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On Field Theory Methods in Singular Perturbation Theory

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Abstract. Singular and supersingular finite rank perturbations of self-adjoint operators are studied using methods from renormalization theory for quantum fields. It is shown that the ideas from dimensional and Pauli–Villars regulatizations can be applied to determine uniquely certain finite rank supersingular perturbations. Approach is based on the regularization of homogeneous singular quadratic forms.

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1. Introduction

In the quantum field theory one needs to take care of formal expressions containing diverging integrals. To handle such expressions one uses the renormalization theory [9] which is the key point of the field theory. The renormalization has to be performed taking into account physical considerations and invariant properties of the theory. A similar problem appears in singular perturbation theory of self-adjoint operators when the perturbed operator is defined using a formal sum of two not subordinated quadratic forms. It appears that one can go beyond the traditional perturbation theory if additional properties of the Hamiltonians like scaling and translational invariance are considered. To determine the perturbed operator, one needs to perform certain renormalizations similar to one used in field theory. This relation was first noticed by F. A. Berezin and L. D. Faddeev who used this analogy to give rigorous mathematical definition for Schrödinger operator with point interaction in $L^2(\mathbf{R}^3)$ [8]. This operator is formally given by the following expression

$$L_\alpha = -\Delta + \alpha\delta \equiv -\Delta + \alpha\langle\delta, \cdot\rangle\delta, \quad \text{in } L^2(\mathbf{R}^3), \tag{1}$$

where $-\Delta$ is Laplace operator and δ is Dirac's delta function. The perturbed operator L_α is one of the self-adjoint extensions of the operator $-\Delta$ restricted to the set of functions vanishing at the origin. All these extensions are described by one real parameter, say γ. But it was not clear what is the exact relation between γ and the coupling parameter $\alpha \in \mathbf{R}$ appearing in (1). It was noticed later that this relation can be established if one takes into account the scaling properties of the Laplace operator and delta function [4]. The main ingredient of this approach is the regularization of homogeneous quadratic forms which leads to the results coinciding with those from the dimensional and Pauli–Villars regularizations [19, 29].

The main subject of this Letter is to apply the methods from quantum field theory to studies of supersingular perturbations of self-adjoint operators. Supersingular perturbations are given by quadratic forms with the domain strictly smaller than the domain of the original operator. Such finite rank perturbations are determined by functionals which are not bounded on the domain of the original operator like it is for L_α. As an example, we consider point perturbations of Laplace operator in $L^2(\mathbf{R}^d)$, $d \geqslant 4$. To determine this operator, one needs to regularize a certain singular homogeneous quadratic form. This regularization in general contains several arbitrary parameters. We show how this regularization can be made unique if one takes into account the scaling properties of the operator and its supersingular perturbation.

The Letter is organized as follows. The approach of Berezin and Faddeev and its generalizations are described in detail in Section 2. Applications to the theory of rank-1 singular perturbations are given. The third section is devoted to renormalization procedures in quantum field theories. The relation between supersingular perturbations of self-adjoint operators and quadratic forms is clarified in Section 4. The last section is devoted to the renormalization of homogeneous quadratic forms and is the central part of the Letter. Necessary conditions for one to be able to extend a homogeneous quadratic form preserving the homogeneity properties is described by Theorem 1. The developed methods are applied to study the Schrödinger operator with a delta potential in $L^2(\mathbf{R}^d), d \geqslant 4$.

2. Singular Perturbations and Renormalization of Homogeneous Integrals

The renormalization problem appears in the operator theory for example when one tries to give a mathematically rigorous definition for the following formal operator:

$$A_\alpha = A + \alpha\langle\varphi, \cdot\rangle\varphi, \tag{2}$$

where A is a positive self-adjoint operator* acting in the Hilbert space $\mathcal{H}$, α is a real coupling constant, and φ is a certain singular vector. If the vector φ belongs to the Hilbert space or is a bounded linear functional on the domain $\mathcal{H}_1$ of the quadratic form of the operator A (so-called $\mathcal{H}_0$ and $\mathcal{H}_{-1}$-perturbations respectively), then

*The case when the operator A is not necessarily semibounded can be studied using similar methods [3, 5].

