005) 73:109–122
DOI 10.1007/s11005-005-0002-1

Reflectionless Potentials and Point Interactions in Pontryagin Spaces

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(Received: 21 February 2005)

Abstract. The problem of constructing generalized point interactions of the second derivative operator in $L^2(\mathbb{R})$ leading to the same scattering data as for reflectionless potentials is considered. It is proved that this problem has a solution only if extensions in Pontryagin spaces are involved. The solution of the inverse scattering problem is not unique, this is illustrated by considering the scattering data for soliton of the Korteweg-de Vries equation.

Mathematics Subject Classifications (2000). 34B25, 47E05, 47B50, 81U40.

Key words. inverse scattering, generalized Nevanlinna function, Pontryagin space.

1. Introduction

The classical Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V(x)$ can be considered as a perturbation of the free second derivative operator $A_0 = -\mathrm{d}^2/\mathrm{d}x^2$ by the operator of multiplication by the real-valued function $V(x)$. It is well known that potentials V from the Faddeev class of locally integrable functions with $\int_{-\infty}^{\infty}(1+|x|)|V(x)|\mathrm{d}x < \infty$ are determined uniquely by the scattering data consisting of one reflection coefficient $R(k)$, the eigenvalues $E_n = -\beta_n^2$, for $n = 1, 2, \ldots, N$ and the corresponding normalizing constants m_n [3, 10] (or see any standard text on Schrödinger scattering theory).

Instead of the above mentioned additive perturbations of A_0 one can also study certain domain perturbations. The method of generalized interactions was first developed by Pavlov [8,9] for operators in Hilbert spaces and recently also in Pontryagin spaces [7].

By restricting the second derivative operator A_0 to a certain symmetric operator with finite deficiency indices and then extending it in a – possibly – larger space to another self-adjoint operator $\mathbb{A}$ one obtains a domain perturbation. If the original Hilbert space has finite defect in the larger space, then the absolutely continuous spectra of the perturbed and the unperturbed operators coincide. Thus one can study the scattering problem for the pair $(A_0, \mathbb{A})$.

The aim of the present paper is to obtain a domain perturbation of A_0, more precisely a generalized point interaction $\mathbb{A}$, leading to the same scattering data as the additive perturbation by a reflectionless potential. In the case of the half line this question has been investigated in [4, 5].

We show that the problem for the whole line always has a solution, but only if extensions in Pontryagin spaces, i.e., spaces with indefinite inner product, are allowed. It turns out that then the model operator $\mathbb{A}$ has the prescribed eigenvalues, but the corresponding eigenspaces are two-dimensional. However, each eigenspace has a positive and a negative subspace, where the positive one corresponds to the physical eigenfunctions. Due to this particular structure one could also split off the non-positive part, but then the model becomes non-local. It appears that different generalized perturbations can determine the same scattering data. An analogy with soliton solutions of KdV equations is used to illustrate this phenomenon.

The paper is organized as follows. In Section 2 generalized interactions supported by one point are considered. The relation between the scattering matrix and energy dependent boundary conditions at this point is discussed. For reflectionless boundary conditions a connection to operator extensions is established via Krein's formula for generalized resolvents. An explicit model of generalized interactions (supported by several points) is given in Section 3. It is shown in Section 4 that the parameters in the model can be chosen such that the scattering matrix is of a prescribed form. Furthermore spectral properties of these operators are discussed. Section 5 is devoted to the corresponding inverse scattering problem, which turns out to be solvable, but not necessarily uniquely.

2. Scattering Matrix and Energy Dependent Boundary Conditions

In this section, we deal with a generalized interaction that is supported by one point, say $x_1 \in \mathbb{R}$, and hence leads to certain energy dependent boundary conditions connecting the boundary values at this point. We discuss the relation to the corresponding scattering matrix.

Consider the one-dimensional Schrödinger equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} f = Ef \tag{1}$$

with the energy dependent boundary conditions

$$\{f\}_{x_1} = q_1(E)[f']_{x_1}, \quad \{f'\}_{x_1} = -q_2(E)[f]_{x_1}, \tag{2}$$

where the functions q_1 and q_2 depend on the energy E and the following notations are used

$$\{f\}_{x_1} := \tfrac{1}{2}\left(f(x_1+0)+f(x_1-0)\right), \quad [f]_{x_1} := \left(f(x_1+0)-f(x_1-0)\right). \tag{3}$$

This form of the boundary conditions guarantees that the energy dependent interaction is invariant with respect to reflection at x_1.