the perturbation term is form-bounded with respect to the quadratic form of the operator A and the perturbed operator A_α can be defined using standard perturbation theory (e.g. the KLMN theorem from [35]). The resolvent of the perturbed operator is given by

$$\frac{1}{A_\alpha - \lambda} = \frac{1}{A-\lambda} - \frac{1}{1/\alpha + \langle \varphi, \frac{1}{A-\lambda}\varphi \rangle} \left\langle \frac{1}{A-\bar{\lambda}}\varphi, \cdot \right\rangle \frac{1}{A-\lambda}\varphi, \quad \lambda \notin \mathbf{R}. \tag{3}$$

One can consider the formal expression (2) for more singular perturbations determined by bounded linear functionals on the domain $\mathcal{H}_2$ of the operators A (so-called $\mathcal{H}_{-2}$-perturbations). The corresponding perturbed operator is not defined uniquely by (2), since the perturbation term is not form-bounded with respect to the operator A [4, 8, 20, 38]. The easiest way to understand this is by analyzing the resolvent formula (3). If φ is not a bounded functional on the domain of the quadratic form of A, then the scalar product $\langle \varphi, (1/(A-\lambda))\varphi \rangle$ appearing in the denominator is not well-defined. In other words, the corresponding integral is diverging. One needs to renormalize this quadratic form by substracting its value at a certain point, say $\lambda = -1$, since the expression

$$\left\langle \varphi, \frac{1}{A-\lambda}\varphi \right\rangle - \left\langle \varphi, \frac{1}{A+1}\varphi \right\rangle \overset{\text{formally}}{=} \left\langle \varphi, \frac{1+\lambda}{A-\lambda}\frac{1}{A+1}\varphi \right\rangle, \tag{4}$$

is well-defined for φ being a bounded linear functional on the domain of the operator A. In that case $(1/(A-\lambda)), \varphi$ belongs to the original Hilbert space $\mathcal{H}$. Formally putting $\langle \varphi, (1/(A+1)), \varphi \rangle = c$, where c is a certain real renormalization parameter, we obtain the renormalized value of the function for all nonreal values of λ

$$\left\langle \varphi, \frac{1}{A-\lambda}\varphi \right\rangle = c + \left\langle \varphi, \frac{1-\lambda}{A-\lambda}\frac{1}{A+1}\varphi \right\rangle. \tag{5}$$

Substituting this renormalized function into (3) we get, for $\lambda \notin \mathbf{R}$,

$$\frac{1}{A_\alpha - \lambda} = \frac{1}{A-\lambda} - \frac{1}{1/\alpha + c + \left\langle \varphi, \frac{1+\lambda}{A-\lambda}\frac{1}{A+1}\varphi \right\rangle} \left\langle \frac{1}{A-\bar{\lambda}}\varphi, \cdot \right\rangle \frac{1}{A-\lambda}\varphi. \tag{6}$$

This formula for each $\alpha \in \mathbf{R}$ determines a one-parameter family of self-adjoint operators in $\mathcal{H}$. Each of these operators is a self-adjoint extension of the operator A restricted to the set of functions u orthogonal to φ: $\langle \varphi, u \rangle = 0$. The restricted operator is a symmetric operator with deficiency indices $(1,1)$ and the domain of the adjoint operator A^{0*} consists of all functions possessing the representation $U = \tilde{u} + u_1(1/(A+1))\varphi$, $\tilde{u} \in \mathrm{Dom}(A), u_1 \in \mathbf{C}$. The self-adjoint extensions of A^0 can be described by one real parameter $\gamma \in \mathbf{R} \cup \infty$ as restriction of A^{0*} to the domain of functions satisfying the boundary condition: $\langle \varphi, \tilde{u} \rangle = \gamma u_1$. Formula (3) describes all such extension if one takes: $\gamma = -(\alpha^{-1} + c)$.

The real renormalization parameter c establishes the relation between the coupling parameter α in (2) and the extension parameter γ. In the general situation, the renormalization parameter c is not defined and can be arbitrarily chosen among real numbers. Hence, the exact relation between α and γ is not defined. To establish this relation, or to calculate the renormalization parameter c, one needs to use additional assumptions. In many applications appearing in mathematical physics, the operator A and the singular vector φ are homogeneous with respect to a certain unitary semigroup $U(s)$

$$U(s)\varphi = s^{\theta}\varphi, \qquad U(s)A = s^{-\beta}AU(s). \tag{7}$$

It was shown in [4] that the unique value of the renormalization constant leading to quadratic forms preserving the homogeneity properties is given by the following function:

$$c = -\frac{1-t^{-\beta}}{1-t^{\beta-2\theta}}\left\langle \frac{1}{A+1}\frac{1}{A+t^{-\beta}}\varphi, \varphi \right\rangle. \tag{8}$$

If the function appearing in (8) does not depend on the parameter t, then this formula determines the unique value of the renormalization parameter c leading to natural scaling properties of the quadratic form.

Similar renormalization problem appears in the field theory where one needs to renormalize diverging integrals. This idea is similar to the renormalization of diverging integrals using scaling and translation invariance, which is explained in more details in the following section. This relation between the two approaches was first pointed out by Faddeev [18].

The relation between these two theories is extremely transparent if one considers the formal operator (1). This is the most widely used example of $\mathcal{H}_{-2}$-perturbations of self-adjoint operators. The quadratic form appearing in the denominator of the formal resolvent can be written using Fourier transform as follows:

$$\left\langle \delta, \frac{1}{-\Delta-\lambda}\delta \right\rangle = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3}\frac{1}{|k|^2-\lambda}\mathrm{d}^3k = \frac{1}{2\pi^2}\int_0^\infty \frac{k^2}{k^2-\lambda}\mathrm{d}k. \tag{9}$$

Similar integrals like (17) appear in different problems of quantum field theory in particular during renormalization of the vacuum expectations of the energy-momentum tensor [9] (see the following section for details). This relation was first notices by Berezin and Faddeev [8], who used this analogy to construct approximations in the strong resolvent sense for the operators from the family corresponding to (1). This method was generalized to include arbitrary $\mathcal{H}_{-2}$-perturbations in [20]. To determine the unique renormalized Nevanlinna function (9), consider the unitary semigroup of scaling transformations

$$S(s)u(x) = s^{3/2}u(sx). \tag{10}$$

Then the homogeneity parameters for the Laplace operator and delta function in $L^2(\mathbf{R}^3)$ are $\beta = 2$, $\theta = -3/2$. The corresponding function in (8) is constant and the parameter c must be taken equal $-(1/4\pi)$. This condition determines the unique self-adjoint operator corresponding to the formal expression (1). The resolvent of this operator is given by

$$\frac{1}{L_\alpha - \lambda} = \frac{1}{L - \lambda} - \frac{1}{\frac{1}{\alpha} + \frac{ik}{4\pi}} \left\langle \frac{e^{-i\bar{k}|x|}}{4\pi|x|}, \cdot \right\rangle \frac{e^{ik|x|}}{4\pi|x|}, \quad k = \sqrt{\lambda},\ \Im\lambda \neq 0. \tag{11}$$

This relation plays an important role in the theory of point interaction [2, 7, 12, 30].

3. Regularization and Renormalization in Quantum Field Theory

One of the main objects in quantum field theory is the vacuum expectation of a product of field operators

$$\langle 0|\varphi(x_1)\varphi(x_2)\cdots\varphi(x_l)|0\rangle, \tag{12}$$

where $|0\rangle$ denotes the vacuum state for the quantized field $\varphi(x)$. Since $\varphi(x)$ is an operator-valued generalized function, expression (12) is not rigorously defined when the arguments x_k and x_n coincide.

Consider the simplest example of noninteracting scalar field in N-dimensional Minkowski spacetime with the mass m and following the equation of motion

$$(g^{ik}\partial_i\partial_k + m^2)\varphi(x) = 0. \tag{13}$$

Then the energy-momentum tensor has the following form

$$T_{ij} = \partial_i\varphi^*\partial_j\varphi + \partial_j\varphi^*\partial_i\varphi - g_{ij}\big(g^{kn}\partial_k\varphi^*\partial_n\varphi - m^2\varphi^*\varphi\big), \tag{14}$$

where

$$i, j, k, n = 0, 1, \ldots, N-1, \qquad g_{ij} = g^{ij} = \operatorname{diag} 1, -1, \ldots, -1)$$