The corresponding scattering matrix can be introduced by considering the two scattering solutions

$$f_{\text{left}} = \begin{cases} e^{ikx} + S_{21}e^{-ikx}, & x<0, \\ S_{22}e^{ikx}, & x>0, \end{cases} \quad \text{and} \quad f_{\text{right}} = \begin{cases} S_{11}e^{-ikx}, & x<0, \\ e^{-ikx}+S_{12}e^{ikx}, & x>0. \end{cases}$$

In this article, we restrict ourselves to reflectionsless scattering matrices, i.e., $S_{12} = S_{21} = 0$ and $S_{11} = S_{22} =: T$, where T is called the *transmission coefficient*. In this case it holds

$$q_2(E) = Eq_1(E). \tag{4}$$

The relation between T and q_1 is given by:

$$q_1(E) = \frac{1}{2ik}\frac{T(k)+1}{T(k)-1} \quad \text{and, conversely,} \quad T(k) = \frac{2ikq_1(E)-1}{2ikq_1(E)+1}, \tag{5}$$

where $E = k^2$. Transmission coefficients corresponding to reflectionless potentials [3] are of the form

$$T(k) = \prod_{n=1}^{N} \frac{ik-\beta_n}{ik+\beta_n} \quad \text{with pairwise distinct } \beta_n > 0, \tag{6}$$

where $-\beta_n^2$ are the eigenvalues of the corresponding Schrödinger operator. Then obviously the functions q_i are rational in the variable k, but more than that the following lemma holds.

LEMMA 1. *The functions q_i for $i=1,2$ appearing in the energy dependent boundary conditions* (2) *and corresponding to reflectionless scattering matrices* (6) *are real rational functions in the variable $E=k^2$.*

Proof. Relations (6) and (5) imply that q_1 is an even function of k and therefore it is a rational function of k^2, as well as q_2. □

Writing the boundary conditions in the form (2) enables us to interpret the functions $q_1(E)$ and $q_2(E)$ as parameters appearing in Krein's formula (7). Really, the formal resolvent corresponding to the differential equation (1) and boundary conditions (2) can be written in the form:

$$R(E) = (A_0 - E)^{-1} - \frac{1}{-\frac{1}{2ik} + q_1(E)} \langle \cdot, (A_0 - \overline{E})^{-1}\delta\rangle (A_0 - E)^{-1}\delta -$$
$$- \frac{1}{\frac{ik}{2} + q_2(E)} \langle \cdot, (A_0 - \overline{E})^{-1}\delta'\rangle (A_0 - E)^{-1}\delta'. \qquad (7)$$

Note that if q_1 and q_2 are Nevanlinna functions (i.e., $\frac{\Im q_i(E)}{\Im E} > 0$ for $E \notin \mathbb{R}$) this implies that $R(E)$ is the compressed resolvent of a self-adjoint extension $\mathbb{A}$ of the symmetric operator

$$A_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\bigg|_{\{f \in W_2^2(\mathbb{R}): f(x_1) = f'(x_1) = 0\}}$$

into a Hilbert space $\mathbb{H}$ with $L_2(\mathbb{R}) \subset \mathbb{H}$. In the situation under consideration q_j are not Nevanlinna functions, but rather belong to some generalized Nevanlinna class [6,2]. Therefore one needs to consider self-adjoint extensions in a Pontryagin space containing the original Hilbert space $L_2(\mathbb{R})$. However, Lemma 1 assures that the extension space can be chosen such that $L_2(\mathbb{R})$ has finite defect.

3. Generalized Point Interactions

In this section, we describe a straightforward modification of the generalized point interaction model to Pontryagain spaces. These models have been suggested first in [8] and described in detail in [1] in the Hilbert space case.

Consider L distinct points x_j, $j=1,2,\ldots,L$ on the real line and with each point x_j associate two finite dimensional Pontryagin spaces $\mathcal{K}_i^j$ for $i=1,2$ and operators A_i^j self-adjoint in $\mathcal{K}_i^j$. Consider the extended Pontryagin space $(\mathbb{H}, [\cdot,\cdot])$, which is given as the orthogonal sum $\mathbb{H} := L^2(\mathbb{R}) \oplus \bigoplus_{j=1}^{L} (\mathcal{K}_1^j \oplus \mathcal{K}_2^j)$ with the induced inner product. Choose in addition $2L$ vectors $\theta_i^j \in \mathcal{K}_i^j$ and $2L$ quadruples of real numbers $(a_i^j, b_i^j, c_i^j, d_i^j) \in \mathbb{R}^4$ satisfying the additional condition $a_i^j d_i^j - b_i^j c_i^j = -1$.

DEFINITION 2. With the above notations the operator $\mathbb{A}$ acting in the Pontryagin space $\mathbb{H}$ is defined on the domain

$$\mathrm{Dom}\,\mathbb{A} := \left\{ \begin{pmatrix} f \\ \begin{pmatrix} f_1^j \\ f_2^j \end{pmatrix}_{j=1}^{L} \end{pmatrix} \in \mathbb{H} : \begin{array}{l} f \in W_2^2(\mathbb{R}\setminus\{x_1,\ldots,x_L\}), \\ \langle f_1^j, \theta_1^j\rangle = c_1^j [f']_{x_j} + d_1^j \{f\}_{x_j}, \forall j \\ \langle f_2^j, \theta_2^j\rangle = c_2^j \{f'\}_{x_j} + d_2^j [f]_{x_j}, \end{array} \right\}$$

by

$$\mathbb{A} \begin{pmatrix} f \\ \begin{pmatrix} f_1^j \\ f_2^j \end{pmatrix}_{j=1}^{L} \end{pmatrix} := \begin{pmatrix} -f'' \\ \begin{pmatrix} (a_1^j [f']_{x_j} + b_1^j \{f\}_{x_j})\theta_1^j + A_1^j f_1^j \\ (a_2^j \{f'\}_{x_j} + b_2^j [f]_{x_j})\theta_2^j + A_2^j f_2^j \end{pmatrix}_{j=1}^{L} \end{pmatrix},$$

where we again use notations (3) for the jump and the average value.