and we use Einstein convention to perform summation over the repeated indices. Then the vacuum expectations of the energy-momentum tensor are given, at least formally, by diverging integrals

$$\langle 0|T_{ij}|0\rangle = \frac{1}{(2\pi)^{N-1}} \int_{\mathbf{R}^{N-1}} \frac{k_i k_j}{\left(m^2 + \sum_{l=1}^{N-1} k_l^2\right)^{1/2}} \, d^{N-1}k, \tag{15}$$

where $d^{N-1}k = dk_1 \cdots dk_{N-1}$. In the case $i = 0$ or $j = 0$, we use the notation $k_0^2 = m^2 + \sum_{l=1}^{N-1} k_l^2$. In quantum field theory, one uses different methods to overcome this divergence, see review [9].

One of the most effective methods is the method of dimensional regularization, proposed by G. 't Hooft and M. Veltman [19] (see also [11] and [39]). They suggested considering formal analytic continuation of the diverging integrals with respect to the dimension of the spacetime making this parameter complex. Rewriting the integral (15) formally in spherical coordinates and integrating over the angle variables, we get the following diverging integrals:

$$\begin{aligned}
\langle 0|T_{00}|0\rangle &= \frac{1}{2^{N-2}\pi^{\frac{N-1}{2}}\Gamma\left(\frac{N-1}{2}\right)}\int_0^\infty r^{N-2}(r^2+m^2)^{1/2}\,\mathrm{d}r;\\
\langle 0|T_{\alpha\beta}|0\rangle &= \frac{\delta_{\alpha\beta}}{2^{N-2}\pi^{\frac{N-1}{2}}(N-1)\Gamma\left(\frac{N-1}{2}\right)}\int_0^\infty \frac{r^N}{(r^2+m^2)^{1/2}}\,\mathrm{d}r;\\
\langle 0|T_{0\alpha}|0\rangle &= 0,
\end{aligned} \tag{16}$$

where $\alpha, \beta = 1, 2, \ldots, N-1$, and Γ denotes the gamma function. To 'calculate' these diverging integrals in accordance with the method of dimensional regularization, let us substitute the dimension parameter N with the complex number $N-2\epsilon$. Then we can use the following identity:

$$\int_0^\infty x^k\,(1+x^2)^{-p}\,\mathrm{d}x = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(p-\frac{k+1}{2}\right)}{2\,\Gamma(p)} \tag{17}$$

valid at least for $-1 < \Re k < 2\Re p - 1$ to determine the integrals

$$\langle 0|T_{ij}|0\rangle = -\frac{m^{N-2\epsilon}}{(4\pi)^{N/2-\epsilon}}\Gamma\left(\epsilon-\frac{N}{2}\right)g_{ik}. \tag{18}$$

To determine integrals (15), one needs to consider the limit $\epsilon \to 0$. This limit is finite for odd values of N and diverges for even values of N. Therefore, one can use this method to calculate diverging integrals for space dimensions $2, 4, \ldots$ For odd space dimensions, one needs to use additional subtractions corresponding to infinite renormalizations of bare Lagrangians [9]. Similar calculations for the curved spacetime are described for example in [33].

4. Supersingular Perturbations and Quadratic Forms

In this section, we return back to the singular perturbation theory. Formal expression (2) can be considered, even in the case when φ does not belong to the space $\mathcal{H}_{-2}$, i.e. is not a bounded linear functional on the domain of the original operator A. Such perturbations are called *supersingular*. To study such perturbations, it is convenient to introduce the scale of Hilbert spaces associated with the positive operator A

$$\begin{array}{ccccc}
 & \mathrm{Dom}(A) & \mathcal{H} & (\mathrm{Dom}(A)) & \\
 & \| & \| & \| & \\
\ldots \subset \mathcal{H}_3 \subset \mathcal{H}_2 \subset & \mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}_{-1} \subset & \mathcal{H}_{-2} \subset \mathcal{H}_{-3} \subset \ldots
\end{array} \tag{19}$$

The norm in each space $\mathcal{H}_s$ is defined by

$$\| U \|^2_{\mathcal{H}_s} = \langle U, (A+1)^s U\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in the original Hilbert space $\mathcal{H}$. The spaces $\mathcal{H}_n$ and $\mathcal{H}_{-n}$ are adjoint to each other with respect to the scalar product in the original Hilbert space $\mathcal{H}$: $\mathcal{H}_n^* = \mathcal{H}_{-n}$. We say that the interaction is from the class $\mathcal{H}_{-n}$ if and only if $\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1}$.

It is natural to associate with the formal expression (2) the following formal resolvent

$$\frac{1}{A_\alpha - \lambda} = \frac{1}{A - \lambda} - \frac{1}{Q_n(\lambda)} \left\langle \frac{1}{A - \bar{\lambda}} \varphi, \cdot \right\rangle \frac{1}{A - \lambda} \varphi, \quad \lambda \notin \mathbf{R}, \tag{20}$$

where $Q_n(\lambda)$ is the renormalized bordered resolvent

$$Q_n(\lambda) = 1/\alpha + \left\langle \varphi, \frac{1}{A - \lambda} \varphi \right\rangle. \tag{21}$$

The expression on the right-hand side of this formula is not defined if $\varphi \notin \mathcal{H}_{-1}$ and one needs to regularize it like in the case $\varphi \in \mathcal{H}_{-2} \backslash \mathcal{H}_{-1}$. If $\varphi \in \mathcal{H}_{-n} \backslash \mathcal{H}_{-n+1}$ then the regularization contains several arbitrary parameters. During this regularization, the formal scalar product is substituted with the expression of the form

$$1/\alpha + \left\langle \varphi, \frac{1}{A - \lambda} \varphi \right\rangle \to p(\lambda) + \left\langle \varphi, \frac{1}{A - \lambda} \frac{q(\lambda)}{q(A)} \varphi \right\rangle, \tag{22}$$

where p and q are polynomials and q has only negative real zeroes. See [14], where this procedure is described in more details. After this regularization, one can see that (20) determines a bounded linear operator acting between $\mathcal{H}_{n-2}$ and $\mathcal{H}_{-n+2}$. It is possible to obtain a self-adjoint operator leading to this resolvent formula. Two approaches have been developed. In [15, 36, 37] an approach involving operators in Pontryagin extensions of the original Hilbert space was suggested. This approach was motivated by the fact that the regularized function Q_n is a generalized Nevanlinna function appearing naturally during the studies of self-adjoint extensions in Pontryagin spaces [21, 22, 23, 28]. Similar methods were used in [13, 16, 17, 31, 32, 34]. Another approach using extended Hilbert spaces only was suggested in [25–27]. Then the resolvent formula (20) appears as a restriction of the resolvent of the operator in the extended space to the original Hilbert space combined with a certain natural embedding.

It is not our aim to repeat this construction here, instead we are going to discuss the renormalization of the singular quadratic form $\langle \varphi, (1/(A-\lambda))\varphi \rangle$ in the case where the operator and the perturbation are homogeneous with respect to the unitary semigroup. It appears that the requirement that the regularized quadratic form possesses the same homogeneity properties allows one to make the regularization process unique. This approach is described in the following section.

5. Renormalization of Homogeneous Quadratic Forms

Let us consider the problem of defining the quadratic form $q_\lambda[\varphi, \varphi] = \langle \varphi, (1/(A-\lambda))\varphi \rangle$ for singular vectors $\varphi \in \mathcal{H}_{-n} \backslash \mathcal{H}_{-n+1}$. In the case $n \geqslant 2$, this quadratic form is not defined, since

$$\frac{1}{A - \lambda} \varphi \in \mathcal{H}_{-n+2} \backslash \mathcal{H}_{-n+3},$$

but φ is a bounded linear functional on $\mathcal{H}_n$ only. To define this quadratic form, one needs to extend the linear bounded functional φ from the space $\mathcal{H}_n$ to a larger linear space, which includes elements like $(1/(A-\lambda))\varphi \subset \mathcal{H}_{-n+2}$. Observe that one does not need to extend φ to the whole space $\mathcal{H}_{-n+2}$: functions $(1/(A-\lambda))\varphi$, $\lambda \in \mathbf{C}\setminus\mathbf{R}_+$ span a finite-dimensional extension of the space $\mathcal{H}_n$. More precisely:

LEMMA 1. *Consider $n-1$ distinct points $a_1, a_2, \ldots, a_{n-1} \in \mathbf{R}_+$. Then for any $\lambda \notin \mathbf{R}_+$, the following representation holds*

$$\frac{1}{A-\lambda}\varphi = \sum_{j=1}^{n-1} u_j \frac{1}{A+a_j}\varphi + U, \tag{23}$$

where $u_j \in \mathbf{C}$, $U \in \mathcal{H}_n$.