PROPOSITION 3. *The operator $\mathbb{A}$ given by Definition 2 is self-adjoint.*

Proof. First we show that $\mathbb{A}$ is symmetric, indeed for arbitrary vectors $\mathbf{f} = (f, \{f_1^j, f_2^j\}_{j=1}^L)^\top$ and $\mathbf{g}=(g, \{g_1^j, g_2^j\}_{j=1}^L)^\top \in \mathrm{Dom}\,(\mathbb{A})$ a straight-forward calculation gives $[\mathbb{A}\mathbf{f}, \mathbf{g}] - [\mathbf{f}, \mathbb{A}\mathbf{g}] = 0$. For the self-adjointness consider the resolvent equation $(\mathbb{A} - E)\mathbf{f} = \mathbf{g} = (g, \{g_1^j, g_2^j\}_{j=1}^L)$, for $E \ll -1$, $\mathbf{g} \in \mathbb{H}$. In components this equation reads as

$$\begin{cases} -f'' - Ef = g \\ (a_1^j [f']_{x_j} + b_1^j \{f\}_{x_j})\theta_1^j + A_1^j f_1^j - E f_1^j = g_1^j \\ (a_2^j \{f'\}_{x_j} + b_2^j [f]_{x_j})\theta_2^j + A_2^j f_2^j - E f_2^j = g_2^j \end{cases}$$

and taking into account the boundary conditions it can be written as an equation on the function $f \in L^2(\mathbb{R})$ only $-f'' - Ef = g$ with energy dependent boundary conditions

$$\begin{cases} \{f\}_{x_j} = q_1^j(E)[f']_{x_j} + \frac{1}{b_1^j D_1^j(E) + d_1^j} \langle (A_1^j - E)^{-1} g_1^j, \theta_1^j \rangle \\ \{f'\}_{x_j} = -q_2^j(E)[f]_{x_j} + \frac{1}{a_2^j D_2^j(E) + c_2^j} \langle (A_2^j - E)^{-1} g_2^j, \theta_2^j \rangle \end{cases}, \tag{8}$$

where the functions q_i^j and D_i^j are given by $D_i^j(E) := \langle (A_i^j - E)^{-1} \theta_i^j, \theta_i^j \rangle$ and

$$q_1^j(E) := -\frac{a_1^j D_1^j(E) + c_1^j}{b_1^j D_1^j(E) + d_1^j}, \quad \text{and} \quad q_2^j(E) := \frac{b_2^j D_2^j(E) + d_2^j}{a_2^j D_2^j(E) + c_2^j}. \tag{9}$$

The solution f of the differential equation, where $E = -\chi^2$ with $\chi > 0$, contains $2L$ arbitrary parameters A_j and B_j

$$f(x) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\chi|x-y|}}{-2\chi} g(y)\mathrm{d}y + \sum_{j=1}^{L} (\mathrm{e}^{\chi|x-x_j|} A_j + \mathrm{sign}(x - x_j)\mathrm{e}^{\chi|x-x_j|} B_j).$$

Substituting this Ansatz into equations (8) yields the linear system

$$\begin{cases} (-\frac{1}{2\chi} + q_1^j(-\chi^2))A_j - \frac{1}{2\chi} \sum_{l \neq j} \mathrm{e}^{\chi|x_j - x_l|}(A_l + \mathrm{sign}\,(x_j - x_l)B_l) = h_1^j \\ (\frac{\chi}{2} + q_2^j(-\chi^2))B_j + \frac{1}{2} \sum_{l \neq j} \mathrm{e}^{\chi|x_j - x_l|}\big(\chi\,\mathrm{sign}(x_j - x_l)A_l + \chi B_l\big) = h_2^j \end{cases},$$

which is solvable for arbitrary $h_1^j, h_2^j \in \mathbb{C}$ and for some $E = -\chi^2$, since the nondiagonal terms vanish exponentially when $\chi \uparrow \infty$. Thus the range of $\mathbb{A} - E$ coincides with $\mathbb{H}$ for a certain E and $\mathbb{A}$ is self-adjoint. $\square$

In the course of the proof we have also shown that the resolvent of the operator $\mathbb{A}$ compressed to the space $L^2(\mathbb{R})$ coincides with the generalized resolvent calculated for the energy dependent boundary conditions (2) and rational functions $q_i^j(E)$ given by (9), that is

$$\{f\}_{x_j} = q_1^j(E)[f']_{x_j} \quad \text{and} \quad \{f'\}_{x_j} = -q_2^j(E)[f]_{x_j} \tag{10}$$

for $j = 1, \dots, L$. But also the converse holds true.