Proof. Applying Hilbert identity

$$\frac{1}{A-\lambda} - \frac{1}{A-\mu} = (\lambda-\mu)\frac{1}{A-\lambda}\frac{1}{A-\mu}$$

n times one can prove that every element $\big(1/(A-\lambda)\big)\varphi$ possesses the representation

$$\frac{1}{A-\lambda}\varphi = \sum_{j=1}^{n-1} w_j \frac{1}{A+a_j}\frac{1}{A+a_{j-1}}\cdots\frac{1}{A+a_1}\varphi + W,$$

where $w_j \in \mathbf{C}$, $W \in \mathcal{H}_n$. The last representation is equivalent to (23). □

Let us introduce the Hilbert space $\mathbf{H} = \mathbf{C}^{n-1} \oplus \mathcal{H}_{n-2}$ the finite-dimensional extension of the space $\mathcal{H}_{n-2}$ together with the natural embedding

$$\begin{aligned} \rho: \qquad & \mathbf{H} \to \mathcal{H}_{-n+2} \\ & (u_1, u_2, \ldots, u_{n-1}, U) \mapsto \sum_{j=1}^{n-1} u_j \frac{1}{A+a_j}\varphi + U. \end{aligned} \tag{24}$$

The problem to define the quadratic form $q_\lambda[\varphi,\varphi]$ can now be seen as the problem to extend the linear bounded functional φ defined originally on $\mathcal{H}_{n-2} \subset \mathbf{H}$ to the whole space $\mathbf{H}$. This extension is defined by $n-1$ independent real parameters c_j by the following equalities:

$$q_{-a_j}[\varphi,\varphi] \equiv \left\langle \varphi, \frac{1}{A+a_j}\varphi \right\rangle = c_j, \quad j = 1, 2, \ldots, n-1. \tag{25}$$

In the general situation, all these real parameters c_j can be chosen independently. Suppose that the operator A and the singular element φ are homogeneous (7). Then it is natural to look for the extensions of the functional φ which are also homogeneous with respect to the same unitary semigroup. For such extensions, the parameters c_j are not independent anymore

$$c_i = \left\langle \varphi, \frac{1}{A + a_i} \varphi \right\rangle = \left(\frac{a_j}{a_i} \right)^{1+\frac{2\theta}{\beta}} \left\langle \varphi, \frac{1}{A + a_j} \varphi \right\rangle = \left(\frac{a_j}{a_i} \right)^{1+\frac{2\theta}{\beta}} c_j. \tag{26}$$

To determine the parameters c_j, it is enough to determine the value of $q_\lambda[\varphi, \varphi]$ for one particular value of λ, for example $\lambda = -1$

$$\left\langle \varphi, \frac{1}{A + 1} \varphi \right\rangle = a_j^{1+\frac{2\theta}{\beta}} \left\langle \varphi, \frac{1}{A + a_j} \varphi \right\rangle. \tag{27}$$

Moreover, the following theorem holds:

THEOREM 1. *Let all normalization points $a_j > 0$ be pairwise different. Then the system of equations*

$$\begin{aligned} &1 + \sum_{i=1}^{n-1} b_i = 0, \\ &1 + \sum_{i=1}^{n-1} b_i a_i = 0, \\ &\qquad \dots \\ &1 + \sum_{i=1}^{n-1} b_i a_i^{(n-2)} = 0. \end{aligned} \tag{28}$$

has the unique solution $b_1, b_2, \dots, b_{n-1}$. Suppose that the homogeneous linear functional φ can be extended to **H** *preserving the homogeneity, then*

- *the extension is unique,*
- *the following sum is different from zero*

$$1 + \sum_{i=1}^{n-1} b_i a_i^{-1-\frac{2\theta}{\beta}} \neq 0; \tag{29}$$

- *the function*

$$\begin{aligned} q_{-1}[\varphi, \varphi] &\equiv \left\langle \varphi, \frac{1}{A + 1} \varphi \right\rangle \\ &= \frac{a_1 a_2 \dots a_{n-1} + \sum_{l=1}^{n-1} a_1 \dots a_{l-1} b_l a_{l+1} \dots a_{n-1}}{1 + \sum_{i=1}^{n-1} b_i a_i^{-1-\frac{2\theta}{\beta}}} \left\langle \varphi, \frac{1}{(A+1) \prod_{i=1}^{n-1} (A + a_i)} \varphi \right\rangle, \end{aligned} \tag{30}$$

  is independent of a_j.

Conversely, if (29) *holds and the function appearing in the right-hand side of* (30) *is independent of a_j, then the unique homogeneous extension is determined by* (30).

Proof. The determinant of the linear system (28)

$$D = \begin{vmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_{n-1} \\ \vdots & \cdots & \vdots \\ a_1^{(n-2)} & \cdots & a_{n-1}^{(n-2)} \end{vmatrix} \neq 0,$$

is a van der Monde determinant and it is different from zero if all a_j are pairwise different. Hence, the system (28) has a unique solution (and it is not trivial). For parameter b_j satisfying (28) using Hilbert identity, we get

$$\frac{1}{A+1} + \sum_{j=1}^{n-1} \frac{b_j}{A+a_j} = \frac{a_1 a_2 \dots a_{n-1} + \sum_{l=1}^{n-1} a_1 \dots a_{l-1} b_l a_{l+1} \dots a_{n-1}}{(A+1)\prod_{i=1}^{n-1}(A+a_i)}.$$

If φ can be extended as a homogeneous functional, then the last identity implies

$$\left\langle \varphi, \frac{1}{A+1}\varphi \right\rangle \left(1 + \sum_{i=1}^{n-1} b_i a_i^{-1-\frac{2\theta}{\beta}} \right)$$
$$= \left(a_1 a_2 \dots a_{n-1} + \sum_{l=1}^{n-1} a_1 \dots a_{l-1} b_l a_{l+1} \dots a_{n-1} \right) \left\langle \varphi, \frac{1}{(A+1)\prod_{i=1}^{n-1}(A+a_j)} \varphi \right\rangle. \tag{31}$$

The expression $a_1 a_2 \dots a_{n-1} + \sum_{l=1}^{n-1} a_1 \dots a_{l-1} b_l a_{l+1} \dots a_{n-1}$ appearing in the right-hand side of the last equality is different from zero. Suppose that it is equal to zero. Then elementary calculations together with (28) imply that b_j satisfy the following linear system

$$\sum_{i=1}^{n-1} b_i = -1, \quad \sum_{i=1}^{n-1} b_i a_i = -1, \dots, \sum_{i=1}^{n-1} b_i a_i^{(n-1)} = -1,$$

which has no solution for pairwise different a_j and we got a contradiction.

In addition for any positive operator A and positive normalizing points a_j the scalar product

$$\left\langle \varphi, \frac{1}{(A+1)\prod_{i=1}^{n-1}(A+a_j)} \varphi \right\rangle$$

is strictly positive. Therefore, if the homogeneous extension of φ exists, then inequality (29) is satisfied. In this case, the value of the quadratic form $q_{-1}[\varphi, \varphi]$ can be calculated using (30).

Conversely formula (30) determines a homogeneous extension of φ provided the function in the right-hand side does not depend on a_j. □

Remark. The theorem does not state that if Equation (29) is satisfied then there is a unique homogeneous extension of φ determined by (30).

COROLLARY 1. *If* $-2\theta = k\beta$, *where* $k = 1, 2, \dots n-1$, *then it is impossible to extend the functional* φ *preserving the homogeneity properties.*

Proof. If $-2\theta = k\beta$, where $k = 1, 2, \dots n-1$, then the sum $1 + \sum_{i=1}^{n-1} b_I a_i^{-1-\frac{2\theta}{\beta}}$ coincides with one of the sums from (28) and therefore is equal to zero. □