PROPOSITION 4. *Let for* $j = 1, 2, \dots, L$ *and* $i = 1, 2$ *functions* $q_i^j(E)$ *be given which are rational in E, symmetric with respect to the real axis, i.e.,* $q(\overline{E}) = \overline{q(E)}$. *Then there exists a model operator* $\mathbb{A}$ (*as in Definition* 2) *leading to the energy dependent boundary conditions* (10).

Proof. According to (9) the functions $D_i^j(E)$ have to be of the form

$$D_1^j(E) = -\frac{d_1^j q_1^j(E) + c_1^j}{b_1^j q_1^j(E) + a_1^j}, \quad D_2^j(E) = -\frac{c_2^j q_2^j(E) - d_2^j}{a_2^j q_2^j(E) - b_2^j}.$$

For every rational q_i^j the real parameters $a_i^j, b_i^j, c_i^j, d_i^j$ can be chosen in such a way that the functions $D_i^j(E)$ vanish at infinity and therefore can be written in the form (9) with a certain Pontryagin space $\mathcal{K}_i^j$, an operator A_i^j, and a vector θ_i^j. □

Next we are describing this model in detail for the reflectionless case.

4. Reflectionless Generalized Interactions

4.1. PONTRYAGIN SPACE MODEL FOR A SINGLE FACTOR

We first discuss the special case where the transmission coefficient is of the form $T(k) = (ik - \beta)/(ik + \beta)$ for some $\beta > 0$. Then the functions q_1 and q_2 in the boundary conditions (2) become $q_1(E) = -2\beta$ and $q_2(E) = -E/2\beta$. Hence a possible choice for the parameters is $a_1 = 0$, $b_1 = c_1 = 1$, $d_1 = 2\beta$ and $a_2 = 1$, $b_2 = c_2 = 0$, $d_2 = -1$ which gives $D_1(E) = 0$ and $D_2(E) = 2\beta/E$. The first boundary condition does not depend on the energy and hence the space $\mathcal{K}_1$ can be chosen trivially, $\mathcal{K}_1 = \{0\}$; the function D_2 can be realized as (9) in a one-dimensional Pontryagin space. Then the model operator $\mathbb{A}$ can be defined in the space $\mathbb{H} = L^2(\mathbb{R}) \oplus \mathbb{C} \ni \mathbf{f} = (f, f_2)^\top$ (where $[\mathbf{f}, \mathbf{g}] := (f, g)_{L^2} - f_2 \overline{g}_2$ denotes the indefinite inner product) by

$$\mathbb{A} : (f, f_2)^\top \mapsto (-f'', \{f'\}_{x_1} \sqrt{2\beta})^\top$$

on

$$\operatorname{dom} \mathbb{A} = \{(f, f_2)^\top \in \mathbb{H} : f \in W_2^2(\mathbb{R} \setminus \{x_1\}), [f']_{x_1} = -2\beta\{f\}_{x_1}, [f]_{x_1} = \sqrt{2\beta} f_2\}.$$

Then $\mathbb{A}$ has only one eigenvalue $E_1 = -\beta^2$, it has multiplicity 2 and the corresponding eigenspace is spanned by $\mathbf{f}_\mathrm{s}$ and $\mathbf{f}_\mathrm{a}$, having symmetric, respectively antisymmetric, with respect to x_1, components $f_\mathrm{s} = \mathrm{e}^{-\beta|x - x_1|}$ and $f_\mathrm{a} = \operatorname{sign}(x - x_1)\, \mathrm{e}^{-\beta|x - x_1|}$ and satisfying

$$[\mathbf{f}_\mathrm{s}, \mathbf{f}_\mathrm{s}] = \frac{1}{\beta} \quad \text{and} \quad [\mathbf{f}_\mathrm{a}, \mathbf{f}_\mathrm{a}] = -\frac{1}{\beta}.$$

Since $\beta > 0$ this means that the symmetric element $\mathbf{f}_s$ is a positive element in the Pontryagin space $\mathbb{H}$ and the antisymmetric element $\mathbf{f}_a$ is negative.

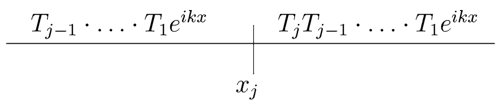
The position of the point x_1 cannot be determined from the scattering matrix and, in analogy with the inverse scattering method for potentials, one needs to introduce normalization constants.

For the eigenvalue $E_1 = -\beta^2$ of the operator $\mathbb{A}$ we define the corresponding normalizing constant $m > 0$ in the following way: consider the first component of the normalized positive eigenelement $1/\sqrt{[\mathbf{f}_s, \mathbf{f}_s]}f_s$, so that this function has the asymptotics $m\mathrm{e}^{-\beta x}$ for $x \to +\infty$. The coefficient m defined like this will be called the *normalizing constant*. In our case this is $m = \sqrt{\beta}\mathrm{e}^{\beta x_1}$. Hence here for given β there is a one-to-one correspondence between the normalizing constant m and the position x_1 of the interaction.

Summed up, in the reflectionless case with one factor as transmission coefficient and for given eigenvalue and normalizing constant we have solved the inverse scattering problem, that is we have constructed a Pontryagin space $\mathbb{H}$ and an operator $\mathbb{A}$ such that its scattering data (scattering matrix and normalization constant) coincide with the prescribed ones. A Hilbert space model can be obtained by reducing the operator $\mathbb{A}$ to a positive invariant subspace, but then the model is not local.

4.2. MODEL FOR SEVERAL FACTORS

In the general situation choose L points $x_1 < x_2 < \cdots < x_L$ and suppose that T is factored as $T = \prod_{j=1}^{L} T_j$, where $T_j(k) := \prod_{n=1}^{N_j}(ik - \beta_n^j)/(ik + \beta_n^j)$, with pairwise disjoint $\beta_1^j > \cdots > \beta_{N_j}$, so that each factor T_j corresponds to a transmission through the point x_j. This means that the scattering solution, which is e^{ikx} for $x < x_1$, locally at the point x_j is of the form:



Define the following functions like in (5) and (4)

$$q_1^j(E) = \frac{1}{2ik}\frac{T_j(k)+1}{T_j(k)-1} \quad \text{and} \quad q_2^j(E) = -\frac{ik}{2}\frac{T_j(k)+1}{T_j(k)-1}. \tag{11}$$

Then, by Proposition 4 there exists a generalized perturbation $\mathbb{A}$, leading to these energy dependent boundary conditons.

PROPOSITION 5. *For the generalized point interaction* $\mathbb{A}$ *the exit space* $\bigoplus_{j=1}^{L}(\mathcal{K}_1^j \oplus \mathcal{K}_2^j)$ *has dimension* $N := \sum_{j=1}^{L} N_j$.

Proof. We omit the index j within this proof since each point x_j can be treated separately. Inserting the functions T explicitly in (11) shows that q_1 can be written in the form $-q_1^{-1}(E) = 2\sum_{l=1}^{L}\beta_l + p_1(E)/q(E)$ where p_1 and q are polynomials with $\deg p_1 < \deg q = n$. The rational function $D_1(E) := p_1(E)/q(E)$ vanishes at ∞ and hence admits a minimal realization $D_1(E) = \langle (A_1 - E)^{-1}\theta_1, \theta_1 \rangle$ in some Pontryagin space $\mathcal{K}_1$ of dimension n with a self-adjoint operator A_1 and an element θ_1. Define

$$D_2(E) := -\frac{1}{q_2(E)} = -\frac{1}{E\, q_1(E)}. \tag{12}$$

Again the explicit analysis shows that for $N = 2n$, we have $D_2 = p_2/q$ with deg $p_2 < \deg q = n$, hence D_2 has a realization in a space $\mathcal{K}_2$ of dimension n. In the second case, $N = 2n+1$ we have $D_2(E) = p_3(E)/(E\, q(E))$, where $p_3(0) \neq 0$, hence D_2 admits a realization in a space $\mathcal{K}_2$ with dimension $n+1$. In both cases the dimensions add up to N. □

It is clear that the continuous spectrum of $\mathbb{A}$ is absolutely continuous and fills up the interval $[0, \infty)$. The following theorem describes the discrete spectrum of $\mathbb{A}$.

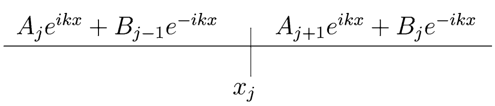
THEOREM 6. *Let* $\mathbb{A}$ *be the self-adjoint operator given in Proposition* 5. *Then its eigenvalues are the numbers* $E_n^j := -(\beta_n^j)^2$ *for* $n = 1, \dots, N_j$ *and* $j = 1, \dots, L$. *The eigenspaces are two-dimensional, indefinite, but nondegenerate.*

Proof. Consider the eigenvalue problem

$$\mathbb{A}\mathbf{f} = E\mathbf{f}, \tag{13}$$

where $\mathbf{f} = (f, (f_1^j, f_2^j)_{j=1}^L)^\top \in \operatorname{Dom} \mathbb{A}$.

Let us first discuss the case $E \in \mathbb{R} \setminus \{0\}$ and set $E =: k^2$. Hence the function f has to be of the form:



Then for $j = 1, \dots, L$ the boundary conditions (10) imply

$$A_{j+1} = T_j(k) A_j \quad \text{and} \quad B_{j-1} = T_j(k) B_j. \tag{14}$$

For $E > 0$ the condition $f \in L^2(\mathbb{R})$ implies $A_1 = A_{N+1} = B_0 = B_N = 0$, and for $E < 0$ without loss of generality we can assume $ik > 0$ and hence

$$A_{N+1} = B_0 = 0. \tag{15}$$

In both cases we see that the function f and hence also the element $\mathbf{f}$ is nontrivial if and only if ik equals one of the β_n^j's for $j = 1, \dots, L$ and $n = 1, \dots, N_j$, which is only possible if $E < 0$. Let us fix $ik = \beta_{n_0}^{j_0}$ and denote it for simplicity by β. From (15) it follows that $A_j = 0$ for $j > j_0$ and $B_j = 0$ for $j < j_0$. Since the β_n^j's are pairwise disjoint the corresponding eigenspace is two-dimensional.