To calculate $q_{-1}[\varphi, \varphi]$ one can use the procedure used in Pauli–Villars regularization [29]. Since the right-hand side of (30) does not depend on the parameters a_j, $j = 1, 2, \dots, n-1$, one can scale these parameters $a_j \to s a_j$ and consider the limit when $s \to \infty$. Formula (30) can be rewritten in the form

$$\left(1 + \sum_{i=1}^{n-1} b_i(s)(sa_i)^{-1-\frac{2\theta}{\beta}}\right) q_{-1}[\varphi, \varphi]$$
$$= \left\langle \varphi, \frac{1}{(A+1)\prod_{i=1}^{n-1}(A + sa_i)} \varphi \right\rangle \times$$
$$\times \left(s^{n-1} a_1 a_2 \dots a_{n-1} + s^{n-2} \sum_{l=1}^{n-1} a_1 \dots a_{l-1} b_l a_{l+1} \dots a_{n-1} \right). \tag{32}$$

Let us determine the dependence of $b_j(s)$ on the scaling parameter s. These coefficients are solution to the following system:

$$\sum_{i=1}^{n-1} c_i = -1, \quad \sum_{i=1}^{n-1} c_i a_i = -\frac{1}{s}, \dots, \sum_{i=1}^{n-1} c_i a_i^{(n-2)} = -\frac{1}{s^{n-2}}, \tag{33}$$

Using Cramer formulas, the solution can be written as

$$b_i = -\sum_{v=0}^{n-2} \frac{b_i^{(v)}}{s^v}, \tag{34}$$

where $b_i^{(v)}$ solve the system

$$\sum_{i=1}^{n-1} b_i^{(v)} = \delta_{0v}, \quad \sum_{i=1}^{n-1} b_i^{(v)} a_i = \delta_{1v}, \dots, \sum_{i=1}^{n-1} b_i^{(v)} a_i^{(n-2)} = \delta_{(n-2)\,v}, \tag{35}$$

which is independent of s. Then the left-hand side of (32) possesses the representation

$$\left(1 + \sum_{l-1}^{n-1} \alpha_l s^{-\frac{2\theta}{\beta}-l} + \mathrm{O}(s^{-1})\right) q_{-1}[\varphi, \varphi], \quad \alpha_l \in \mathbf{R}. \tag{36}$$

The same representation holds for the right-hand side of (32). To calculate $q_{-1}[\varphi, \varphi]$, it is enough to determine the term of order zero in this expansion. This term can be calculated using different methods, for example residue calculus.

Let us consider the special case corresponding to the formal operator

$$L_\alpha = -\Delta + \alpha\langle \delta, \cdot \rangle \delta, \qquad \text{in } L^2(\mathbf{R}^d), \quad d = 3, 4, \dots \tag{37}$$

The perturbation is from the class $\mathcal{H}_{-n}$, $n = (d+1)^2$ and is supersingular if $d \geqslant 4$. The homogeneity parameters with respect to the group of scaling transformations are $\theta = -d/2,\ \beta = 2$. Using Fourier transform the diverging quadratic form can be written as

$$q_{-1}[\delta, \delta] = \left\langle \delta, \frac{1}{-\Delta + 1}\delta \right\rangle = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} \frac{1}{|k|^2 + 1}\,\mathrm{d}^d k.$$

Then the right-hand side of (32) is

$$\frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\left(s^{n-1}a_1 a_2 \ldots a_{n-1} + s^{n-2}\sum_{l=1}^{n-1} a_1 \ldots a_{l-1} b_l a_{l+1} \ldots a_{n-1}\right)\times$$
$$\times \int_0^\infty \frac{r^{\mathrm{d}-1}}{(r^2+1)(r^2+sa_1)\cdots(r^2+sa_{n-1})}\,\mathrm{d}r.$$

Expanding this function with respect to $s \to \infty$, we calculate the unique renormalized value of the quadratic form for odd values of d

$$\left\langle \delta, \frac{1}{-\Delta + 1}\delta \right\rangle = \frac{(-1)^{(d-1)}/2}{2^d \pi^{(d/2)-1}\Gamma(d/2)}. \tag{38}$$

For even values of d the functional δ cannot be extended preserving the homogeneity and our approach does not work. The method of dimensional regularization gives the same result. In particular, $q_{-1}[\delta, \delta] = -1/(4\pi)$ for $d = 3$ as we noted earlier.

The methods developed here can be applied to describe finite rank perturbations like it was done in [1, 6, 10, 24].

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