What is left is to calculate the inner product $[\mathbf{f}, \mathbf{f}]$ in the space $\mathbb{H}$. For the internal components u_i^j equation (13) implies

$$f_1^j = -(a_1^j [f']_{x_j} + b_1^j \{f\}_{x_j})(A_1^j + \beta^2)^{-1}\theta_1^j = -\{f\}_{x_j}(A_1^j + \beta^2)^{-1}\theta_1^j$$
$$f_2^j = -(a_2^j \{f'\}_{x_j} + b_2^j [f]_{x_j})(A_2^j + \beta^2)^{-1}\theta_2^j = -\{f'\}_{x_j}(A_2^j + \beta^2)^{-1}\theta_2^j.$$

Note $-\beta^2 \in \varrho(A_i^j)$. Indeed, the eigenvalues of A_i^j are the poles of D_i^j and hence the zeros of q_i^j. But it can be seen that the functions q_i^j do not vanish on the negative real line. Then for $j < j_0$ it holds

$$\begin{aligned}\langle u_1^j, u_1^j\rangle + \langle u_2^j, u_2^j\rangle &= \frac{1}{4}(A_{j+1} + A_j)^2 e^{2\beta x_j} \times \\ &\quad \times \big(\langle (A_1^j + \beta^2)^{-2}\theta_1^j, \theta_1^j\rangle + \beta^2 \langle (A_2^j + \beta^2)^{-2}\theta_2^j, \theta_2^j\rangle\big) \\ &= \frac{1}{4}(A_{j+1} + A_j)^2 e^{2\beta x_j}\big(D_1^{j\prime}(-\beta^2) + \beta^2 D_2^{j\prime}(-\beta^2)\big)\end{aligned}$$

and likewise for $j > j_0$

$$\langle u_1^j, u_1^j\rangle + \langle u_2^j, u_2^j\rangle = \frac{1}{4}(B_{j-1} + B_j)^2 e^{-2\beta x_j}\big(D_1^{j\prime}(-\beta^2) + \beta^2 D_2^{j\prime}(-\beta^2)\big).$$

Here we have used the identity

$$\langle (A_i^j - E)^{-2}\theta_i^j, \theta_i^j\rangle = \frac{d}{dE} D_i^j(E) = D_i^{j\prime}(E).$$

Relations (12) and (4) imply $D_1^{j\prime}(E) - E\, D_2^{j\prime}(E) = -\frac{1}{E\, q_1^j(E)}$, and hence

$$\begin{aligned}[\mathbf{f}, \mathbf{f}] = \frac{1}{2\beta}\Bigg[&\sum_{j=1}^{j_0-1}\Big(A_j^2 - A_{j+1}^2 - (A_{j+1} + A_j)^2 \frac{T_j(\beta) - 1}{T_j(\beta) + 1}\Big) e^{2\beta x_j} + \\ &+ \sum_{j=j_0+1}^{L}\Big(B_j^2 - B_{j-1}^2 - (B_{j-1} + B_j)^2 \frac{T_j(\beta) - 1}{T_j(\beta) + 1}\Big) e^{-2\beta x_j}\Bigg] + \\ &+ \frac{1}{2\beta}\big(A_{j_0}^2 e^{2\beta x_{j_0}} + B_{j_0}^2 e^{-2\beta x_{j_0}}\big) + \langle u_1^{j_0}, u_1^{j_0}\rangle + \langle u_2^{j_0}, u_2^{j_0}\rangle,\end{aligned}$$

here the summands in the square brackets vanish by (14).

In what follows we focus on two particular eigenelements for which the external component f locally at x_{j_0} is symmetric (and antisymmetric) with respect to the point x_{j_0}. Denote by $\mathbf{f}_s$ the symmetric (and $\mathbf{f}_a$ the antisymmetric) element, i.e., $A_{j_0} = e^{-\beta x_{j_0}}$, $B_{j_0} = e^{\beta x_{j_0}}$ (and $A_{j_0} = -e^{-\beta x_{j_0}}$, $B_{j_0} = e^{\beta x_{j_0}}$, respectively). Observing the identity

$$\frac{q_1^{j\prime}}{(q_1^j)^2}(-k^2) = \frac{1}{k}\frac{\tau_j(k) - 1}{\tau_j(k) + 1} + \frac{2\tau_j'(k)}{(\tau_j(k) + 1)^2}$$

with the real functions $\tau_j(k) := T_j(k/i)$ it follows

$$
\begin{aligned}
[\mathbf{f}_\mathrm{s}, \mathbf{f}_\mathrm{s}] &= \frac{1}{\beta} + \frac{q_1^{j\,\prime}}{(q_1^j)^2}(-\beta^2) = 2\tau_{j_0}'(\beta) \\
[\mathbf{f}_\mathrm{a}, \mathbf{f}_\mathrm{a}] &= \frac{1}{\beta} - \frac{q_1^{j\,\prime}}{(q_1^j)^2}(-\beta^2) - \frac{1}{\beta^2 q_1^{j_0}(-\beta^2)} = -2\tau_{j_0}'(\beta).
\end{aligned}
\tag{16}
$$

Hence for each eigenvalue either the symmetric or the antisymmetric element is positive (negative, respectively). Moreover, a similar calculation yields that they are orthogonal, i.e., $[\mathbf{f}_\mathrm{s}, \mathbf{f}_\mathrm{a}] = 0$. Hence the eigenspace is nondegenerate.

In the same way one can treat $E = 0$ and obtain that it cannot be an eigenvalue of $\mathbb{A}$. Since the dimension of the extension space is $N = \sum_{j=1}^{L} N_j$ it follows that the number of negative squares (and hence also the dimension of every non-positive subspace) of the Pontryagin space $\mathbb{H}$ is bounded by N. But we have shown that there exists an N-dimensional negative subspace spanned by eigenelements for real eigenvalues. Thus there cannot be a non-real eigenvalue, since its eigenelement had to be neutral. □

Equation (16) also implies that for each point x_j, the symmetric and antisymmetric eigenfunctions interchange their type so that the symmetric eigenfunction corresponding to the lowest eigenvalue $-(\beta_1^j)^2$ is always of positive type.

5. Inverse Problem for Reflectionless Operator Extensions

5.1. INVERSE PROBLEM: EXISTENCE OF THE SOLUTION

In the model described – following the idea in Section 4.1 – we define normalizing constants $m(\beta_j, x_j)$ in the following way:

DEFINITION 7. Let $E = -\beta^2$ be an eigenvalue of the operator $\mathbb{A}$. Denote by $\mathbf{f}$ the symmetric eigenelement if it is positive (otherwise take the antisymmetric). We call a number $m > 0$ *normalizing constant* related to the eigenvalue $E = -\beta^2$, if the external (i.e., the first) component of $\frac{1}{\sqrt{[\mathbf{f},\mathbf{f}]}}\mathbf{f}$ may be chosen equal to $m\mathrm{e}^{-\beta x}$ for $x > x_L$.

For given scattering data $(E_i, m_i)_{i=1}^{n}$ it is not immediately clear how to reconstruct the positions of the interactions. In fact, here a normalizing constant does not only depend on the corresponding point, but also on its position with respect to the other points. We will show that the inverse scattering problem can always be solved, however its solution need not be unique.

Let us first consider single points, i.e., at the point x_j there is the interaction according to the transmission coefficient $T_j(k) = (ik - \beta_j)/(ik + \beta_j)$. With (14) and (16) one finds that in this case the normalizing constants are given by:

$$m(\beta_j, x_j) = \sqrt{\beta_j} \prod_{x_n > x_j} \left| \frac{\beta_j + \beta_n}{\beta_j - \beta_n} \right| \mathrm{e}^{\beta_j x_j}.$$

Denote $y(\beta, m) := 1/\beta \ln(m/\sqrt{\beta})$, which gives the position of a single basic interaction with the energy $E = -\beta^2$ and normalizing constant m. The following theorem gives an answer to the inverse scattering problem, and in addition describes one of the solutions more precisely.

THEOREM 8. *For any scattering data consisting of the eigenvalues* $E_1 < E_2 < \cdots < E_n$ *and the corresponding normalizing constants* $m_j \in \mathbb{R}^+$ *for* $j = 1, \ldots, n$ *there exist distinct real points* $x_1, \ldots, x_n$ *such that* $m_j = m(\beta_j, x_j)$ *and* $x_j \leqslant \max_{i=1,\ldots,n} y_i$, *where* $y_i := y(\beta_i, m_i)$.

Proof. We use the fact that changing the model by adding a single interaction to the right of the other points changes the normalizing constants only by a factor, which is less than 1 and then we use induction. For $n = 1$ the point x_1 equals the point y_1. So let us assume that the claim already holds for $n - 1$. By N denote an index for which $y_N = \max_{i=1,\ldots,n} y_i$ and define the numbers $\widetilde{m}_j := \left|(\beta_j - \beta_N)/(\beta_j + \beta_N)\right| m_j$. Then according to the assumption for the scattering data $(E_j, \widetilde{m}_j)_{j \in \{1,\ldots,n\} \setminus \{N\}}$ there exist distinct points x_j for $j \in \{1, \ldots, n\} \setminus \{N\}$ such that $\widetilde{m}_j = m(\beta_j, x_j)$. We show that the points x_j for $j \in \{1, \ldots, n\} \setminus \{N\}$ and $x_N := y_N$ give a solution to the problem. Indeed, by assumption and since $y(E, m)$ is strictly increasing in m, it holds

$$x_j \leqslant \max_{j \in \{1,\ldots,n\} \setminus \{N\}} y(E_j, \widetilde{m}_j) < \max_{j \in \{1,\ldots,n\} \setminus \{N\}} y(E_j, m_j) \leqslant y_N.$$

Hence the points are distinct and the additional inequality is satisfied. Moreover, $m(\beta_j, x_j) = \left|(\beta_j + \beta_N)/(\beta_j - \beta_N)\right| \widetilde{m}_j = m_j$, which finishes the proof. □

In the proof only one possible choice for the positions x_j is given. Although for some scattering data it is unique, it can also happen that different configurations of the x_j's lead to the same scattering data. This phenomenon occurs when the singular points are 'close' and is described in the following subsection.

5.2. INVERSE PROBLEM: NON-UNIQUENESS

In this section we consider the time-dependent scattering data

$$\beta_i(t) = \beta_i(0), \quad m_i(t) = m_i(0) \mathrm{e}^{4\beta_i^3 t} \tag{17}$$

and analyze the corresponding model operator $\mathbb{A}$ and its time evolution.

Remark 9. Note that, with reflection coefficients $R_i(t) = 0$ for every fixed t the inverse scattering method gives for the scattering data (17) a potential $V(x, t)$. This function is a soliton of the Korteweg-de Vries equation.

Our consideration can easily be generalized to an arbitrary number of solitons but we prefer to restrict ourselves to the case of two in order to make the presentation more transparent.

The operator $\mathbb{A}$ can be constructed with two (may be coinciding) singular points x_1 and x_2. We recall the three possibilities that can occur. Without loss of generality suppose $E_1 < E_2$, with $E_j = -\beta_j^2$. Denote the corresponding singular points by x_1 and x_2, or X_1 and X_2. In the case of a double singular point we denote its coordinate by x_0.

1. **Double point**: x_0 and the corresponding transmission coefficient is

$$T(k) = \frac{ik - \beta_1}{ik + \beta_1} \frac{ik - \beta_2}{ik + \beta_2}.$$

Then the symmetric eigenelement $\mathbf{f}_s$ corresponding to β_1 is positive and the normalizing constant is

$$m_1 = \sqrt{\beta_1 D_0} e^{\beta_1 x_0}, \quad \text{with } D_0 = \frac{\beta_1 + \beta_2}{\beta_1 - \beta_2}.$$

The corresponding antisymmetric eigenfunction for $E_2 = -\beta_2^2$ has the normalizing constant

$$m_2 = \sqrt{\beta_2 D_0} e^{\beta_2 x_0}.$$

2. **Two single points**: $x_1 < x_2$. The positive eigenfunctions are locally symmetric (near the corresponding singular points) and the normalizing constants are given by:

$$m_1 = \sqrt{\beta_1} D_0 e^{\beta_1 x_1} \quad \text{and} \quad m_2 = \sqrt{\beta_2} e^{\beta_2 x_2}.$$

3. **Two single points**: $X_1 > X_2$. Again the eigenfunctions are locally symmetric, but the normalizing constants are given by:

$$m_1 = \sqrt{\beta_1} e^{\beta_1 X_1} \quad \text{and} \quad m_2 = \sqrt{\beta_2} D_0 e^{\beta_2 X_2}.$$

In order to reconstruct the model, one has in particular to reconstruct the singular points from the scattering data. Note that in (17) only the normalizing constants depend on the time, hence in the model only the positions of the singular points are time-dependent.

For $t \ll -1$ the second model has to be used and hence it holds

$$x_1(t) = \frac{1}{\beta_1} \ln \frac{m_1(0)}{\sqrt{\beta_1} D_0} + 4\beta_1^2 t \quad \text{and} \quad x_2(t) = \frac{1}{\beta_2} \ln \frac{m_2(0)}{\sqrt{\beta_2}} + 4\beta_2^2 t.$$

Here the precise upper bound t_B, for which these formulas are allowed is determined by the condition $x_1(t) < x_2(t)$.

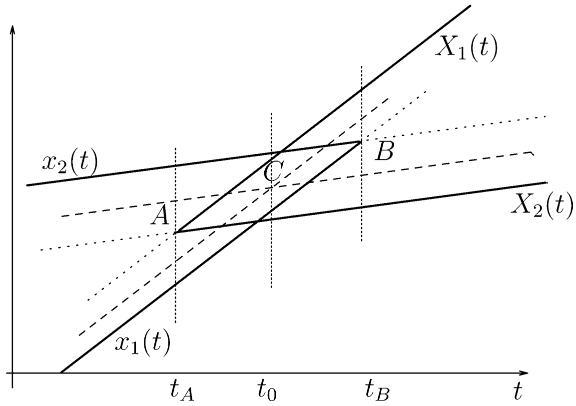


Figure 1.

For $t \gg 1$ the third model has to be used, which gives

$$X_1(t) = \frac{1}{\beta_1} \ln \frac{m_1(0)}{\sqrt{\beta_1}} + 4\beta_1^2 t \quad \text{and} \quad X_2(t) = \frac{1}{\beta_2} \ln \frac{m_2(0)}{\sqrt{\beta_2} D_0} + 4\beta_2^2 t.$$

In Fig. 1 the time dependence is illustrated.

For $t < t_A$ the points have to move along the lines of x_1 and x_2, since the inequality $X_2 < X_1$ is not satisfied. Since x_1, the point to the left, is moving faster, their distance is getting smaller. For $t_A < t < t_B$ both inequalities are satisfied and hence both cases (two and three) are possible (note $D_0 > 1$). For $t > t_B$ the points have to be on the lines X_1 and X_2. So somewhere inbetween they have to change the lines, and therefore they have to jump, it is impossible to find a continous movement of the singular points.

Acknowledgement

The author, Annemarie Luger gratefully acknowledges the support of the "Fond zur Förderung der wissenschaftlichen Forschung" (FWF, Austria), grant number P15540-N05.

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